An initial value problem in the theory of baroclinic instability

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ABSTRACT

A discussion of the initial value problem for the baroclinic model of Eady is presented. The completeness of the eigen-solutions is discussed. It is shown that a continuous spectrum of waves is required to supplement the normal mode solutions in order to represent the evolution of an arbitrary disturbance, but that the stability properties of the flow are properly described by the normal mode solutions.

The analysis of the stability of baroclinic flows is usually based on a search for wave solutions, or normal modes, of the perturbation quasi-geostrophic potential vorticity equation (see for example CHARNEY (1947), EADY (1949), CHARNEY and STERN (1962)). The problem is considered an eigenvalue problem, where c, the phase speed, is considered the eigenvalue. If a solution is found for some values of the external parameters for which c is complex, the flow is considered unstable. The implicit assumption is made that *any* arbitrary initial disturbance can be represented by a synthesis of these normal modes, so that the stability of the flow to an arbitrary disturbance can be directly related to the stability properties of the normal modes. In certain stability problems, such as Bernard convection, such an assumption is valid, for the normal modes do form a complete set and can thus be used to represent an arbitrary initial disturbance.

Generally speaking, in the investigation of the stability of shear flows to inviscid disturbances, the normal modes which correspond to a discrete spectrum of c eigenvalues are not complete. The mathematical difficulty of these problems usually precludes a discussion of the completeness properties of the eigensolutions. On the other hand, if there are solutions which do not correspond to the discrete normal mode spectrum, it is clear that the study of the behavior and stability of this additional part of the spectrum, which is generally a continuous one, is essential to a complete understanding of the stability problem for two reasons. First and most obvious, we wish to know whether these new additional modes are stable or unstable, especially in parameter ranges where the normal modes are stable, for if they are unstable in regions where the normal modes are stable, a stability diagram based only on the normal mode solutions would be extremely misleading. Secondly, if the normal modes are not complete we must investigate in what manner it will be necessary to supplement the discrete spectrum in order to represent the time evolution of an arbitrary disturbance, and to see the relationship between the eigen-solutions and the initial value problem.

In fact the most straightforward way to find these additional solutions is to consider the initial value problem for an arbitrary initial condition and identify that portion of the solution which cannot be represented by the normal modes as the additional solution.

A simple problem in baroclinic instability which illustrates these points and yet is simple enough to clearly demonstrate the relation between the initial value problem and the eigenvalue problem can be found in the model proposed by EADY (1949) to study baroclinic instability. He discussed the stability of a current with no horizontal shear, constant vertical shear, constant density and static stability, with respect to disturbances independent of the horizontal direction perpendicular to the mean current. He also ignored the planetary vorticity gradient. Furthermore the flow is confined between two rigid horizontal surfaces.

The perturbation potential vorticity equation in non-dimensional variables, can be written with these assumptions as (CHARNEY and STERN, 1962):

$$\left(\frac{\partial}{\partial t}+u(z)\frac{\partial}{\partial x}\right)(\Phi_{xx}+\varepsilon\Phi_{zz})=0. \qquad (1\,a)$$

Φ is the perturbation stream function, $u(z)$ is the basic zonal current, and ε, which is a constant, is a measure of the static stability and in the following will be taken, without loss of generality, equal to unity.

The boundary conditions are:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\Phi_z-u_z\Phi_x=0 \quad \text{at} \quad z=0,1 \qquad (1\,b)$$

and $u(z)$ is taken to be equal to λz.

In the usual manner Eady searched for wave solutions in the form

$$\Phi=\varphi(z)\,e^{ik(x-ct)} \qquad (2)$$

leading to the eigenvalue problem:

$$\varphi_{zz}-k^2\varphi=0, \qquad (3\,a)$$

$$(\lambda z-c)\varphi_z-\lambda\varphi=0 \quad \text{at} \quad z=0,1. \qquad (3\,b)$$

The solutions of (3) may be written

$$\varphi=\sinh kz+B(c)\cosh kz \qquad (4)$$

with the relations

$$B(c)=-kc/\lambda, \qquad (5\,a)$$

$$B(c)\{\lambda\cosh k-(\lambda-c)\,k\sinh k\}$$
$$=\{(\lambda-c)\,k\cosh k-\lambda\sinh k\}, \qquad (5\,b)$$

which yield a quadratic equation for c. For each value of k Eady found two solutions, corresponding to values of c given by

$$-ikc_1=-\frac{ik\lambda}{2}+\lambda\sqrt{\left(\frac{k}{2}-\tanh\frac{k}{2}\right)\left(\coth\frac{k}{2}-\frac{k}{2}\right)}, \qquad (6\,a)$$

$$-ikc_2=-\frac{ik\lambda}{2}-\lambda\sqrt{\left(\frac{k}{2}-\tanh\frac{k}{2}\right)\left(\coth\frac{k}{2}-\frac{k}{2}\right)}. \qquad (6\,b)$$

The relations (5) determine $B(c)$ with the use of (6) for the two modes. Thus for $k<k_c$ the first solution corresponding to c_1 will be unstable growing exponentially with time, while the other will exponentially decay; k_c being defined by the relation

$$\frac{k_c}{2}=\coth\frac{k_c}{2}. \qquad (7)$$

For $k>k_c$ both solutions will be neutral.

Now it is clear that since there are only two solutions for each value of k, the normal mode solutions (4) cannot form a complete set and thus can obviously not represent an arbitrary initial disturbance. To find the complete solution we shall formulate the problem as an initial value problem in line with our earlier remarks. Returning to (1) we introduce the Fourier transform variable:

$$\Psi=\int_{-\infty}^{\infty}\Phi e^{-ik(x)}\,dx=\Psi(k,z,t) \qquad (8)$$

and the Laplace transform

$$\varphi=\int_0^{\infty}e^{-pt}\,\Psi\,dt=\varphi(k,z,p). \qquad (9)$$

Using (1) it follows that φ must satisfy the equation:

$$\varphi_{zz}-k^2\varphi=\frac{-v(k,z)}{p+ik\lambda z}, \qquad (10\,a)$$

$$(p+\lambda zik)\,\varphi z-ik\lambda\varphi=m(kz) \quad \text{at} \quad z=0,1 \qquad (10\,b)$$

where $$v(k,z)=(k^2\Psi-\Psi_{zz})_{t=0}, \qquad (10\,c)$$

$$m(k,z)=(\Psi_z)_{\left\{\substack{z=0,\,t=0\\ z=1,\,t=0}\right\}}. \qquad (10\,d)$$

To solve (10) we introduce Green's function $G(z,z')$ defined by

$$G_{zz}-k^2G=-\delta(z-z'), \qquad (11\,a)$$

$$(p+\lambda ikz)G_z-\lambda ikG=0 \quad \text{at} \quad z=0,1. \qquad (11\,b)$$

The solution of (10) can then be written in terms of $G(z,z')$ as:

$$\varphi(z)=\int_0^1\frac{G(z,z')\,v(z')\,dz'}{(p+\lambda iz'k)}+\left.\frac{G(z,z')\,m(z')}{(p+\lambda z'ik)}\right]_{z'=0}^{z'=1}. \qquad (12)$$

Green's function can be found by standard methods and can be shown to be

$$G(z,z') = \frac{p(p+\lambda ik) \times \left[\cosh kz_< + \frac{i\lambda}{p}\sinh kz_<\right]\left[\cosh k(z_> - 1) + \frac{i\lambda}{p+ik}\sinh k(z_> - 1)\right]}{(k\sinh k)(p-p_1)(p-p_2)}, \tag{13}$$

where $z_<$ is the lesser of z and z' while $z_>$ is the larger and where

$$\left.\begin{aligned} p_1 &= -\frac{i\lambda k}{2} + \lambda\sqrt{\left(\frac{k}{2} - \tanh\frac{k}{2}\right)\left(\coth\frac{k}{2} - \frac{k}{2}\right)}, \\ p_2 &= -\frac{i\lambda k}{2} - \lambda\sqrt{\left(\frac{k}{2} - \tanh\frac{k}{2}\right)\left(\coth\frac{k}{2} - \frac{k}{2}\right)}. \end{aligned}\right\} \tag{14}$$

A comparison of (14) with (6) shows that Green's function has simple poles in the p plane which will yield, upon inversion, contributions to the solution whose time dependence will be precisely that of the Eady normal modes. The function Ψ can be written

$$\Psi = \frac{1}{2\pi i}\int_{-\infty i+}^{\infty i+} e^{pt}dp \times \left[\int_0^1 \frac{G(z,z')\,v(z')\,dz'}{(p+\lambda ikz')} + \left[\frac{G(z,z')\,m(z')}{(p+\lambda ikz')}\right]_{z'=0}^{z'=1}\right]. \tag{15}$$

If the p integration is performed first we observe that the integration can be replaced by the sum of contribution of three poles. Two of these are at the "intrinsic" poles of the Green's function, which correspond to the Eady modes, while the third is a contribution from the pole at

$$p = -ik\lambda z'. \tag{16}$$

Now since z' is a continuous variable on the interval $(0,1)$ there are really an infinite number of such contributions, continuously distributed, which will be included by the integration over z'. The Laplace inversion yields

$$\begin{aligned} \Psi = e^{p_1 t}\vartheta\Bigg\{ & \frac{\int_0^1 dz'(p_1)(p_1+i\lambda k)\left(\cosh kz_< + \frac{i\lambda}{p_1}\sinh kz_<\right)\left(\cosh k(z_> - 1) + \frac{i\lambda}{p_1+\lambda ik}\sinh k(z_> - 1)\right)v(z')}{(k\sinh k)(p_1-p_2)(p_1+\lambda iz'k)} \\ & + \left[\frac{(p_1)(p_1+i\lambda k)\left(\cosh kz_< + \frac{i\lambda}{p_1}\sinh kz_<\right)\left(\cosh k(z_> - 1) + \frac{i\lambda}{p_1+\lambda ik}\sinh k(z_> - 1)\right)m(z')}{(k\sinh k)(p_1-p_2)(p_1+\lambda iz'k)}\right]_{z'=0}^{z'=1}\Bigg\} \\ + e^{p_2 t}\Bigg\{ & \frac{\int_0^1 dz'(p_2)(p_2+i\lambda k)\left(\cosh kz_< + \frac{i\lambda}{p_2}\sinh kz_<\right)\left(\cosh k(z_> - 1) + \frac{i\lambda}{p_2+\lambda ik}\sinh k(z_> - 1)\right)v(z')}{(k\sinh k)(p_2-p_1)(p_2+\lambda iz'k)}\Bigg\} \\ & + \left[\frac{(p_2)(p_2+i\lambda k)\left(\cosh kz_< + \frac{i\lambda}{p_2}\sinh kz_<\right)\left(\cosh k(z_> - 1) + \frac{i\lambda}{p_2+i\lambda k}\sinh k(z_> - 1)\right)m(z')}{(k\sinh k)(p_2-p_1)(p_2+\lambda iz'k)}\right]_{z'=0}^{z'=1} \\ & + \frac{\int_0^1 dz'e^{-i\lambda kz't}\lambda^2k^2z'(1-z')\left(\cosh kz_< - \frac{\sinh kz_<}{kz'}\right)\left(\cosh k(z_> - 1) + \frac{\sinh k(z_> - 1)}{k(1-z')}\right)v(z')}{(k\sinh k)(p_1+i\lambda z'k)(p_2+i\lambda z'k)} \end{aligned} \tag{17}$$

Thus the time evolution of an arbitrary initial disturbance can be represented by the sum of three terms. Since

$$\begin{aligned} & \left(\cosh kz_< + \frac{i\lambda}{p_1}\sinh kz_<\right) \times \left(\cosh k(z_> - 1) + \frac{i\lambda}{p_1+i\lambda k}\sinh k(z_> - 1)\right) \\ & = (B(c_1)\cosh kz + \sinh kz) \times (B(c_1)\cosh kz' + \sinh kz') \times \left(\cosh k - \frac{i\lambda\sinh k}{p_1+\lambda ik}\right)\lambda^2/k^2c^2 \end{aligned} \tag{18}$$

by (5) and (14), it follows that the first term of (17) is a simple multiple of the first Eady normal mode, while in the same manner the second term is also a constant multiple of the second Eady mode. The constants are determined by the projection of the particular initial conditions upon these two normal modes. The third term corresponds to a superposition of an infinite number of waves, each one moving with the velocity of the zonal current at some point z'. This is the continuous spectrum that is necessary to supplement the two discrete normal modes. The two discrete normal modes, aside from a multiplicative constant, are seen to be given by the residues of Green's function at its poles p_1 and p_2.

For a discussion of the asymptotic character of the solution it is convenient to discuss separately the cases $k < k_c$ and $k > k_c$.

Case I $k < k_c$

For this long wave part of the spectrum the Eady normal modes are unstable, that is p_1 and p_2 are complex by (14). The p_1 mode exponentially grows while the p_2 mode exponentially decays. Now since p_1 and p_2 are complex, the integrands in all the integrals for Ψ', and particularly for the continuous spectrum are non singular on the range of integration so that by the Riemann–Lebesgue Lemma (see for example JEFFREYS and JEFFREYS, 1953) the asymptotic character of the contribution from the continuous spectrum must decay algebraically at least as fast as t^{-1} for large t. Thus, for that part of the k spectrum for which the Eady normal modes are unstable, the continuous wave spectrum is stable and algebraically decays for large t. For large t, only the unstable mode will be observed. However, the continuous spectrum of waves will in general make a larger contribution than the p_2 mode. Of course, both are negligible compared to the unstable p_1 mode.

Ccse II $k > k_c$

For this part of the zonal wave number spectrum the normal modes are neutral since p_1 and p_2 are purely imaginary. Each of the individual integrals in (17) has a singular integrand. The first at $z_1' = -p_1/ik\lambda$, the second at $z_2' = -p_2/i\lambda k$, while the continuous spectrum contribution has pole singularities at z_1' and z_2'. It is not difficult to show that z_1' and z_2' always lie in the range $(0,1)$. On the other hand, it can be shown that *the integral as a whole is non-singular*, for even though z_1' and z_2' lie within $(0,1)$ the residues at z_1' and z_2' of the continuous spectrum cancel the residues at z_1' and z_2' of the Eady normal modes. To see this explicitly it is convenient to define the functions:

$$Z_1(z,z') = \frac{(p_1)(p_1+\lambda ik)\left[\cosh kz_< + \frac{i\lambda}{p_1}\sinh kz_<\right]\left[\cosh k(z_>-1) + \frac{i\lambda \sinh k(z_>-1)}{(p_1+\lambda ik)}\right]}{(k\sinh k)(p_1-p_2)} \tag{19 a}$$

$$Z_2(z,z') = \frac{(p_2)(p_2+\lambda ik)\left[\cosh kz_< + \frac{i\lambda}{p_2}\sinh kz_<\right]\left[\cosh k(z_>-1) + \frac{i\lambda \sinh k(z_>-1)}{(p_2+\lambda ik)}\right]}{(k\sinh k)(p_2-p_1)} \tag{19 b}$$

$$Z_3(z,z') = \frac{\lambda^2 k^2 z'(1-z')\left[\cosh kz_< - \frac{\sinh kz_<}{kz'}\right]\left[\cosh k(z_>-1) + \frac{\sinh k(z_>-1)}{k(1-z')}\right]}{(k\sinh k)} \tag{19 c}$$

Then Ψ' can be written

$$\Psi' = e^{p_1 t}\int_0^1 \frac{Z_1(z,z')\,v(z')\,dz'}{(p_1+i\lambda z'k)} + e^{p_2 t}\int_0^1 \frac{Z_2(z,z')\,v(z')\,dz'}{(p_2+i\lambda z'k)} + \int_0^1 \frac{e^{-i\lambda kz't}\,Z_3(z,z')\,v(z')\,dz'}{(p_1+i\lambda z'k)(p_2+i\lambda z'k)} + \left.\frac{e^{p_1 t}Z_1(z,z')\,m(z')}{(p_1+i\lambda z'k)}\right]_{z'=0}^{z'=1} + \left.\frac{e^{p_2 t}Z_2(z,z)\,m(z')}{(p_2+i\lambda z'k)}\right]_{z'=0}^{z'=1}. \tag{20}$$

It can be easily verified that the following relations hold:

$$Z_1\left(z,\frac{ip_1}{\lambda k}\right) = -\frac{Z_3(z,ip_1/\lambda k)}{(p_2-p_1)} = Z_1(z,z_1'), \tag{21 a}$$

$$Z_2\left(z,\frac{ip_2}{\lambda k}\right) = -\frac{Z_3(z,ip_2/\lambda k)}{(p_1-p_2)} = Z_2(z,z_2'), \tag{21 b}$$

which express the non-singular character of the total integrand which can be explicitly demonstrated by using (21) to rewrite Ψ as:

$$
\begin{aligned}
\Psi &= e^{p_1 t}\int_0^1 dz' \frac{(Z_1(z,z')\,v(z') - Z_1(z,z_1')\,v(z_1'))}{(p_1 + i\lambda z' k)} \\
&+ e^{p_2 t}\int_0^1 dz' \frac{(Z_2(z,z')\,v(z') - Z_2(z,z_2')\,v(z_2'))}{(p_2 + \lambda i z' k)} \\
&+ \int_0^1 dz' e^{-ik\lambda z' t} \frac{(Z_3(z,z')\,v(z') - Z_3(z,z_1')\,v(z_1'))}{(p_2 - p_1)(p_1 + i\lambda z' k)} \\
&+ \int_0^1 dz' e^{-ik\lambda z' t} \frac{(Z_3(z,z')\,v(z') - Z_3(z,z_2')\,v(z_2'))}{(p_1 - p_2)(p_2 + i\lambda z' k)} \\
&+ \frac{e^{p_2 t} m(z')\,Z_2(z,z')}{(p_2 + i\lambda z' k)}\Bigg]_{z'=0}^{z'=1} + \frac{e^{p_1 t} m(z')\,Z_1(z,z')}{(p_1 + i\lambda z' k)}\Bigg]_{z'=0}^{z'=1} \\
&+ \int_0^1 v(z_1)\,Z_1(z,z_1')\left\{\frac{e^{p_1 t} - e^{-i\lambda k z' t}}{(p_1 + i\lambda k z')}\right\} dz' \\
&+ \int_0^1 v(z_2)\,Z_2(z,z_2')\left\{\frac{e^{p_2 t} - e^{-i\lambda k z' t}}{(p_2 + i\lambda k z')}\right\} dz' \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \qquad (22)
\end{aligned}
$$

Now the integrals I_1 and I_2 which are non-singular oscillate with the appropriate neutral Eady frequencies. I_3 which is again non-singular (the poles have been extracted) will decay algebraically for large t as fast as t^{-1}. I_4 and I_5 are the (non-singular) boundary contributions which also oscillate with the frequency of the Eady modes. To examine I_6 we write it in the form

$$I_6 = \frac{e^{p_1 t} v(z_1)\,Z_1(z,z')}{i\lambda k}\int_0^1 \frac{1 - e^{ik\lambda t(z_1 - z')}}{z' - z_1}\,dz'. \qquad (23)$$

Introducing the variable

$$\eta = tk\lambda(z' - z_1), \qquad (24)$$

$$I_6 = \frac{e^{p_1 t} v(z_1)\,Z_1(z,z_1')}{i\lambda k}\int_{-k\lambda z_1 t}^{\lambda t k(1-z_1)} \frac{1 - e^{-i\eta}}{\eta}\,d\eta. \qquad (25)$$

The integral I_6 has the asymptotic value:

$$\lim_{t\to\infty} I_6 = \frac{e^{p_1 t} v(z_1)}{\lambda z}\,Z_1(z,z_1')\,\pi. \qquad (26)$$

I_7 is the same as I_6 with subscripted variables 1 replaced by 2. Thus I_6 and I_7 also oscillate with the neutral Eady frequencies for $k > k_c$. In that part of the spectrum for which the normal modes are neutral the continuous spectrum is also stable. There is one important difference between the cases $k < k_c$ and $k > k_c$. In the latter it was not possible to separate completely the asymptotic contributions of the continuous and discrete part of the spectrum, for as we saw, the singularities of the continuous spectrum canceled in an essential way the singularities in the contributions of the normal modes of the discrete spectrum. Thus, although part of the continuous spectrum still algebraically decays (I_3), a portion of the continuous spectrum combines with the discrete spectrum to provide a non-vanishing oscillation as $t \to \infty (I_6, I_7)$. Thus for $k > k_c$ (stable waves) the continuous spectrum is required for a proper asymptotic description, while for $k < k_c$ (unstable waves) only the unstable normal mode is necessary for a correct asymptotic representation.

To summarize then, we have found that the normal mode solutions, in this perhaps simplest baroclinic problem, give the correct stability criteria, but that a detailed picture of an arbitrary initial disturbance requires the inclusion of a stable continuous spectrum of waves. For the region $k < k_c$ the unstable normal mode is capable of describing alone the solution for large t while for $k > k_c$ the continuous spectrum must be added to the discrete spectrum, even for the asymptotic solution. The time dependence of the solution is given, however, for large t by the Eady discrete frequencies.

Since the normal modes have an essentially non-oscillatory character in z, and since the complete asymptotic solution always has the *form* of the normal modes, we can draw the conclusion that regardless of the fine structure that may be present in the vertical structure of the initial disturbance, the disturbance will lose this character and extend throughout the atmosphere, in the non-oscillatory form of the normal modes for large t.

The initial value approach indicates that, at least for this baroclinic modél, that the standard normal mode analysis, while not capable of describing the detailed evolution of a disturbance, does yield the correct stability criteria for the flow, even though these normal mode solutions do not form a complete set.

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