A direct method of stream-function computation[1]

By R. M. ENDLICH and R. L. MANCUSO,
Stanford Research Institute, Menlo Park, California

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ABSTRACT

An iterative method is described that fits a stream function (ψ) to wind observations. The technique was designed for use in tropical regions where errors in height data and other difficulties preclude use of the geostrophic or balance equations to obtain ψ; however, the technique is believed to be sufficiently general to have other applications. Since the calculations may be carried out rapidly by an electronic computer, the technique might serve as a replacement for the laborious streamline-isotach method of tropical analysis.

The equations basic to the computations are given. If the stream function is defined to have the units of height (i.e., meters), it is convenient to use observed heights as the initial-guess field. Several iterations are made until the computed values of ψ stabilize within a pre-determined limit. Explicit calculations of vorticity and of boundary values of ψ, as required in a stream function obtained by solution of a Poisson equation, are not needed. Analyses of computed values of ψ in the Caribbean are compared with independent, subjective analyses of the same data, and show excellent agreement. Autocorrelations show that computed values of stream function were considerably more persistent than height data for the period studied.

1. Introduction

Computer methods of analysis and prediction, which have become standard in temperate latitudes, have not yet been widely applied in tropical meteorology. The essential difficulty is that tropical weather presents several special problems. Among these problems are the weak circulations, the vanishing of the Coriolis parameter, and the disturbing effects of local circulations and diurnal variations. Other difficulties of a practical nature include errors of measurement, and the absence of data over large regions.

The most common objective tools of extratropical meteorology involve use of the geostrophic or balance equations to obtain vorticity from height data. These two equations have not been applied in the tropics because of errors in height data as well as the uncertainties concerning their validity. Their failure has in turn prevented the application in the tropics of standard vorticity models that assume that one or the other of the two equations is correct. In view of these difficulties, theoreticians concerned with hemispheric models have usually viewed the tropics as a more-or-less quiescent boundary region. On the other hand, many synoptic meteorologists have found that tropical meteorology requires special methods of analysis, such as the streamline-isotach method of wind representation (cf., RIEHL, 1954; PALMER, 1955).

The present investigation was undertaken in an attempt to develop computer methods of analysis (and later, of forecasting) generally applicable to tropical regions. The "direct method" of computing a stream function was the main result of this work. The reader interested in other applications of computers that were investigated should consult the final report (ENDLICH and MANCUSO, 1963). The methods were tested on Caribbean data because

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the network of stations is relatively dense there, and because the pertinent upper-air observations were available on IBM cards.

In spite of the present emphasis on tropical meteorology, it is believed that the stream-function computation is sufficiently general that it can be applied also in extratropical regions. In particular, analysis in the stratosphere (where height data are much less reliable than winds) presents problems quite similar to those of tropical analysis.

2. Comments on several possible methods of objective analysis

As mentioned earlier, it does not appear feasible to compute a stream function in the tropics by use of the geostrophic or balance equations. One might consider as an alternative the standard *streamline-isotach method of analysis*, which is currently carried out subjectively. However, the streamlines (drawn everywhere tangent to observed wind vectors) are not spaced inversely proportional to wind speed, as is true for the contours of mid-latitudes. In addition, streamlines do not have numerical values so that a computer formulation of this method does not appear to be feasible. A third alternative is to determine the vorticity of the observed winds. Then if the stream function along the boundary of the region is known, the interior values of stream function can be obtained by solution of a Poisson equation. This method has been used experimentally in mid-latitudes by BROWN and NEILON (1961), CHARNEY (1962), and others. Application of this technique to restricted regions of the tropics would be hampered by the difficulty of obtaining wind components at grid points and by uncertain boundary conditions. The fourth alternative (described in Sec. 3) is to determine a stream function from observed winds without the necessity of explicitly calculating vorticity and without requiring a separate computation of boundary conditions. This method is similar in concept to a technique of hand analysis used by SCOTT (1958).

[1] Since f approaches zero as the equator is approached, it is desirable to avoid the use of f when this becomes critical. This can be done by letting f be a constant, or by placing a fictitious lower limit upon it.

3. The direct method of stream-function computation

From the well-known Helmholtz equation, the horizontal wind vector $\mathbf{V}$ may be represented as the sum of two components, the first being nondivergent and the second being irrotational

$$\mathbf{V} = (\mathbf{k} \times \nabla\psi) + \nabla\chi = \mathbf{V}_\psi + \mathbf{V}_\chi. \qquad (1)$$

The relative vorticity and divergence of $\mathbf{V}$ are then given by the two Poisson equations $\zeta = \nabla^2\psi$ and $D = \nabla^2\chi$. We wish to obtain a stream function (ψ) that represents the components of translation, vorticity and deformation, while excluding divergence which is to be represented by the velocity potential (χ). It is known from empirical investigations that divergent components are relatively small. We will assume for the time being that they can be suppressed by an areal averaging process performed during the stream function computation. From equation (1).

$$u_\psi = -(\partial\psi/\partial y), \; v_\psi = (\partial\psi/\partial x). \qquad (2)$$

Also, we may write the differential of ψ between two points separated by distance increments δx and δy as

$$\delta\psi = (\partial\psi/\partial x)\,\delta x + (\partial\psi/\partial y)\,\delta y = v_\psi\,\delta x - u_\psi\,\delta y \qquad (3)$$

These relationships are sufficient for the computation of ψ which can then be used in well-known prediction equations. However, in this formulation, ψ has the units m^2 sec^{-1} and it is therefore not directly comparable to any other meteorological quantity. For purpose of comparing values of stream-function with observed heights, it is convenient to define

$$\overset{*}{\psi} = (f/g)\,\psi \qquad (4)$$

where f is the Coriolis parameter[1] and g is gravity so that $\overset{*}{\psi}$ has the units of height (e.g., meters). Then if heights (Z) and winds were measured without error, and if the flow were geostrophic, values of $\overset{*}{\psi}$ and Z would be identical. Conversely, differences between $\overset{*}{\psi}$ and Z may be produced by errors in height and wind data as well as by geostrophic departures.

The differential in $\overset{*}{\psi}$ between a particular station 0 and a nearby station 1 is $\delta\overset{*}{\psi} = \overset{*}{\psi}_0 - \overset{*}{\psi}_1$, or by using equations (3) and (4) with the average wind components taken as $(u_0 + u_1)/2$ and $(v_0 + v_1)/2$, it is

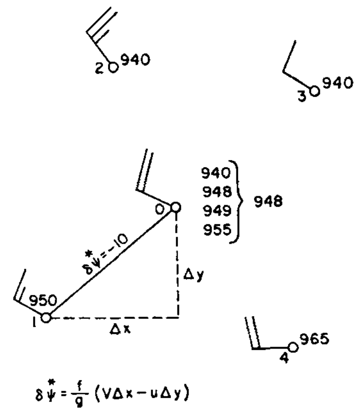

FIG. 1. Illustrating the computation of the stream function at station "0".

$$\overset{*}{\psi}_0 = \overset{*}{\psi}_1 + \frac{(f_0+f_1)}{2g}\left[\frac{(v_0+v_1)(x_0-x_1)}{2} - \frac{(u_0+u_1)(y_0-y_1)}{2}\right] \tag{5}$$

where subscripts denote values at the two stations. Let us assume that we have an estimate of $\overset{*}{\psi}_1$ and that the wind components are known. Then equation (5) can be solved to give $\overset{*}{\psi}_0$. If station 0 has n neighbors, each of them can be used to give an estimate of $\overset{*}{\psi}_0$. Presumably, the best estimate of $\overset{*}{\psi}_0$ will be an average of the n separate estimates, i.e.,

$$\overset{*}{\psi}_0 = \frac{1}{n}\sum_{i=1}^{n}\overset{*}{\psi}_i + \frac{1}{n}\sum_{i=1}^{n}\delta\overset{*}{\psi}_i. \tag{6}$$

An illustration of the computation of $\overset{*}{\psi}_0$ (the value of $\overset{*}{\psi}$ at an arbitrary station) is shown in Fig. 1. The estimates of $\overset{*}{\psi}$ at nearby stations are listed. Using the value of $\overset{*}{\psi}_1$ (950) and considering the winds at stations 0 and 1, we obtain an estimate of $\overset{*}{\psi}_0$ of 940. Similarly, using the other stations in turn, we obtain the estimates within the bracket. The average value of the estimates is 948. This value is then retained (as in a Liebmann iteration) while attention is focussed on the next station. All stations are considered in an arbitrary order. For convenience, the original estimates of $\overset{*}{\psi}$ are taken equal to reported values of height. Moreover, two estimates of $\overset{*}{\psi}_0$ are obtained. The first estimate considers all stations lying within a radius of 300 miles (5° latitude) of the station of interest while the second estimate considers stations lying between 300 and 550 miles away. These two estimates are then combined as

$$\overset{*}{\psi}_0 = \mu_1(\overset{*}{\psi}_0)_1 + \mu_2(\overset{*}{\psi}_0)_2 \tag{7}$$

where μ_1 and μ_2 are weighting factors presently set at 0.8 and 0.2. Thus the observations are weighted more or less inversely with distance from the point of interest as is done in mid-latitudes. The inclusion of winds over a fairly large area also has the characteristic of tending to suppress the smaller-scale divergent wind components. Since observed heights are used as the initial values of $\overset{*}{\psi}_i$, the average value of stream function for the entire region remains within a few meters of the average value of the observed heights. A block diagram of the computational proceedure is shown in Fig. 2.

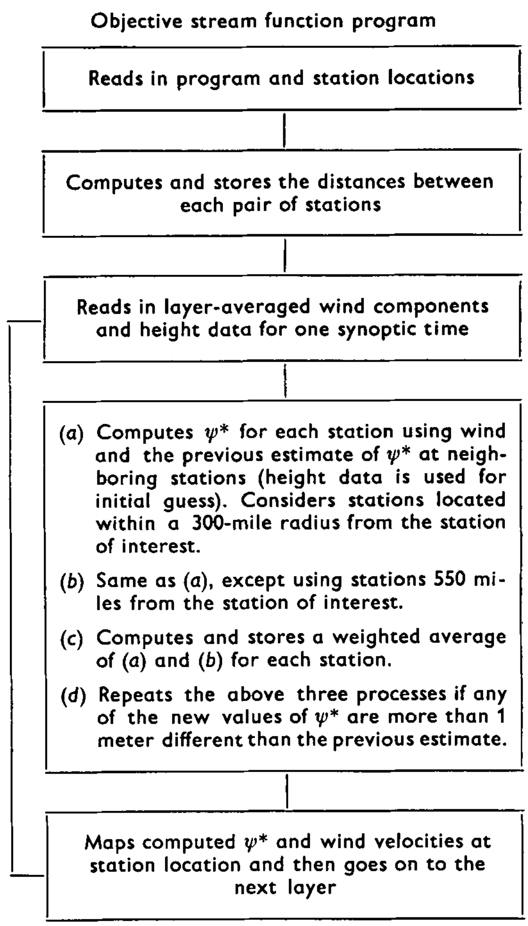

FIG. 2. Flow diagram of the computation of the stream function.

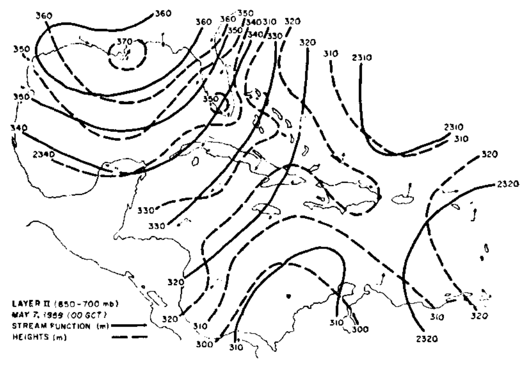

FIG. 3. A typical example of observed heights (dashed lines) and values of stream function (solid lines) in the lower troposphere in the Caribbean. Conventions for wind arrows are that a half barb equals 2.5 m sec^{-1}, a full barb equals 5 m sec^{-1}, and a pennant equals 25 m sec^{-1}.

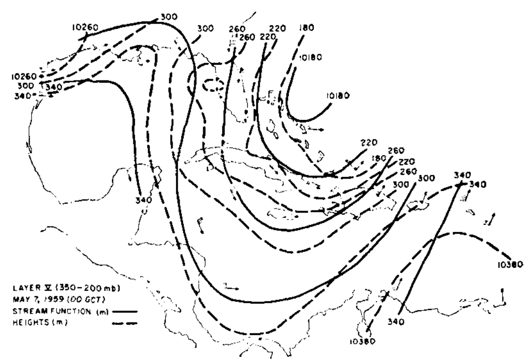

FIG. 4. A typical example of observed heights and values of stream function in the upper troposphere.

It should be noted that in the present formulation, values of stream function are calculated at *station locations*, and *not at grid points*. This choice was made to facilitate comparisons of stream function and observed heights, and also to avoid introducing uncertainties of grid point analyses into the computations. However, further insight into the workings of the stream-function procedure may be gained by applying equation (5) to a hypothetical group of stations arranged in a square grid—Station 0 is at the center, Station 1 lies a distance d to the east, Station 2 lies similarly to the north, 3 to the west, and 4 to the south. This equation is used for relating Station 0 with each of the four neighbors ($i = 1, 2, 3, 4$). For example, if $i = 1$ and f is considered constant

$$\overset{*}{\psi}_0 = \overset{*}{\psi}_1 + \frac{f}{2g}(v_0 + v_1)(-d). \qquad (8)$$

The four relations for $i = 1, 2, 3, 4$ are than summed giving

$$4\overset{*}{\psi}_0 = \sum_{i=1}^{4} \overset{*}{\psi}_i - \frac{fd^2}{g}\left[\frac{(v_1 - v_3)}{2d} - \frac{(u_2 - u_4)}{2d}\right]. \qquad (9)$$

The quantity in brackets is a finite-difference expression for vorticity (obtained from wind components) and $\sum \overset{*}{\psi}_i - 4\overset{*}{\psi}_0$ is a finite-difference form of a Laplacian of the stream function. Viewed in this way, the finite difference form of the direct stream-function procedure becomes identical to the finite-difference form of a Poisson equation. Therefore, when the direct stream function procedure scans over a dense, evenly spaced network, it may be expected to give results very close to those that would be obtained by solution of a Poisson equation. Otherwise, the solution will depend on equation (5) and on the arrangement of stations in a thinly covered region or at regional boundaries.

As mentioned earlier, we have chosen to concentrate attention on $\overset{*}{\psi}$ rather than ψ since the former values can be compared directly with height data, and also permit analyses to be continued from mid-latitudes into the tropics. However, in the former case, one should note that the vorticity is

$$\zeta = f^{-1}(g\nabla^2\overset{*}{\psi} + \beta u) \qquad (10)$$

where $\beta = \partial f/\partial y$. Similarly, the divergence is

$$D = f^{-1}(g\nabla^2\overset{*}{\chi} - \beta v) \qquad (11)$$

Thus, the stream-function field $\overset{*}{\psi}$ contains the divergence $-f^{-1}(\beta v)$ which is on the order 10^{-6} sec^{-1}. In the context of numerical forecasting, it may be simpler to utilize ψ rather than $\overset{*}{\psi}$. Either or both quantities can be obtained by a simple option in the computer program.

The direct stream-function computation described above appears to have several desirable

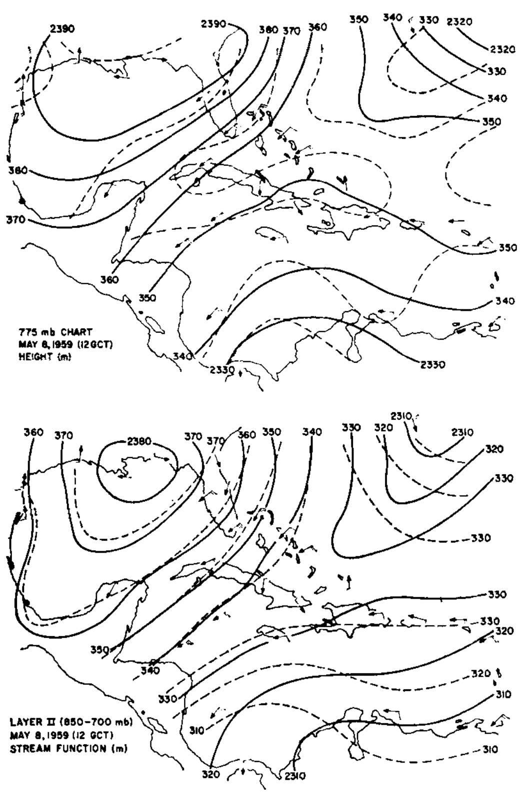

FIG. 5. Observed heights, smoothed heights, and stream function in the lower troposphere. Upper: Dashed lines are drawn to observed heights. Solid lines are height analysis made by Portig. Lower: Dashed lines are drawn to stream function values computed in the first scan. Solid lines are final values of stream function.

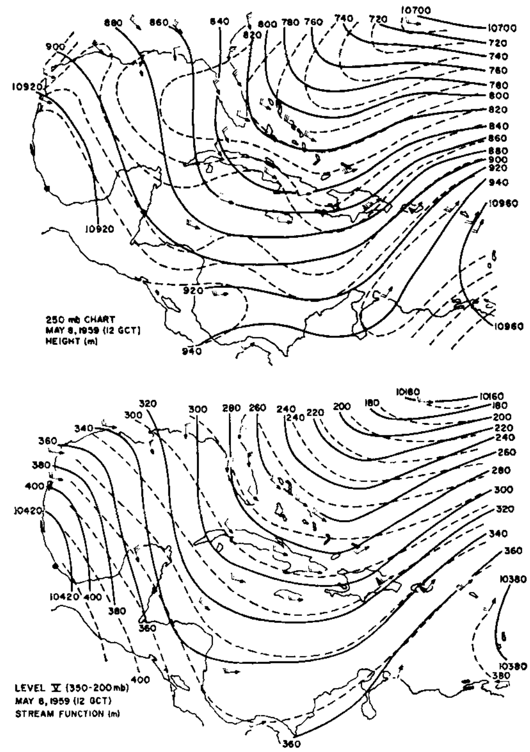

FIG. 6. Observed heights, smoothed heights, and stream function in the upper troposphere. Upper: Dashed lines are drawn to observed heights. Solid lines are smoothed height analysis made by Portig. Lower: Dashed lines are drawn to stream function values computed in the second scan. Solid lines are final values of stream function.

attributes. The general approach is simple in principle. The use of explicit boundary conditions is avoided. It is not necessary to transform to a regularly-spaced grid. The method is not subject to a limitation on the computational stability (such as the ellipticity condition $\zeta_g > -f/2$ which applies to the balance equation). Also, the stream-function computation is fast, rather insensitive to data distribution, operates quite well when data are missing, and could easily incorporate aircraft, constant-level balloon, or satellite wind data. At present, total machine time for computing $\overset{*}{\psi}$ in each of six layers for the Caribbean and printing out the computed values in mapped form is about 8 minutes using the Burroughs 220 computer, and has been reduced to about 20 seconds using the IBM 7090.

A limitation on the method is imposed by the assumption (in Eq. 5) that the average wind components between two stations are simply $(u_0 + u_1)/2$ and $(v_0 + v_1)/2$. The accuracy of this assumption naturally decreases as the distance between stations increases. Our experience indicates that the stream-function computation gives reasonable results for distances between stations up to approximately 500 to 600 miles.

4. Examples of computed fields of stream function

Examples of the stream-function fields in Layer II (lower troposphere) and Layer V (upper troposphere) are shown in Figs. 3, 4, 5,

TABLE 1. *Values of stream function versus scan number, showing the convergence of the computational procedure. Values are in meters, first digit omitted in layer II and first two digits omitted in layer V.*

Station no.	Scan no. (layer II)							Scan no. (layer V)							
	0[a]	1	2	3	4	5	6	0[a]	1	2	3	4	5	6	7
250	370	358	355	355	357	358	358	360	350	369	379	386	389	391	391
251	345	361	358	357	558	360	360	285	329	343	356	364	369	371	372
240	380	371	375	373	375	376	377	322	280	308	318	329	334	337	338
232	376	378	381	382	383	384	384	289	295	311	320	325	328	329	329
221	Msg	377	375	380	382	382	382	Msg	292	292	303	306	307	308	308
206	376	365	368	372	373	373	373	289	264	270	277	278	278	278	277
211	375	365	371	373	373	374	374	269	277	284	287	287	287	287	286
794	357	361	364	367	367	367	368	264	258	268	721	272	272	271	271
202	345	351	357	357	357	358	358	273	273	279	279	279	279	278	278
063	343	348	352	352	352	353	353	256	260	266	266	266	265	264	264
076	338	341	342	341	342	342	342	236	261	264	262	262	261	260	260
089	334	339	335	335	336	336	336	248	271	265	266	265	264	263	262
325	330	352	352	354	354	354	355	286	302	297	299	298	298	297	296
644	357	360	362	363	364	365	365	340	347	356	354	356	356	356	356
692	Msg	362	360	360	362	363	364	Msg	408	426	430	346	438	439	439
501	323	338	334	337	338	339	340	325	345	328	326	324	323	322	322
383	343	334	339	338	339	340	340	334	315	315	310	309	308	308	307
355	346	338	337	338	338	339	339	325	294	293	292	291	290	289	289
397	324	320	330	330	331	331	331	327	313	311	309	308	307	307	306
367	339	332	331	332	332	333	333	293	298	295	295	293	292	291	291
118	355	336	330	331	331	331	331	299	287	286	285	283	282	282	281
467	340	331	329	329	328	328	328	319	318	320	316	315	314	314	313
525	330	332	330	329	328	328	327	342	351	346	344	343	342	341	340
866	337	331	330	328	328	327	327	386	370	364	364	363	362	361	360
897	320	324	321	320	319	319	319	392	383	380	381	379	378	377	377
001	325	311	316	319	320	321	321	391	357	349	344	343	342	341	341
805	294	305	311	314	315	315	316	369	371	360	357	356	355	354	354
988	323	318	314	313	313	312	312	391	358	371	365	365	364	363	363
967	318	311	308	308	307	307	307	321	381	380	378	378	377	376	375

[a] These values are reported heights taken as the initial-guess field.

and 6. The advantage of using layer-averaged data has been described previously (ENDLICH and CLARK, 1963). In addition to the stream function, the height contours and wind vectors are also portrayed. In Fig. 3 (representing a portion of the lower troposphere), the height contours (dashed lines) show many features not in accord with the wind measurements. However, the lines of stream function (solid lines) represent the flow direction and speed as intended. The difficulties of using height data are even more apparent in the upper troposphere, as indicated in Fig. 4. Again, the lines of stream function appear to represent the wind flow adequately. These two figures are typical of a number of others that were also examined during the investigation.

The computed values of stream function were also compared with independent subjective analyses of the same data made by Portig of the University of Texas.[1] It is our understanding that the analysis technique was based on heights corrected by use of thickness charts and time continuity, and also considered winds. The subjective contour analyses for the 775 mb and 250 mb levels—in close but not identical correspondence to the Layers II and V—are illustrated in Figs. 5 and 6. As can be seen by comparing the solid lines in the upper and lower portions of each figure, there is excellent agreement, while both are significantly in disagreement with the observed heights (dashed lines in the upper portions). Other comparisons of the two methods of analysis have given the

[1] These analyses were made available to us upon completion of the stream-function computations through the courtesy of Dr. W. H. Portig.

same impression. The discrepancies between the two sets of solid lines can be attributed to the fact that Portig's analyses included height data while ours were based upon winds, to differences between layer-averaged and constant-level data, and to the subjectivity of drawing lines in areas of sparse data.

The convergence of the stream-function computation is shown by the progression of values at each station presented in Table I. As the computation proceeds, the alterations in $\overset{*}{\psi}$ become smaller and the computation terminates when all changes (from the preceding scan) become smaller than a preassigned number. In these experiments, this limit was arbitrarily set at one meter. The dashed lines in the lower portions of Figs. 5 and 6 show values of stream function at an early stage of the computation. It is apparent in Fig. 5 that a single scan alters the original guess values (i.e., the observed heights) into reasonable correspondence with the winds. In the upper troposphere (Fig. 6), two scans smooth out many small features of the original guess field. Further scans tend to eliminate the heights (except their average value as mentioned earlier) in favor of the winds. In order to test the influence of the first guess on the computation, a trial was made wherein the initial guess was taken as a constant value at each station. After several more scans than are usually required, the same gradients were obtained as when reported heights were used as the initial guess. Thus, the final result is not sensitive to the initial values.

In evaluating the stream function, it was believed of importance to investigate a quality generally sought in meteorological quantities—persistence—that is, that the quantity remain unchanged for a certain period of time at a particular station and thus, to a degree, serve as a forecast. In order to investigate the persistence of the stream-function and of reported heights, the root-mean-square changes [i.e., the RMS values of $(Z_{t+\Delta t} - Z_t)$ and $(\overset{*}{\psi}_{t+\Delta t} - \overset{*}{\psi})$] and the autocorrelations $[r(Z_{t+\Delta t}, Z_t)$ and $r(\overset{*}{\psi}_{t+\Delta t}, \overset{*}{\psi}_t)]$ over various time intervals (Δt) were computed. The autocorrelation is particularly revealing because it eliminates consideration of the variations of the mean fields. The results for Layers II and V derived from the eight synoptic times that were investigated are tabulated in Table II. In the lower troposphere (Layer II) RMS changes in height and stream function for the various intervals were quite similar; however, values of stream function were considerably more persistent as shown by the higher correlation coefficients. In the upper troposphere, the stream function was clearly superior both in terms of smaller RMS changes, and higher correlation coefficients.

TABLE 2. *Comparison of height (Z) versus stream function ($\overset{*}{\psi}$) for 5–8 May 1959.*

(30 Caribbean Stations)

Layer	Period (hours)	Root-mean-square change (meters) Z	ψ^*	Correlation Z	ψ^*
II					
(850–700 mb)	12	13	10	0.80	0.92
	24	16	16	0.81	0.87
	36	23	24	0.75	0.80
	48	30	33	0.72	0.73
V					
(350–200 mb)	12	32	19	0.73	0.96
	24	40	27	0.86	0.96
	36	51	35	0.79	0.94
	48	55	43	0.80	0.92

5. Concluding remarks

The writers believe that the "direct method" of stream-function computation is a relatively simple and rapid method for reducing wind data to a form suitable for kinematical and dynamical forecasting. The method does not require use of a regularly-spaced grid nor calculation of boundary conditions. It is appropriate for analysis in the tropics where height observations are relatively unreliable, and may be useful also in other regions where wind observations are available.

In further work, it would be desirable to compare stream-function fields resulting from the direct method with results obtained by solution of a Poisson equation. In mid-latitudes it would be interesting to compare solutions given by the above two methods (which utilize wind data) with stream functions obtained by conventional usage of the geostrophic or balance equations (which utilize height data to obtain ψ). It is also possible that a modification of the stream-function technique can be made that will permit computation of the velocity potential.

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