SHORTER CONTRIBUTION

An overlooked aspect of the baroclinic stability problem

By NORMAN A. PHILLIPS, *Massachusetts Institute of Technology*

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The behavior of small disturbances of a stratified heavy fluid in nonuniform rotation is well known to be a complex subject, even if additional physical complications such as compressibility, non-plane boundaries, and irreversible effects are ignored. But because of its relevance to atmospheric motions, the subject has been studied extensively and a considerable literature exists. All such studies known to the writer have however involved assumptions beyond those of the type mentioned above.

The most fruitful assumption has been that introduced by CHARNEY (1947) and EADY (1949), wherein attention is restricted to basic flows characterized by quasi-uniform rotation and a large stable density stratification. In the mathematical analysis, this appears formally as a requirement that the Rossby number Ro be small (so the horizontal motion is nearly geostrophic) and the Richardson number Ri be large. These conditions are generally satisfied in the extra-tropical atmosphere, and the theoretical results have therefore been of great utility in understanding the general circulation and in numerical weather prediction. Interpretation of laboratory experiments with heated rotating basins has also been aided by this theory (LORENZ, 1962; BARCILON, 1964).

An additional type of assumption is frequently employed, however, in baroclinic stability studies which are not restricted to quasi-geostrophic flow. This is the purely *ad-hoc* neglect of perturbation dependence on the horizontal coordinate y normal to the basic current $\bar{u}(z)$. Some recent examples are GATES (1961), ARAKAWA (1962), WIIN-NIELSEN (1963), and ÁRNASON (1963). The purpose of this note is to suggest that this is an extremely bad approximation, one which under normal circumstances will ignore the most important non-geostrophic effects.

To show this most clearly, let us consider the stability problem treated by EADY (1949), in which the compressibility of the atmosphere and the spherical shape of the earth is neglected. This may also be interpreted as duplicating the conditions encountered in laboratory experiments where a rotating annulus of rectangular cross section is heated and cooled at its outer and inner radii (HIDE, 1958; FULTZ, 1959). If the mean radius is large compared to the width W of the annulus, Cartesian geometry is a reasonable approximation.

The simplest baroclinic stability problem under these circumstances is essentially that treated by Eady, in which the basic current $\bar{u}$ relative to the annulus (in the x-direction) is a linear function only of z, varying from $-V/2$ at the bottom of the annulus to $+V/2$ at the top. The basic temperature field $\bar{T}$ is a linear function of y and z, such that $\partial \bar{T}/\partial y$ is in geostrophic balance with $d\bar{u}/dz$. (It is convenient here to use the incompressible Boussinesq approximation, in which density variations are important only in the buoyancy force, and the (variable) density is set equal to αT, α being the temperature expansion coefficient.)

EADY (1949) has derived the partial differential equation for the vertical velocity w (assuming hydrostatic balance). We will express this equation in non-dimensional form, using horizontal and vertical length scales of W and H (representing the width and height of the annulus) and the time scale, W/V. The constant B measures the static stability:

$$B = \frac{gH^2}{f^2 W^2}\alpha \frac{\partial \bar{T}}{\partial z}$$

and the Rossby number is

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$$\mathrm{Ro} = \frac{V}{fW}.$$

$f = 2\Omega$ is the Coriolis parameter. Let D and ∇^2 represent the (non-dimensional) operators

$$D = \partial/\partial t + \bar{u}\,\partial/\partial x = \partial/\partial t + z\,\partial/\partial x,$$

and $$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

The non-dimensional linearized form of Eady's equation (II, 7) is

$$D\left[(1+\mathrm{Ro}^2D^2)\frac{\partial^2 w}{\partial z^2} + B\nabla^2 w + 2\mathrm{Ro}\frac{\partial^2 w}{\partial z\,\partial y}\right] - 2\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial z} + \mathrm{Ro}\frac{\partial w}{\partial y}\right] = 0. \quad (1)$$

The *approximate* solution obtained by Eady results from discarding the terms containing Ro and Ro^2:

$$D\left[\frac{\partial^2 w}{\partial z^2} + B\nabla^2 w\right] - 2\frac{\partial^2 w}{\partial z\,\partial x} = 0, \quad (2)$$

an equation equivalent to Eady's (II, 21).[1]

An important feature of (2), in distinction to (1), is that it is readily separable in y and z, after the usual assumption of exponential variation in x and t is introduced. The eigenvalue problem then reduces to the solution of an ordinary differential equation in z. Furthermore, the relation between w and the transverse (y) velocity v is simple enough (to this order of accuracy) that trigonometric variation in y suffices to satisfy the boundary condition of $v = 0$ at $y = \pm\frac{1}{2}$. The x- and y-wave numbers then appear in the equation for the phase speed c only as the sum of their squares, showing that the y-variation plays only a *quantitative* role in (2) (i.e. if it were ignored in (2), only a quantitative error in the frequency equation would result).

A completely different problem exists if we wish a more accurate solution of (1). Since the geophysical contexts giving rise to the problem of baroclinic instability are characterized by Ro less than one, it is appropriate to proceed by considering the next higher order terms in Ro. For example, if the energetically consistent form of the "balance equations" is applied to this problem (Lorenz, 1960; Charney, 1962, pages 136–142), the equivalent of (1) becomes

$$D\left[\frac{\partial^2 w}{\partial z^2} + B\nabla^2 w + 2\mathrm{Ro}\frac{\partial^2 w}{\partial y\,\partial z}\right] - 2\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial z} + \mathrm{Ro}\frac{\partial w}{\partial y}\right] = 0, \quad (3)$$

i.e. only the Ro^2 term disappears from (1). The y-variation is now much more significant than it was for equation (2). In fact, we see that if we ignore the y-variation in (3), the Ro terms disappear and the equation reduces to (2)!

If instead of using the balance equations, we were to introduce the assumption $\partial/\partial y = 0$ directly into (1), the Ro terms would disappear while the Ro^2 term would remain. Some non-geostrophic effects would thereby be retained. However, the Ro terms are presumably more important than the Ro^2 term for motions of meteorological interest.

In addition to this Rossby number argument for retaining y dependence, the kinematic restraint imposed by lateral (y) limitation of the motion is undoubtedly important. This constraint is very sharp in the present example of baroclinic instability, where the width (W) of the annulus has even been used to define the Rossby number. In the atmospheric situation, latitudinal variation of the basic current $\bar{u}$ and the spherical geometry of the earth will both impose important lateral constraints which should not be ignored. These too will require that lateral dependence of the perturbation solutions be recognized.

We are therefore led to the following uncomfortable conclusion:

Non-geostrophic effects in the baroclinic stability problem, beyond those already incorporated in the quasi-geostrophic equations, depend critically on a correct treatment of the transverse coordinate.

Although this conclusion is based on a very simple physical model, we may expect that it also holds if compressibility, spherical boundaries, and dependence of $\bar{u}$ on y are added to the problem. Further analysis of non-geostrophic baroclinic instability therefore presents mathematical problems of a nature not likely to be solved by methods now in use.

[1] An alternate derivation of Eady's solution is to use the quasigeostrophic potential vorticity equation (Phillips, 1963, pages 149–150).

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