Low frequency oscillations trapped near the equator

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ABSTRACT

Two simple models illustrate the physical mechanism behind the trapping within a narrow band near the equator of low frequency, infinitesimal oscillations for which the motion is independent of longitude. A thin layer of liquid confined between two rotating spherical shells is in the first case homogeneous, in the second case stably stratified. In the equatorial ocean, one such oscillation would have a period of about ten days and be confined between 6° N and 6° S.

1. Introduction

In a recent paper, STERN (1963) discussed inertial waves in a rotating, homogeneous liquid confined between two concentric spheres of nearly equal radii, and showed that there exist standing oscillations, independent of longitude, of frequency $\Omega(nh/R)^{\frac{1}{2}}$, where Ω is the rate of rotation of the system, h the thickness of the liquid, R the mean radius of the spheres, and n a positive integer. His analysis assumed that $h/R \ll 1$, and he showed that the wave amplitude decreases to zero outside a band of order $(h/R)^{\frac{1}{2}}$ of latitude. The mathematics necessary to describe these modes is somewhat complex, and it is not easy to relate the phenomenon to more familiar concepts. Given below are two distinct simple models which, it is hoped, may help to provide a little insight into the mechanisms involved. The second model is strictly applicable only to a gravitationally stratified fluid but may be expected to apply qualitatively to the homogeneous case discussed by Stern. In fact, the stratified case is of more interest for geophysical applications, and the possibility may exist of such a standing oscillation in the equatorial ocean confined between about 6° N and 6° S with a period of around ten days. There remain a number of questions about this oscillation as yet unresolved, associated with the longitudinal boundaries and the equatorial countercurrent, and its possible existence remains speculation.

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2. Model 1—Ray Theory

As is well known, in a homogeneous liquid rotating with uniform angular velocity $\mathbf{\Omega}$, plane waves of vector wave number $\mathbf{k}$ can propagate with frequency ω given by

$$\omega = \frac{1}{k} 2\,\mathbf{\Omega} \cdot \mathbf{k}.$$

This frequency is independent of the magnitude k of $\mathbf{k}$ (i.e., of the wavelength) but does depend on the angle θ between the rotation vector and the plane of the wavefront (Fig. 1):

$$\omega = 2\Omega \sin \theta.$$

The group velocity $\mathbf{c}$ of such waves is perpendicular to $\mathbf{k}$, and thus to the phase velocity, and is given by (WHITHAM, 1960) the vector derivative of $\omega(\mathbf{k})$ in wave number space,

$$\mathbf{c} = \nabla_{\mathbf{k}}\, \omega(\mathbf{k}) = \frac{1}{k}\left\{2\,\mathbf{\Omega} - \frac{1}{k}(2\,\mathbf{\Omega}\cdot\mathbf{k})\,\mathbf{k}\right\}.$$

Though for a strictly infinite plane wavefront points of constant phase move perpendicular to it, the energy associated with the wavefront, together with slight non-uniformities in it, propagates in the plane of the front along the projection of the rotation vector, with speed $2\Omega \cos \theta / k$.

We now idealize the equatorial regions of the spherical shell by part of two infinitely long concentric cylinders, and consider only plane waves with wave number perpendicular to the cylinder axis. The liquid motion is thus in-

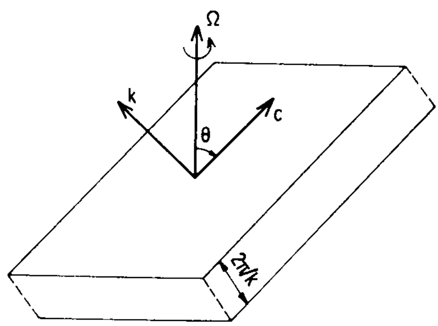

FIG. 1. Wave number, **k**, and group velocity, **c**, for a plane inertial wave in a homogeneous rotating liquid.

dependent of the distance x along the generators of the cylinder (i.e., independent of longitude). We are here neglecting the curvature of latitude circles, and are replacing them by straight lines.

In the first model, we consider only waves of wavelength $2\pi/k$ very much less than the difference h between the radii of the cylinders. A ray theory is then appropriate to discuss the energy propagation and reflection of such waves, the direction of the ray always being parallel to the local group velocity. The lowest order trapped mode is described by the ray, $ABCD$ (Fig. 2). The ray AB makes an angle θ with the rotation axis, and is associated with a wavefront in the plane formed by AB and a line through A parallel to the cylinder axis, but of arbitrary wavelength. This wave has frequency $2\Omega \sin \theta$. On reaching the outer wall at B, the wave is reflected and its frequency is unchanged. Hence, the reflected wave BC also

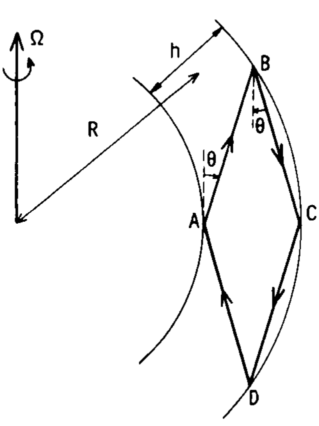

FIG. 2. Ray path for trapped oscillation of lowest order.

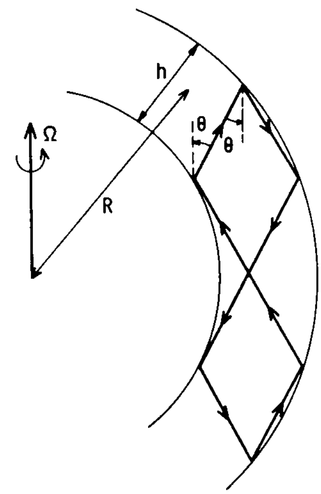

FIG. 3. Ray path for the mode $n = 3$.

makes the same angle θ with the rotation axis. C, like A, lies in the plane of the equator, so that $AB = BC$. At C and D the wave is reflected again in a symmetrical manner to return to A. The distance of the point B from the equatorial plane is the distance to the horizon from a point $h/2$ above the surface of the earth (at least if $h/R \ll 1$). This is $(hR)^{\frac{1}{2}}$, and is in qualitative agreement with the north-south penetration described by Stern. Likewise, the frequency is $2\Omega \sin \theta = \Omega(h/R)^{\frac{1}{2}}$, and is exact. The rays for another, more complex mode are shown in Fig. 3. It may be verified that this corresponds to $n = 3$.

It is remarkable that this analysis gives quantitatively comparable results. The ray paths shown coincide with characteristics in the coordinate system used by Stern. However, the assumption that $kh \gg 1$ is apparently essential here, whereas the previous work is valid for $kh \sim 1$. In general, a wave travelling round the ray path shown in Figure 2 will not return to the point A with its original phase. However, the time of travel is proportional $kh(h/R)^{-\frac{1}{2}}\Omega^{-1}$, whereas the wave period $\sim (h/R)^{\frac{1}{2}}\Omega^{-1}$. Thus, if kh is large, a small proportional change in k is needed to make the travel time change by as much as a complete period, and there will be an asymptotically dense set of suitable values of k. That these modes are among those previously described by Stern may be seen by replacing h by h/N, and n by Nn, where N is a large positive integer. Allowing for a change of notation, reference to Stern's analysis will show that the necessary equations are all satisfied in and on the boundaries of a liquid layer of thickness h, as well as for the thickness h/N for which the analysis was designed. However, the frequency of the corresponding very high order mode is independent of N, and that is why the agreement is apparently exact even when $N=1$.

In the second model, we consider a mode of oscillation which has only one node in the radial direction. This corresponds to a long wave approximation, whereas in this section we have considered short waves. However, the correspondence with a special case of Stern's analysis is no longer exact. A rather crude approximation is required which can be justified only in a qualitative way. However, it is felt that the model is of interest both by way of illustration and in its own right, although it is not entirely novel.

3. Model 2—Long Waves

We consider at first long internal waves in the interface between two liquids of equal depth but slightly different densities confined between two rigid level surfaces a distance h apart at the surface of the earth. Assuming as before that $h/R \ll 1$, we will develop the equatorial regions of these level surfaces onto planes. If $\Omega^2 h/g^* < 1$, where g^* is the reduced gravity appropriate to the two liquids (the true gravitational acceleration multiplied by the ratio of the density difference to the mean density), then only the vertical component $\frac{1}{2}f$ of the rotation vector Ω is of significance, and the stratification is "strong". The physical meaning of this condition is that the baroclinicity vector causes changes in the horizontal component of vorticity much larger than those associated with rotation; but because the vertical component of the baroclinicity vector vanishes identically, the rotation may not be neglected when considering horizontal gradients of horizontal velocity. We will, however, allow the Coriolis parameter f to vary with latitude, $f=2\Omega y/R$.

For waves in which the motion and the pressure field are independent of x (Fig. 4), a fluid particle moved horizontally parallel to Oy through a distance Y_1 is accelerated in the Ox direction by the Coriolis force until it has a velocity fY_1. This, in turn, is associated with a

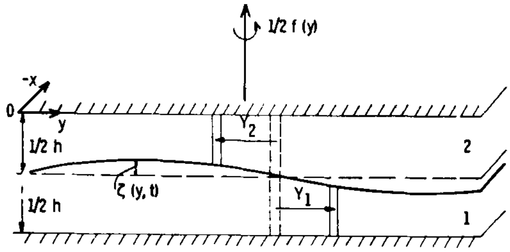

Fig. 4. Two layer stably stratified model confined between two horizontal planes.

force $-f^2 Y_1$ tending to restore it to its original latitude (this will hereafter be termed the rotation inertial force). The equations of motion for the particles in the lower and upper layers are thus:

$$\frac{d^2 Y_1}{dt^2} + f^2 Y_1 + \frac{\partial p_1}{\partial y} = 0$$

$$\frac{d^2 Y_2}{dt^2} + f^2 Y_2 + \frac{\partial p_2}{\partial y} = 0,$$

where $p_1(y,t)$, $p_2(y,t)$ are the departures from equilibrium hydrostatic pressure divided by the liquid density (in this context assumed the same for each layer according to the usual Boussinesq approximation). As usual for long (shallow water) waves, the pressure difference $p_1 - p_2$ is given solely in terms of the instantaneous displacement of the interface $\zeta(y,t)$

$$p_1 - p_2 = g^* \zeta.$$

The last equation required is that of continuity

$$\frac{\partial \zeta}{\partial t} = -\frac{h}{2}\frac{\partial}{\partial y}\left(\frac{dY_1}{dt}\right) = \frac{h}{2}\frac{\partial}{\partial y}\left(\frac{dY_2}{dt}\right).$$

For infinitesimal waves, these five equations are easily combined into a single wave equation in terms of the difference $\eta = Y_1 - Y_2$ in horizontal displacement between the layers;

$$\frac{\partial^2 \eta}{\partial t^2} + f^2(y)\,\eta - \frac{g^* h}{4}\frac{\partial^2 \eta}{\partial y^2} = 0. \qquad (1)$$

It should be noted that at no stage has even a relative constancy of $f(y)$ been assumed. Equation (1) is valid for disturbances on any scale much larger than h, even if $f(y)$ changes substantially over that scale, provided only that they are independent of longitude x.

To describe the trapping of waves near the

the equator in a strongly *stratified* fluid, we take $f(y) = 2\Omega y/R$, and consider oscillations of frequency ω, $\eta = \hat{\eta}(y)e^{-i\omega t}$. Then ω is determined as an eigenvalue, for which the equation

$$\frac{g^* h}{4}\frac{d^2\hat{\eta}}{dy^2} + \left\{\omega^2 - 4\frac{\Omega^2}{R^2}y^2\right\}\hat{\eta} = 0 \qquad (2)$$

has solutions which tend to zero as $y \to \pm\infty$. The solutions of Equation (2) are oscillatory in character if $|y| < R\omega/2\Omega$, and exponential in character outside this range. The waves will be trapped between the two parallels of latitude outside which the Coriolis parameter becomes greater than the wave frequency. A simple explanation for this is that both gravitational forces and Coriolis forces tend to restore a particle to its original latitude. There is thus a minimum frequency for the oscillation at a given latitude, namely, that of purely horizontal motions under the action of Coriolis forces. This frequency equals the Coriolis parameter $f(y)$. A wave travelling northwards or southwards must conserve its frequency, and cannot penetrate into a region where the Coriolis parameter exceeds it, and so the wave is reflected and a standing oscillation is possible. To put Equation (2) in dimensionless form, we write

$$c^2 = g^* h/4.$$

c is then the velocity of long waves at the equator. Also, if

$$y^* = \left(\frac{2\Omega R}{c}\right)^{\frac{1}{2}}\frac{Y}{R}, \quad S = \frac{1}{2}\frac{R\Omega}{c}\frac{\omega^2}{\Omega^2},$$

then

$$\frac{d^2\eta}{dy^{*2}} + (S - y^{*2})\,\eta = 0, \qquad (3)$$

is the governing equation for the interface. This equation has solutions which vanish at $y^* = \pm\infty$ if, and only if, S is an odd integer $2n+1$. The solutions are then Hermite functions. The frequency of the equatorial oscillations is determined by

$$\omega = 2\Omega\{(n+\tfrac{1}{2})\,c/R\Omega\}^{\frac{1}{2}}. \qquad (4)$$

The lowest order mode, $n = 0$, is described by

$$\eta = e^{-\frac{1}{2}y^{*2}}.$$

In a *homogeneous* liquid, it is no longer permissible to neglect the horizontal component of rotation. The dynamical effects of the vertical and horizontal components are closely interrelated, but the following argument suggests that in a crude qualitative manner they may be treated as if they were separate. There is then a tendency for a particle displaced through a vertical distance Z, to be restored to its original position with an acceleration $4\Omega^2 Z$, due to the horizontal component of rotation. A layer of liquid thus behaves as if it were stably stratified, with stability frequency 2Ω, and long waves will propagate in a north-south direction with speed $2\Omega h/\pi$. If we substitute this value for c in the previous analysis, everything proceeds exactly as before, leading to oscillations of frequency

$$\omega = \Omega\left\{\frac{4}{\pi}(2n+1)\frac{h}{R}\right\}^{\frac{1}{2}}, \qquad (5)$$

trapped between latitudes of order $(h/R)^{\frac{1}{2}}$. The correspondence with Stern's result is no longer exact, but the variation with the parameter h/R is the same.

As far as it goes, the justification for separating the horizontal and vertical parts of the motion is seen by considering the restoring force on a particle displaced a distance Y northwards and Z vertically. Provided y/R is small, this force has a horizontal component

$$4\Omega^2\{Y(y/R) - Z\}y/R$$

and a vertical component

$$4\Omega^2\{-Yy/R + Z\}.$$

Now continuity requires that, at least for low values of the order n, the terms Yy/R and Z shall be comparable in magnitude, and preferential neglect of a different one in each component does not appear reasonable. However, it should be noticed that the oscillations in the two layer model depend only on the difference in horizontal displacement between the upper and lower layers, and on the mean of the vertical displacements. The mean of the horizontal displacements and the difference of the vertical ones vanishes by continuity. On performing these mean and difference operations on the restoring force in the homogeneous rotating liquid, *insofar as the motion may be regarded as that of two distinct layers*, we are left with $4\Omega^2(y^2/R^2)(Y_1 - Y_2)$ for the effective horizontal component, and $4\Omega^2\frac{1}{2}(Z_1 + Z_2)$ for the vertical one. This is exactly what has been assumed in the two layer model above.

That the results of the above analysis are not strictly accurate implies that the horizontal and vertical displacements cannot justifiably be separated in this way. This conclusion will not surprise anyone who has worked with the exact equations, but nevertheless there is tendency towards separability in this case. We see that there is no longer a minimum restoring force on a displaced particle which is greater than or equal to the square of the Coriolis parameter. If $Z/Y = y/R$ the restoring force vanishes, but is easily seen that as y/R increases such displacements by most particles become steadily less compatible with the constraints of the horizontal bounding surfaces. Insofar as the particles are constrained to move almost horizontally away from the equator, the oscillation will be trapped between those latitudes for which the Coriolis parameter is less than its frequency. But the possibility of vertical motion near the equator gives a definite wave velocity for northward or southward moving waves. These two considerations determine the frequency and latitudinal penetration.

4. Trapped Oscillations in a General Stratified Medium

The percipient reader will already have guessed that Equation (2) is a special case of Laplace's tidal equation. The necessary approximations for its derivation are first that the pressure is everywhere instantaneously given by the hydrostatic value, and second that the horizontal component of the earth's rotation may be ignored. These conditions are very familiar and are, in general, valid even at the equator at least for waves of horizontal scale much larger than their vertical scale in a medium for which the Brunt-Väisälä frequency is much larger than the rotation rate. The full linearized equations of motion are then separable, and the horizontal distribution of velocities is determined by the tidal equation, into which enters the wave velocity c for waves in the corresponding non-rotating medium. The zonal oscillations described here have been given previously in this context by Eckart (1960). They are not similar to Rossby waves, in spite of their long period, because the variation in the Coriolis parameter plays no part in the restoring force, only in the trapping of the oscillations.

It is instructive to insert orders of magnitude suitable for geophysical applications. The relevant parameter is $c/2\Omega R$, the ratio of the wave speed to twice the tangential velocity of the earth at the equator. In the atmosphere $c \sim 300$ m/s, the parameter is near unity, and the oscillation would extend over most of the earth. Under such circumstances, the above analysis is scarcely applicable. However, if we consider baroclinic internal waves in the equatorial ocean, and take a rough value of c as 2 m/sec, then $c/2\Omega R \sim 1/400$, the period of the lowest order oscillation would be ten days, and it would be confined between about latitudes 6° N and 6° S. At this stage the influence of longitudinal boundaries must be considered. Equation (2) is a good approximation provided the longitudinal scale of variation of the motion is large compared to the latitudinal one. This is easily satisfied, but, in general, variations in the amplitude of the oscillation will propagate round the equator with the group velocity. Although a strong equatorial countercurrent does not affect Equation (2), being perpendicular to all the velocity gradients, it will interact with the group velocity. Finally, variations in the depth of the thermocline and the ocean will cause longitudinal changes in the wave velocity c. These questions need to be answered before any final answer can be given to the possibility of a standing oscillation. They require more sophisticated mathematics which would be outside the spirit of this paper. However, assuming slow longitudinal variations, they appear to be tractable, and investigations are continuing.

In conclusion, the author would like to express the feeling of insight which a study of these simple models has conveyed to him. He must also acknowledge a great debt to Mr. Blyth Hughes, of the University of Cambridge, who originally made him aware of the reflection at a critical latitude of internal gravity waves of period longer than twelve hours, and stimulated him into explaining it in simpler terms.

REFERENCES

Eckart, C., 1960, *Hydrodynamics of Oceans and Atmospheres*, Pergamon, New York, p. 258.

Stern, M. E., 1963, Trapping of low frequency oscillations in an equatorial boundary layer, *Tellus*, Vol. **15**, pp. 246–250.

Whitham, G. B., 1960, A note on group velocity, *J. Fluid Mech.* **9**, p. 347.