Interaction of a hurricane with the steering flow and its effect upon the hurricane trajectory

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ABSTRACT

If the steering flow is defined at all times as the residual after subtraction of an axially symmetric vortex that moves with variable speed but invariant shape, then interaction terms appear in the prediction equation that governs the steering flow. The vortex is taken as the one that minimizes the kinetic energy of the steering flow in the initial data. The propagation velocity of the vortex is defined in such a way as to minimize interactions between vortex and steering flow, and is found to have a second-order dynamic part proportional to the gradient of steering-flow absolute vorticity. It also is shown that in a uniform steering flow, interaction terms give rise to an acceleration of cyclonic vortices in the direction of increasing steering-flow absolute potential vorticity.

Results are presented for numerical computations made from 500-mb data on hurricane Betsy 1956 with the proposed prediction model with and without the interaction terms. The interaction terms remove the major part of a systematic rightward bias found in earlier hurricane-trajectory forecasts based upon steering-flow models.

1. Introduction

Among the technical difficulties involved in the numerical prediction of hurricane movement is the fact that one has to deal with two different scales of motion of the atmosphere. A hurricane is an intense vortex embedded in a large-scale flow which influences its movement. If one wants to describe hurricane scales of motion with acceptable resolution, a network with mesh interval as small as 30 kilometers may be needed. On the other hand, the use of a 300-km mesh appears to be satisfactory for large-scale motions.

One approach to this problem is to use a finer mesh over the storm. This is feasible technically if a computer is used that has sufficient speed and capacity; BIRCHFIELD (1960, 1961), for example, has made barotropic hurricane forecasts using a 150-km mesh length. Another approach lays more emphasis upon the hurricane trajectory than upon evolution of the hurricane circulation. The method consists essentially of removing the hurricane vortex from the total pattern and then making a trajectory calculation based upon evolution of the residual flow. This idea was suggested originally by SASAKI & MIYAKODA (1954) and SASAKI (1955). We shall refer to this approach as the "steering-flow" method.

During the course of research on trajectory forecasts with the steering-flow method, we found that a rightward deflection of the predicted trajectory relative to the actual path appeared as a systematic bias. This bias was conspicuous not only in barotropic predictions for hurricanes Diane 1955 and Betsy 1956 (KASAHARA, 1957 and 1959), but also in two-level baroclinic forecasts for hurricanes Betsy 1956 (KASAHARA, 1960) and Audrey 1957 (JONES, 1961). HUBERT (1959), who performed a number of 500-mb barotropic predictions with a steering-flow model under operational conditions, attempted to interpret the bias in question as a tendency for hurricanes to move to the left of the geostrophic wind at that level, but was unable to confirm this hypothesis because of large dispersion of the forecasts errors. BIRCHFIELD (1961) made equivalent-barotropic forecasts for hurricane Betsy 1956 with a 150-

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km grid network over the entire forecast region. His model included implicitly the effect of interaction between the vortex and its environment. He reported the mean error in the predicted direction of movement to be less than the one obtained by KASAHARA (1959) (although errors in the prediction of displacement are approximately the same for the two sets of forecasts).

The question of interaction between the vortex and its environment was taken up by MORIKAWA (1960) in an investigation based upon representation of the hurricane vortex as the singular solution of the potential-vorticity equation. MORIKAWA (1962) made two sets of 500-mb forecasts, one with and one without the interaction effect for the same hurricane Betsy 1956 cited above. He found that there is a systematic tendency for the forecast tracks without interaction to be deflected rightward relative to those with the interaction effect. This interesting comparison raises the general question of the role of interaction between the vortex circulation and its large-scale environmental flow, on the vortex movement.

The main purpose of this paper is to investigate some of the consequences of dynamical interaction between an axially symmetric vortex and its environment. The basis for this investigation is the potential-vorticity equation derived in section 2. A comparative study of the two prediction schemes given by Kasahara and by Morikawa (cited above) leads to an alternative approach described in detail in section 3. In this approach, the steering flow is defined at all times as the residual after subtraction of an axially symmetric vortex that moves with variable speed but invariant shape. Under these conditions, interaction terms appear in the steering-flow prediction equation. The corresponding vortex propagation velocity and acceleration are derived in sections 3 and 4, respectively. A forecasting procedure with this scheme is presented in section 5, and the results of numerical computations with 500-mb data for hurricane Betsy are described in section 6.

2. The dynamical model

For a dynamical model of the atmosphere we consider a homogeneous, incompressible fluid of depth $D+h$, where D is the depth of fluid when there is no motion, and h is the fluctuation of depth produced by motion of the fluid. (If topography of the earth's surface is ignored, D is uniform.) The motion with which we are concerned is determined by the potential-vorticity equation (ROSSBY *et al.*, 1939)

$$\left.\begin{aligned} \frac{d(f+\zeta)}{dt} - \frac{f+\zeta}{D+h}\frac{d(D+h)}{dt} &= 0, \\ d/dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla, & \end{aligned}\right\} \tag{2.1}$$

where $\mathbf{v}$ is the horizontal velocity, ∇ is the horizontal gradient operator, f is the absolute vorticity when there is no motion relative to the earth, and ζ is the relative vorticity. (If curvature of the earth's surface is ignored, f is uniform.) The main assumptions behind (2.1) are that the motion is quasi-hydrostatic (vertical (z) component of inertial acceleration neglected) and quasi-horizontal (horizontal velocity components independent of z, and vertical velocity component a linear function of z).

On the basis of a scale analysis such as given by CHARNEY & STERN (1962) or PHILLIPS (1963), we can approximate (2.1) by the following quasi-geostrophic "potential vorticity" equation

$$\left.\begin{aligned} \partial\gamma/\partial t &= -\mathbf{v}\cdot\nabla(f+\gamma), \\ \mathbf{v} &= \mathbf{k}\times\nabla\psi, \\ \gamma &\equiv (\nabla^2-\lambda^2)\psi, \\ \lambda &\equiv f_0/(gD_0)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.2}$$

where f_0 and D_0 denote mean values of f and D; g, the acceleration of gravity; $\mathbf{k}$, vertical unit vector; and ψ, the stream function for the rotational part of the velocity field. The main assumptions behind (2.2) are (1) that the motion is quasi-geostrophic (by this we mean (a) that the time scale of the motion is of the order of the ratio of characteristic length L and velocity scale U, and (b) that the ratio of horizontal inertial acceleration to Coriolis acceleration $= U/(f_0L) \equiv R$, the Rossby number, is small); (2) that the Froude number, the ratio of U to the propagation speed of long gravity waves $(gD_0)^{\frac{1}{2}}$ is of the order of R, and (3) that the ratio of L to the radius of the earth is also of the order of R. For a more detailed discussion of this system of approximations, the reader should consult the papers cited above.

In (2.2), the term involving λ^2 expresses vorticity changes induced by horizontal divergence associated with deformations of the upper boundary (free surface). If $\lambda \neq 0$, we refer to the model as "divergent"; if $\lambda = 0$ (as is nearly the case for sufficiently large D_0), we refer to the model as "non-divergent".

3. Definition and velocity of vortex

Let $\mathbf{R}_0(t)$ denote the position vector from a fixed origin to the moving center of the vortex: our aim is to predict $\mathbf{R}_0(t)$. If the vortex were retained as an integral part of the predicted flow—as in the "total-flow" method—one would obtain $\mathbf{R}_0(t)$ as an immediate by-product of the computation, by following the location of the point of minimum pressure (or extreme vorticity). However, in the "steering-flow" method we regard the total flow at each instant as divided into two parts: a vortex flow and a steering flow. (The manner in which this partition is made will be discussed presently.) Separate dynamical equations are written for each part, but step-wise numerical integration is carried out only for the steering flow. One therefore obtains $\mathbf{R}_0(t)$ through step-wise numerical integration of the equation

$$d\mathbf{R}_0/dt = \mathbf{C}, \tag{3.1}$$

where the velocity $\mathbf{C}$ of the vortex center depends upon the steering flow in a manner determined by the method of partitioning the total flow.

Some general remarks can be made without specifying the method of partitioning. Let $\mathbf{v}$, $\mathbf{V}$ denote vortex-flow and steering-flow velocities, so that $\mathbf{v} + \mathbf{V}$ is the total flow (for which a special symbol will not be needed). Formally, the potential-vorticity equation (2.2) for the total flow can be restated in the notation

$$\left.\begin{aligned} &(\partial\gamma/\partial t + \mathbf{v}\cdot\nabla f) + (\partial\Gamma/\partial t + \mathbf{V}\cdot\nabla f) \\ &\quad = -\mathbf{v}\cdot\nabla\gamma - \mathbf{V}\cdot\nabla\gamma - \mathbf{v}\cdot\nabla\Gamma - \mathbf{V}\cdot\nabla\Gamma, \\ &\mathbf{v} = \mathbf{k}\times\nabla\psi, \quad \mathbf{V} = \mathbf{k}\times\nabla\Psi, \\ &\gamma \equiv (\nabla^2 - \lambda^2)\psi, \quad \Gamma \equiv (\nabla^2 - \lambda^2)\Psi, \end{aligned}\right\} \tag{3.2}$$

where ψ, Ψ are the vortex-flow and steering-flow stream functions, and γ, Γ the corresponding (relative) potential vorticities. The second and third advection terms on the right in (3.2) evidently may be regarded as expressing "interaction" between the two parts of the total flow.

If one assumes that at the instant $t = 0$ a partition between vortex and steering flow is made in some reasonable manner, the advection terms of (3.2) are then, in principle, individually completely determined. However, in general one does not know *a priori* how the aggregate of these terms is to be partitioned between vorticity *changes* in the two fields; that is, the partition between the vortex and the steering flow must be defined for $t > 0$ as well as for $t = 0$. *This is one of the main problems in devising a steering-flow model.* In principle, there is an unlimited number of ways in which the partition may be defined, inasmuch as Nature does not herself provide a unique means for separating the vortex dynamically from its environment.

It has already been pointed out that an essential difficulty of the hurricane-trajectory prediction problem is the fact that the scale length that characterizes the vortex is distinctly smaller (by as much as one order of magnitude) than that which typifies the steering flow. Moreover, the characteristic *velocity* scale of the vortex is distinctly larger (by as much as one order of magnitude) than that of the steering flow. Thus, if U_v, U_s and L_v, L_s denote scale velocities and lengths for vortex and steering flow, an elementary scale analysis of (3.2) leads to the result that, after division by $(U_s/L_s)^2$ the advection terms on the right in (3.2) have *relative* orders of magnitude as follows:

$$0(a^2b^2), \quad 0(ab^2), \quad 0(a), \quad 0(1), \tag{3.3}$$

where $a \equiv U_v/U_s > 1$ and $b \equiv L_s/L_v > 1$. In (3.3) the terms appear in the same sequence as shown on the right side of (3.2). Since a and b have, roughly, the same order of magnitude, (3.3) shows that the third, second and first terms on the right in (3.2) are respectively roughly one, three and four orders of a, b, greater than the last term. For this reason, equation (3.2) as it stands is not suitable for conventional numerical integration.

The difficulty implicit in (3.3) also may be characterized by means of a comparison between R_v and R_s, the Rossby numbers of vortex and steering flow. With $R \equiv U/f_0L$ it follows from the definitions of a and b that

$$R_v = R_s \cdot a \cdot b.$$

The numbers involved here doubtless vary over a wide range, but as typical values one might take $ab \approx 50$ and $R_s \approx 0.1$; this makes $R_v \approx 5$. It follows that validity of the quasi-geostrophic potential-vorticity equation (2.2) is doubtful in the vortex region, since the derivation of this equation depends upon approximations that result from the assumption $R < 1$. We have not attempted to deal with this difficulty in a direct or systematic manner, but have sought to deflect it partly by use of the gradient wind (rather than geostrophic wind) in the vortex, as explained later (section 5), and partly by means of the partition between vortex and steering flow.

In order to clarify the relation between the method used here and earlier methods, we now discuss briefly the partitions used previously by KASAHARA (1957) and by MORIKAWA (1960)—designated, respectively, method I and method II in the sequel. Certain features of each of these methods then are combined to form the partition used in the present investigation.

a. PARTITION USED IN METHOD I

In the previous work by KASAHARA (1957) the partition of (3.2) was made as follows:

$$\partial\gamma/\partial t + \mathbf{v}\cdot\nabla f = -\mathbf{v}\cdot\nabla\gamma - \mathbf{V}\cdot\nabla\gamma - \mathbf{v}\cdot\nabla\Gamma, \quad (3.4\text{a})$$

$$\partial\Gamma/\partial t + \mathbf{V}\cdot\nabla f = -\mathbf{V}\cdot\nabla\Gamma. \quad (3.4\text{b})$$

It is evident from (3.4b) that in this partition *the vortex has no influence upon the evolution of the steering flow.* (Accordingly, in method I the "interaction" terms—the last two on the right in (3.4a)—are more properly described as "coupling" terms.) The fact that (3.4) does not permit "feedback" from the vortex to the steering flow clearly is the result of the *definition* of the partition: *it does not arise from an approximation.*

In the partition defined by (3.4), the scale difficulties of our problem are lodged in the vortex equation (3.4a). The orders of magnitude of the advection terms on the right in this equation are, relative to the middle term,

$$0(a), \quad 0(1), \quad 0(b^{-2}).$$

The middle term expresses the principal effect of "coupling" between vortex and steering flow. If interest in the vortex is restricted to prediction of the trajectory of the vortex center, one can adopt the point of view that the only information we must extract from (3.4a) is a determination or an estimate of the center velocity $\mathbf{C}$ needed for numerical integration of (3.1).

Although the vortex may be assumed axially symmetric initially, (3.4a) will in general cause it to evolve asymmetrically. Nevertheless, the procedure of method I is based upon the approximation that for the purpose of estimating $\mathbf{C}$ from (3.4a), *the vortex may be assumed invariant (in shape and intensity) and axially symmetric.* If one defines the vortex center as the point at which the vortex velocity vanishes ($\nabla\psi = 0$), the assumption of axial symmetry makes it possible to show that[1]

$$\mathbf{C} = -\frac{2}{\zeta_0}\left(\nabla\frac{\partial\psi}{\partial t}\right)_0, \quad (3.5)$$

where $\zeta \equiv \nabla^2\psi$ is the vortex vorticity and the zero subscript signifies evaluation at the vortex center. The determination of $(\nabla\partial\psi/\partial t)_0$ is made from (3.4a), and will be given here for the non-divergent model. In this case $\gamma = \nabla^2\psi(\equiv\zeta)$ and $\Gamma = \nabla^2\Psi(\equiv Z)$, so (3.4a) may be written

$$\left.\begin{aligned} \nabla^2\partial\psi/\partial t &= -A, \\ A &\equiv \mathbf{V}\cdot\nabla\zeta + \mathbf{v}\cdot\nabla(Z+f) \\ &= \mathbf{V}\cdot\nabla\zeta - \mathbf{P}\cdot\nabla\psi, \end{aligned}\right\} \quad (3.6)$$

where $\mathbf{P} \equiv \mathbf{k}\times\nabla(Z+f)$, the first term on the right in (3.4a) having been dropped on the basis of the assumption of axial symmetry. With aid of the Poisson integral representation for the solution $\partial\psi/\partial t$ of (3.6), one finds from (3.5) that

$$\left.\begin{aligned} \mathbf{C} &= \frac{1}{\pi}\int A\nabla\chi\, dS, \\ \chi &\equiv -\zeta_0^{-1}\ln r, \end{aligned}\right\} \quad (3.7)$$

where r is radial distance from the vortex center, dS is an area element, and the region of integration extends to the outer limit of the vortex. By expansion of $\mathbf{V}$ and $\mathbf{P}$ in the advection function A as Taylor series on the vortex center, the integral in (3.7) may be evaluated formally; the result is (see footnote)

[1] For the derivation of this equation, the reader may consult Appendix A of a technical report by KASAHARA & PLATZMAN (1963).

$$\left.\begin{aligned} \mathbf{C} &= \sum_{k=0}^{\infty} \sigma_k (\nabla^{2k}\mathbf{V})_0 + \sum_{k=0}^{\infty} \tau_k (\nabla^{2k}\mathbf{P})_0 \\ \sigma_k &\equiv -\frac{\zeta_0^{-1}}{k!(k+1)!}\int_0^{\infty} (\tfrac{1}{2}r)^{2k}\frac{d\zeta}{dr}dr, \\ \tau_k &\equiv +\frac{\zeta_0^{-1}}{k!(k+1)!}\int_0^{\infty} (\tfrac{1}{2}r)^{2k}\frac{d\psi}{dr}dr, \end{aligned}\right\} \quad (3.8)$$

subscript zero signifying evaluation at the vortex center.

In (3.8) the coefficients σ_k and τ_k are constants completely determined by the structure of the vortex. On the other hand, $(\nabla^{2k}\mathbf{V})_0$ and $(\nabla^{2k}\mathbf{P})_0$ depend only upon the steering-flow velocity $\mathbf{V}$ and vorticity-gradient $\mathbf{P} \equiv \mathbf{k} \times \nabla(Z+f)$, through Laplacians of various orders evaluated at the vortex center. A scale analysis leads without difficulty to the results that[1]

$$\sigma_k(\nabla^{2k}\mathbf{V})_0 = U_s \cdot 0(b^{-2k})$$
$$\tau_k(\nabla^{2k}\mathbf{P})_0 = U_s \cdot 0(b^{-2k-2}),$$

where U_s is a scale velocity for the steering flow and $b \equiv L_s/L_v$; therefore

$$\mathbf{C} = \sigma_0\mathbf{V}_0 + \sigma_1(\nabla^2\mathbf{V})_0 + \tau_0\mathbf{P}_0 + 0(b^{-4}).$$

From the definitions in (3.8), it is easy to see that $\sigma_0 = 1$ and $\sigma_1 = 0$, while $\tau_0 = -\zeta_0^{-1}\psi_0$; hence the preceding equation reduces to

$$\mathbf{C} = \mathbf{V}_0 + \tau\mathbf{k} \times [\nabla(Z+f)]_0 + 0(b^{-4}) \qquad (3.9)$$
$$\tau \equiv -\psi_0/\zeta_0.$$

This is the form used by Kasahara (1957), and also is contained in the work of Sasaki (1955). Note that $\tau_0 > 0$ for both cyclonic and anticyclonic vortices, since ψ_0 and ζ_0 have opposite signs in either case.

The first term on the right in (3.9)—the steering-flow velocity at the vortex center—is the principal kinematical effect of the steering flow on propagation of the vortex. The second term, of order b^{-2} times the first, is a coupling term dependent upon the gradient of steering-flow absolute vorticity in a manner completely analogous to the dependence of Rossby waves upon the planetary vorticity gradient. This term may be divided into two parts: a *Coriolian* part $\tau\mathbf{k} \times (\nabla Z)_0$ proportional to the steering-flow (relative) vorticity gradient, and a *planetary* part $\tau\mathbf{k} \times (\nabla f)_0$ proportional to the planetary vorticity gradient. The former gives a contribution to propagation of the vortex parallel to isopleths of steering-flow (relative) vorticity, with low vorticity to the left of the propagation vector; the latter gives a similar contribution parallel to latitude circles, with low latitude (low planetary vorticity) to the left of the propagation vector (that is, westward).

[1] In the second statement here, scale considerations affecting f have been ignored.

b. Partition used in method II

An interesting partition of the total flow has been developed by Morikawa (1960, 1962) for application to the hurricane-trajectory problem. Morikawa's procedure is to define the vortex stream function for $t \geq 0$ as the axially symmetric singular solution of

$$\gamma \equiv (\nabla^2 - \lambda^2)\psi = 0 \qquad (3.10\text{a})$$

so that the vortex is a *filament* of potential vorticity ($\gamma = 0$ everywhere except at the vortex center). It follows that the velocity of the vortex center is exactly $\mathbf{V}_0$ (since the filament otherwise could not conserve its infinite potential vorticity); equation (3.9) therefore is replaced by the simpler statement

$$\mathbf{C} = \mathbf{V}_0, \qquad (3.11)$$

although it should be kept in mind that the leading term of (3.9) is the dominant one in any case.

Equation (3.10a) replaces (3.4a); the equation corresponding to (3.4b) for prediction of the steering flow is obtained from (3.2) with $\gamma = 0$:

$$\partial\Gamma/\partial t + (\mathbf{V}+\mathbf{v})\cdot\nabla f = -\mathbf{v}\cdot\nabla\Gamma - \mathbf{V}\cdot\nabla\Gamma. \quad (3.10\text{b})$$

In accordance with (3.3), the relative orders of magnitude of the two terms on the right are

$$0(a), \quad 0(1).$$

The first of these corresponds to the interaction term $\mathbf{v}\cdot\nabla\Gamma$; the effect of the other interaction term $\mathbf{V}\cdot\nabla\gamma$ is expressed implicitly through (3.11).

The required solution of (3.10a) is $\psi = \bar{\psi}K_0(\lambda r)$, where K_0 is the zero-order exponentially decreasing Bessel function with logarithmic singularity. In Morikawa's procedure the parameters $\bar{\psi}$ and λ are selected empirically to give the best representation of vortex intensity and scale, with the result that $\lambda^{-1} = 0(L_v)$; for example, λ^{-1} ranged between 300 and 400 miles for Betsy 1956 (MORIKAWA, 1962).

From the point of view that we have adopted, λ should be determined from the scale of the steering flow rather than that of the vortex: $\lambda^{-1} = 0(L_s)$. If this were done in method II, the additional parameter that must be available to express the scale of the vortex can be introduced by removing the *core* of the singular vortex and replacing it with an "eye" having non-zero but uniform potential vorticity.[1] The steering-flow prediction equation (3.10b) will not be changed, because the term $\mathbf{V} \cdot \nabla\gamma$ again vanishes owing to $\nabla\gamma = 0$; but now the terms expressing advection of steering-flow and planetary vorticity by the vortex remain finite throughout the vortex. In the core of this "combined" vortex the stream function will be given by the exponentially increasing Bessel function $I_0(\lambda r)$, and the two regions will be joined by requiring continuity of ψ and of $d\psi/dr$. The result may be stated

$$\left.\begin{aligned} \psi &= \psi_0 \cdot \begin{Bmatrix} (1+p) - pI_0(\lambda r) & (r \leqslant l) \\ qK_0(\lambda r) & (r \geqslant l) \end{Bmatrix} \\ \begin{Bmatrix} p \\ q \end{Bmatrix} &\equiv \frac{\lambda l}{1 - \lambda l K_1(\lambda l)} \begin{Bmatrix} K_1(\lambda l) \\ I_1(\lambda l) \end{Bmatrix}, \end{aligned}\right\} \quad (3.12)$$

where l is the radius of the core, which now serves as the additional parameter needed to fix the scale of the vortex, while ψ_0 (the central value of ψ) now is finite and fixes the vortex intensity. The latter also may be expressed in terms of $\gamma = -\lambda^2(1+p)\psi_0$.

If the foregoing modification is made in method II, one must introduce an additional statement concerning changes of shape and intensity of the vortex core. For example, the core may be defined as an axially symmetric vortex with invariant and uniform potential vorticity. Determination of the vortex propagation velocity also must be reconsidered, because of possible nonuniformity of steering-flow velocity in the region of the core. These questions lead to our discussion of method III.

c. Partition used in present investigation (method III)

The principal weakness of method I lies in the approximation that the vortex as defined by the partition (3.4) is assumed invariant in shape and intensity. On the other hand, in method II the use of a prescribed vortex having only two disposable parameters puts restrictions on the accuracy of representation of real data. Our aim now is to relax these assumptions as much as possible.

Consider a *nonsingular* vortex ($\gamma \neq 0$) which by *definition* (rather than by approximation, as in method I) is axially symmetric and invariant in both shape and intensity. (The specification of this vortex from initial data will be discussed in section 5.) It follows that (3.2) transforms to

$$\partial\gamma/\partial t = -\mathbf{C} \cdot \nabla\gamma, \quad (3.13\text{a})$$

$$\partial\Gamma/\partial t + (\mathbf{V} + \mathbf{v}) \cdot \nabla f = -(\mathbf{V} - \mathbf{C}) \cdot \nabla\gamma - \mathbf{v} \cdot \nabla\Gamma - \mathbf{V} \cdot \nabla\Gamma, \quad (3.13\text{b})$$

where $\mathbf{C}$, the propagation velocity of the vortex, remains to be assigned. We regard (3.13a) as the analogue of (3.4a). The prediction equation for the steering flow, (3.13b), is obtained by inserting (3.13a) into (3.2), the term $\mathbf{v} \cdot \nabla\gamma$ having been omitted owing to the condition of axial symmetry.

In accordance with (3.3), the relative orders of magnitude of the advection terms on the right in (3.13b) are

$$0(ab^2), \quad 0(a), \quad 0(1),$$

assuming that $|\mathbf{C}| = 0(U_s)$. Now in general, in the steering-flow prediction equation (3.13b) the effect of any term that involves advection by the vortex or of the vortex, is to introduce distortions of steering flow on the scale of the vortex. However, if a significant amount of energy were to enter the steering flow through such distortions, the dynamical and numerical justifications for treating the steering-flow prediction equation quasi-geostrophically and with the conventional "coarse" mesh (of order 300 km), would eventually be vitiated. We therefore arrange (3.13b) with all "distortion" terms on the right,

$$\left.\begin{aligned} \partial\Gamma/\partial t + \mathbf{V} \cdot \nabla(\Gamma + f) &= \mathbf{C} \cdot \nabla\gamma - A, \\ A &\equiv \mathbf{V} \cdot \nabla\gamma + \mathbf{v} \cdot \nabla(\Gamma + f) \\ &= \mathbf{V} \cdot \nabla\gamma - \mathbf{P} \cdot \nabla\psi, \end{aligned}\right\} \quad (3.13\text{c})$$

where $\mathbf{P} \equiv \mathbf{k} \times \nabla(\Gamma + f)$, and we *define* the vortex

[1] The result is completely analogous to "Rankine's combined vortex" in the case of the logarithmic potential.

propagation velocity **C** (as yet unspecified) in such a way that at each instant the gross distortion expressed by

$$\int (\mathbf{C}\cdot\nabla\gamma - A)^2 dS$$

is minimized with respect to all possible choices of **C**.

Taking the variation of this integral with respect to **C**, we come to the condition

$$\int (\mathbf{C}\cdot\nabla\gamma)\nabla\gamma dS = \int A\nabla\gamma dS.$$

Since γ, the potential vorticity of the vortex, is assumed to be an axially symmetric function, and from the fact that **C** is not a function of space, the integral on the left in the preceding equation can be transformed to $\frac{1}{2}\mathbf{C}\int(\nabla\gamma)^2 dS$, and accordingly one finds

$$\left.\begin{aligned} \mathbf{C} &= \frac{1}{\pi}\int A\nabla\chi\, dS,\\ \chi &\equiv \gamma \Big/ \frac{1}{2\pi}\int (\nabla\gamma)^2 dS. \end{aligned}\right\} \quad (3.14)$$

This representation of **C** is formally identical to that stated in (3.7), and can be evaluated by expansion of **V** and **P** in the advection A in a Taylor series on the vortex center. The result is (see footnote, page 324)

$$\left.\begin{aligned} \mathbf{C} &= \sum_{k=0}^{\infty} \sigma_k(\nabla^{2k}\mathbf{V})_0 + \sum_{k=0}^{\infty}\tau_k(\nabla^{2k}\mathbf{P})_0,\\ \sigma_k &\equiv +\frac{1}{k!(k+1)!}\int_0^\infty (\tfrac{1}{2}r)^{2k}\frac{d\gamma}{dr}\frac{d\chi}{dr} r\,dr,\\ \tau_k &\equiv -\frac{1}{k!(k+1)!}\int_0^\infty (\tfrac{1}{2}r)^{2k}\frac{d\psi}{dr}\frac{d\chi}{dr} r\,dr, \end{aligned}\right\} \quad (3.15)$$

subscript zero signifying evaluation at the vortex center.

The scale analysis of (3.15) is identical to that of (3.8), so

$$\mathbf{C} = \sigma_0\mathbf{V}_0 + \sigma_1(\nabla^2\mathbf{V})_0 + \tau_0\mathbf{P}_0 + 0(b^{-4})$$

as before. Moreover, $\nabla^2\mathbf{V} = \mathbf{k}\times\nabla Z$ and $\mathbf{P}\equiv \mathbf{k}\times\nabla(Z+f)$, where Z is the *steering-flow* (relative) vorticity and $\Gamma = Z - \lambda^2\Psi$ the potential vorticity. Hence, with the aid of definitions in (3.15), it is easy to see that the preceding equation for **C** reduces to

$$\left.\begin{aligned} \mathbf{C} &= \mathbf{V}_0 + \sigma\mathbf{k}\times(\nabla Z)_0 + \tau\mathbf{k}\times[\nabla(\Gamma+f)]_0\\ &\quad + 0(b^{-4}),\\ \sigma &\equiv \frac{1}{8}\int_0^\infty \left(\frac{d\gamma}{dr}\right)^2 r^3 dr \Big/ \int_0^\infty \left(\frac{d\gamma}{dr}\right)^2 r\,dr,\\ \tau &\equiv -\int_0^\infty \frac{d\psi}{dr}\frac{d\gamma}{dr} r\,dr \Big/ \int_0^\infty \left(\frac{d\gamma}{dr}\right)^2 r\,dr. \end{aligned}\right\} \quad (3.16)$$

The second term on the right in (3.16) is of order $U_s b^{-2}$ and arises from advection of vortex vorticity by the steering flow; the corresponding term is absent in (3.9). The third term, also of order $U_s b^{-2}$, arises from advection of steering-flow absolute potential vorticity by the vortex, and is analogous to the τ-term in (3.9).

It should be noted that γ in σ and τ of (3.16) is the potential vorticity of the vortex: $\gamma = \zeta - \lambda^2\psi$. Now, in accordance with the scale analysis of section 2, we have $\lambda = 0(L_s^{-1})$, so it follows that

$$\gamma = \zeta\cdot[1 + 0(b^{-2})],$$

since $\zeta \equiv \nabla^2\psi = \psi\cdot 0(L_v^{-2})$. This means that replacement of γ by ζ in σ and τ will affect **C** only to $0(b^{-4})$, because the σ- and τ-terms are themselves of $0(b^{-2})$. Therefore, it suffices to use the simpler expressions

$$\left.\begin{aligned} \sigma &= \frac{1}{8}\int_0^\infty \left(\frac{d\zeta}{dr}\right)^2 r^3 dr \Big/ \int_0^\infty \left(\frac{d\zeta}{dr}\right)^2 r\,dr,\\ \tau &= -\int_0^\infty \frac{d\psi}{dr}\frac{d\zeta}{dr} r\,dr \Big/ \int_0^\infty \left(\frac{d\zeta}{dr}\right)^2 r\,dr\\ &= \int_0^\infty \zeta^2 r\,dr \Big/ \int_0^\infty \left(\frac{d\zeta}{dr}\right)^2 r\,dr \end{aligned}\right\} \quad (3.17)$$

in place of those given in (3.16).

Formula (3.16) has been used in the present study to predict the hurricane trajectory and evolution of the steering flow by joint integration of (3.1) and (3.13b). Since in (3.16) we have $\mathbf{C} = \mathbf{V}_0 + 0(b^{-2})$, the scale magnitude of the leading term on the right in (3.13b) is approximately $0(a)$, if one ignores the slight variations of **V** within the region of flow dominated by the vortex. Implicit effects of (3.16) must also still further reduce distortion of the steering flow through the first two terms of (3.13b).

In previous work by KASAHARA (1957), the hurricane trajectory and steering-flow evolution were predicted by joint integration of (3.1) and (3.4b), with (3.8) for $\mathbf{C}$. In that work it was *assumed* that the vortex is axially symmetric and invariant in shape and intensity, so that detailed integration of (3.4a) is not necessary; the latter equation, in fact was used only to obtain the formula for $\mathbf{C}$. By definition of the steering flow, the vortex equation absorbed all "interaction" terms, the effects of which appeared explicitly only in the dynamic terms of $\mathbf{C}$. In the present method the vortex is *defined* as axially symmetric and invariant, and all "interaction" terms are lodged in the steering-flow equation, for which a detailed integration is made. One may expect, therefore, that interaction effects should be better represented in the present method than in the earlier one, as would be inferred from the work of MORIKAWA (1960). In *principle*, of course, both methods should lead to the same results if all equations were treated without approximation.

As a particular application of (3.16), it is interesting to consider the special case of the "combined" vortex discussed in connection with method II: an axially symmetric vortex having uniform potential vorticity in a core of radius l, and zero potential vorticity outside this core. If the integrals for σ and τ in (3.16) are evaluated in this case, with due regard for the discontinuous change of γ at $r = l$, one finds easily that $\sigma = l^2/8$ and $\tau = 0$; thus

$$\mathbf{C} = \mathbf{V}_0 + \tfrac{1}{8}l^2\mathbf{k} \times (\nabla Z)_0 + 0(b^{-4}). \qquad (3.18)$$

The dynamic term here vanishes for a uniform flow; it is, of course, of second order in any case. Moreover, the planetary effect does not appear explicitly (compare with (3.9), method I).

4. Acceleration of vortex

Each of the three partitions discussed in the preceding analysis leads to a characteristic vortex-propagation velocity

$$\mathbf{C} = \mathbf{V}_0 + \sigma\mathbf{k} \times (\nabla Z)_0 + \tau\mathbf{k} \times [\nabla(\Gamma + f)]_0. \qquad (4.1)$$

In method I we found $\sigma = 0$ and τ as in (3.9) (in the non-divergent case[1]); in method II we have $\sigma = l^2/8$ and $\tau = 0$ (with $l = 0$ in the case considered by MORIKAWA); while in method III we found σ and τ as in (3.16). Since

$$\mathbf{k} \times \nabla Z = \nabla^2\mathbf{V}, \qquad (4.2\text{a})$$

$$\mathbf{k} \times \nabla\Gamma = (\nabla^2 - \lambda^2)\mathbf{V}, \qquad (4.2\text{b})$$

one may write (4.1) in the form

$$\mathbf{C} = (1 - \lambda^2\tau)\mathbf{V}_0 + (\sigma + \tau)(\nabla^2\mathbf{V})_0 + \tau\mathbf{k} \times (\nabla f)_0.$$

If the steering flow is uniform and rectilinear, this reduces to

$$\mathbf{C} = (1 - \lambda^2\tau)\mathbf{V}_0 + \tau\mathbf{k} \times (\nabla f)_0.$$

Therefore, apart from the planetary effect, all three methods give the same *direction* of propagation, namely that of the steering flow.[2]

The vortex velocity given by (4.1) must change with time, in accordance with changes in the steering flow. The latter changes are governed by the steering-flow prediction equation, which in method I is given by (3.4b), in method II by (3.10b), and in method III by (3.13b). These equations can be written

$$\partial\Gamma/\partial t = \mathbf{P} \cdot \nabla\Psi, \qquad (4.3)$$

$$\partial\Gamma/\partial t = \mathbf{P} \cdot \nabla\Psi + \mathbf{P} \cdot \nabla\psi + (\mathbf{C} - \mathbf{V}) \cdot \nabla\gamma, \quad (4.4)$$

where $\mathbf{P} \equiv \mathbf{k} \times \nabla(\Gamma + f)$. Equation (4.3) governs method I, while (4.4) governs methods II and III on the understanding that $\nabla\gamma = 0$ in method II, so the third term on the right is absent in this case. Thus, methods II and III have one interaction term in common, $\mathbf{P} \cdot \nabla\psi$, which represents advection of steering-flow absolute potential vorticity by the vortex. Method III has an additional interaction term, $(\mathbf{C} - \mathbf{V}) \cdot \nabla\gamma$, which represents advection of vortex potential vorticity by the steering flow relative to propagation velocity of the vortex.

Since in method I the steering flow was defined without interaction, characteristic differences must be expected in *changes* of steering flow when method I is compared with methods II and III; and these differences must lead to corresponding differences in changes of vortex propagation velocity, through (4.1). In the following analysis we investigate this problem by deriving a formula for *acceleration* of the vortex.

[1] This is the case to which method I was applied by KASAHARA (1957). If $\lambda^2 \neq 0$, the analysis can be carried out along similar lines and leads again to a result expressible in the form (4.1).

[2] The speeds are slightly different: in particular, the retarding effect of (quasi-geostrophically induced) divergence should be noted.

With the aid of (4.2) one can write (4.1) in the form

$$\mathbf{C} = (1+\lambda^2\sigma)\mathbf{V}_0 + (\sigma+\tau)\mathbf{P}_0 - \sigma\mathbf{k}\times(\nabla f)_0$$

(where zero subscript signifies evaluation at vortex center), so the vortex acceleration is

$$\frac{d\mathbf{C}}{dt} = (1+\lambda^2\sigma)\left(\frac{d\mathbf{V}}{dt}\right)_0 + (\sigma+\tau)\left(\frac{d\mathbf{P}}{dt}\right)_0,$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{C}\cdot\nabla,$$

if variations of ∇f are neglected. The subsequent analysis can be simplified greatly by means of the reasonable assumption that the steering flow is uniform, instantaneously. Then with the aid of (4.2),

$$\mathbf{P} = -\lambda^2\mathbf{V} + \mathbf{k}\times\nabla f, \tag{4.5}$$

so under the conditions assumed, both $\mathbf{V}$ and $\mathbf{P}$ are instantaneously uniform vectors, and the preceding formula for acceleration reduces to

$$\frac{d\mathbf{C}}{dt} = (1+\lambda^2\sigma)\left(\frac{\partial\mathbf{V}}{\partial t}\right)_0 + (\sigma+\tau)\left(\frac{\partial\mathbf{P}}{\partial t}\right)_0. \tag{4.6}$$

Evaluation of the tendencies required here proceeds from (4.3) or (4.4), as follows.

First, since the steering flow is uniform instantaneously, the middle term of (4.1) is zero, so instantaneously $\mathbf{C} = \mathbf{V} + \tau\mathbf{P}$. (Subscript zero is superfluous here because both $\mathbf{V}$ and $\mathbf{P}$ are uniform.) If this expression is substituted on the right in (4.4), the result is

$$\partial\Gamma/\partial t = \mathbf{P}\cdot\nabla\Psi + \mathbf{P}\cdot\nabla\omega,$$

$$\omega \equiv \psi + \tau\gamma.$$

Since $\mathbf{P} \equiv \mathbf{k}\times\nabla(\Gamma+f)$, application of $\mathbf{k}\times\nabla$ to this equation yields

$$\partial\mathbf{P}/\partial t = \mathbf{k}\times\nabla(\mathbf{P}\cdot\nabla\omega) \tag{4.7}$$

because $\mathbf{P}\cdot\nabla\Psi$ is uniform. It can be shown that when evaluated at the vortex center, (4.7) reduces to[1]

$$(\partial\mathbf{P}/\partial t)_0 = \tfrac{1}{2}(\nabla^2\omega)_0\mathbf{k}\times\mathbf{P}, \tag{4.8}$$

owing to axial symmetry of ω. Equation (4.8) completes the determination of the second term in (4.6) for methods II and III. (In the former, $\tau=0$ so $\omega\equiv\psi+\tau\gamma=\psi$.) In method I, application of $\mathbf{k}\times\nabla$ to (4.3) makes $\partial\mathbf{P}/\partial t=0$, because $\mathbf{P}\cdot\nabla\Psi$ is uniform. Thus, if we take

$$\left.\begin{aligned} &\text{method I: } \omega = 0,\\ &\text{method II: } \omega = \psi,\\ &\text{method III: } \omega = \psi+\tau\gamma,\end{aligned}\right\} \tag{4.9}$$

then (4.7) and (4.8) may be regarded as applicable to all three methods.

Coming now to the first term in (4.6), note that in accordance with (4.2b):

$$(\nabla^2-\lambda^2)\frac{\partial\mathbf{V}}{\partial t} = \frac{\partial\mathbf{P}}{\partial t}. \tag{4.10}$$

With $\partial\mathbf{P}/\partial t$ given as in (4.7), this differential equation may be solved for $\partial\mathbf{V}/\partial t$. It can be shown that when evaluated at the vortex center, the solution may be stated (see footnote)

$$\left.\begin{aligned} &(\partial\mathbf{V}/\partial t)_0 = \tfrac{1}{2}(1-\lambda^2\delta)\,\omega\mathbf{k}\times\mathbf{P},\\ &\delta \equiv \omega_0^{-1}\int_0^\infty K_0(\lambda r)\,\omega r\,dr,\end{aligned}\right\} \tag{4.11}$$

where K_0 is the zero-order exponential Bessel function, and δ is a parameter determined by the structure of the vortex.

With (4.11) for the first term in (4.6), and (4.8) for the second, the vortex acceleration may be expressed finally in the form

$$\left.\begin{aligned} &d\mathbf{C}/dt = -\tfrac{1}{2}G\nabla(\Gamma+f),\\ &G \equiv (1+\lambda^2\sigma)(1-\lambda^2\delta)\,\omega_0 + (\sigma+\tau)(\nabla^2\omega)_0,\end{aligned}\right\} \tag{4.12}$$

since $\mathbf{k}\times\mathbf{P} = -\nabla(\Gamma+f)$. Thus, *the acceleration is proportional to the gradient of steering-flow absolute potential vorticity.* The proportionality factor G is fixed by the structure of the vortex. In method I this factor vanishes since $\omega=0$ by (4.9), so in that method *the vortex acceleration is zero* (when the steering flow is instantaneously uniform). On the other hand, we will show that in the two other methods G carries the sign of ψ_0 and thus is negative for cyclonic vortices, so *cyclonic vortices accelerate in the direction of increasing absolute potential vorticity of the steering flow.* This property of atmospheric

[1] For this derivation, the reader may consult Appendix B of a technical report by KASAHARA & PLATZMAN (1963).

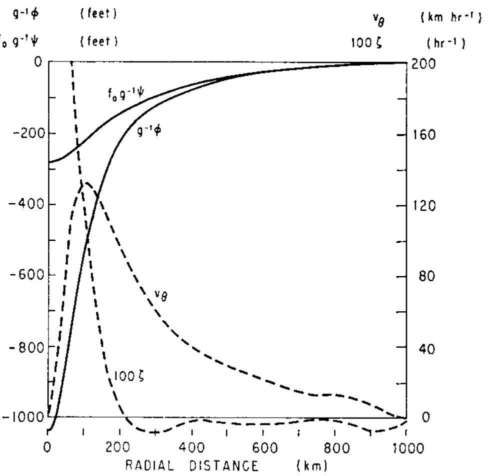

FIG. 1. Vortex data at 500 mb for hurricane Betsy, 1500 GCT 16 August 1956. Solid lines are $g^{-1}\phi$ and $g^{-1}f_0\psi$ in units of feet (ordinate on left side of figure); broken lines are tangential wind speed v_θ in units km hr^{-1} and relative vorticity $\zeta \times 100$ in units hr^{-1} (ordinate on right side). Abscissa: radial distance from vortex center in km.

vortices has been discussed by many authors, notably by ROSSBY (1949). In his analysis—which was from a point of view rather different from that of this paper—only the planetary effect was considered, and as a result, cyclonic vortices were found to have northward acceleration (see also ADEM, 1956).

In addition to the planetary gradient of potential vorticity in the acceleration formula (4.12), we must consider the steering-flow gradient $\nabla\Gamma$. For a uniform flow,

$$\nabla\Gamma = -\lambda^2\nabla\Psi = \lambda^2\mathbf{k}\times\mathbf{V},$$

and we find from (4.12) *a transverse acceleration, the direction of which is toward the left of the steering flow for cyclonic vortices* ($G<0$). In connection with his investigation of what we have called method II, MORIKAWA (1962) has shown (by numerical integration of the steering-flow equation) that there is a similar transverse deflection of a singular vortex embedded in an initially uniform flow.

The potential-vorticity gradient responsible for the transverse acceleration just noted is created by the transverse mass-field gradient that must accompany the quasi-geostrophic (and quasi-hydrostatic) pressure field associated with the steering flow. In a strictly nondivergent model, such mass-field gradients are not produced, and the effect is absent.

In order to assess the sign of G in (4.12), it is sufficient to consider only zero-order terms. Thus, since $\tau = 0(L_v^2)$ and $\lambda^2 = 0(L_s^{-2})$, one finds that $\omega \equiv \psi + \tau\gamma$ reduces to

$$\omega = \psi + \tau\nabla^2\psi + 0(b^{-2}),$$

where $b \equiv L_s/L_v$. Moreover, since $\sigma = 0(L_v^2)$ and $\delta = 0(L_v^2)$, it is easy to see that when second-order terms are omitted, the expression for G in (4.12) reduces to

$$G = \psi_0 + (\sigma + 2\tau)(\nabla^2\psi)_0 + \tau(\sigma+\tau)(\nabla^4\psi)_0. \quad (4.13)$$

In method II we have seen that $\sigma = l^2/8$ and $\tau = 0$; moreover, in the core,

$$(\nabla^2 - \lambda^2)\psi = -\lambda^2(1+p)\psi_0,$$

where p is the function of λl defined in (3.12). It is now easy to see that

$$G = \psi_0\left(1 - \frac{z^3K_1(z)/8}{1 - zK_1(z)}\right),$$

where $z \equiv \lambda l$. If the second term in parentheses is examined as a function of z, it will be found to have a maximum value of 0.2. Therefore, G must carry the sign of ψ_0 in method II.

A rigorous proof in the more general case (method III), where neither σ nor τ is zero, seems very difficult. Instead, consider an idealized vortex

$$\psi = \psi_0\exp(-r^2/4L_v^2).$$

An elementary calculation leads to

$$\left.\begin{aligned} L_v^2\psi_0^{-1}(\nabla^2\psi)_0 &= -1, \quad L_v^4\psi_0^{-1}(\nabla^4\psi)_0 = 2,\\ L_v^{-2}\sigma &= 1/3, \quad L_v^2\tau = 2/3, \end{aligned}\right\} \quad (4.14)$$

the latter having been obtained by evaluating the integrals in (3.17). Substitution of (4.14) into (4.13) yields $G = \psi_0 \cdot \frac{4}{9}$, which confirms that G carries the sign of ψ_0 for the idealized vortex given above. We have also evaluated the four parameters needed in (4.13) from a mean profile of the 500-mb vortex of hurricane Betsy 1956 (see Fig. 1):

$$\left.\begin{aligned} L_v^2\psi_0^{-1}(\nabla^2\psi)_0 &= -1, \quad L_v^4\psi_0^{-1}(\nabla^4\psi)_0 = 4.07,\\ L_v^{-2}\sigma &= 0.45, \quad L_v^{-2}\tau = 0.64. \end{aligned}\right\} \quad (4.15)$$

(Here $L_v = 103$ km has been chosen to make the first of these parameters exactly -1, as in (4.14).) Substitution of (4.15) into (4.13) yields $G = 2.13\psi_0$, again confirming that G carries the sign of ψ_0.

5. Forecasting procedure

We describe now the numerical procedure used to obtain forecasts of the trajectory of hurricane Betsy 1956 which are discussed in the next section.

The basic prediction equations are the vortex trajectory equation (3.1), the steering-flow potential vorticity equation (3.13b) and formula (3.16) for the vortex propagation velocity. The vortex is defined from initial stream-function data by the circular averaging process

$$\left.\begin{aligned} \psi(r) &= \tilde{\Psi}(r) - \tilde{\Psi}(\varrho), \\ \tilde{\Psi}(r) &\equiv \frac{1}{2\pi}\int_0^{2\pi} \hat{\Psi}(r,\theta)\,d\theta. \end{aligned}\right\} \qquad (5.1)$$

Here r, θ are polar coordinates with respect to the vortex center; $\hat{\Psi}(r,\theta)$ is the initial stream function of the *total* flow, $\tilde{\Psi}(r)$ is the circular average of $\hat{\Psi}(r,\theta)$, and ϱ is the radius of a circle at a great distance from the center, where the vortex circulation vanishes.[1]

The initial stream function for the steering flow is defined by

$$\Psi(r,\theta) \equiv \hat{\Psi}(r,\theta) - \psi(r); \qquad (5.2)$$

as shown elsewhere (KASAHARA, 1957), the definition (5.1) of the vortex field—compared with any other axially symmetric function—minimizes the kinetic energy of the residual flow (5.2). In the present work the operations required by (5.1) and (5.2) were performed graphically; however, it is possible to program these steps for a computer, as has been done, for example, by JONES (1961).

In practice, only the geopotential field $\hat{\Phi}(r,\theta)$ (for the total flow) is given initially. Operations (5.1) and (5.2) therefore are performed on $\hat{\Phi}$ and yield the geopotential profile $\phi(r)$ for the vortex, and the steering-flow geopotential field $\Phi(r,\theta)$. We have constructed the vortex stream-function profile from $\phi(r)$ by solving the gradient-wind equation:

[1] The magnitude of ϱ is about 1000 km in the case of hurricane Betsy 1956.

$$\psi(r) = \int_r^{\varrho} \{\tfrac{1}{2}f_0 r - [(\tfrac{1}{2}f_0 r)^2 + r\partial\phi/\partial r]^{\frac{1}{2}}\}\,dr, \qquad (5.3)$$

where ϱ is the radius of the circle on which the vortex circulation vanishes, and f_0 is the value of f at the vortex center.

Once the profile $\psi(r)$ is determined, the corresponding tangential circulation v_θ and potential vorticity γ defined in (3.2) are evaluated from

$$v_\theta = \partial\psi/\partial r, \qquad (5.4\,\text{a})$$

$$\gamma = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \lambda^2\psi, \qquad (5.4\,\text{b})$$

by numerical differentiation; the coefficients σ and τ needed in (3.16) then are obtained by numerical quadrature (typical values are given in section 4). In these calculations, an interval $\Delta r \approx 60$ km was used.

Figure 1 shows the vortex profile for 500-mb data, hurricane Betsy, 1500 GCT 16 August 1956. Included in this figure are: a geopotential profile $g^{-1}\phi(r)$ obtained by circular averaging in the manner of (5.1); a stream function $g^{-1}f_0\psi(r)$ obtained from (5.3); the tangential wind v_θ from (5.4a); and the vorticity ζ from (5.4b) with $\lambda = 0$.

The prediction equation (3.13b) for the steering flow may be stated more explicitly as

$$\left.\begin{aligned} (\nabla^2 - \lambda^2)\frac{\partial\Psi}{\partial t} &= \frac{\partial(\Gamma + f, \Psi)}{\partial(x,y)} + I, \\ I \equiv v_\theta &\left[\frac{\partial(\Gamma+f)}{\partial x}\sin\theta - \frac{\partial(\Gamma+f)}{\partial y}\cos\theta\right] \\ &- \frac{\partial\gamma}{\partial r}[(U - C_x)\cos\theta + (V - C_y)\sin\theta], \end{aligned}\right\} \qquad (5.5)$$

with respect to coordinates x, y and r, θ with origin at the instantaneous position of the vortex center. The expression I includes all interaction terms, and depends upon the structure of the vortex through v_θ and $\partial\gamma/\partial r$. These quantities are stored as one-dimensional tables at intervals $\Delta r \approx 60$ km, from the computation based upon (5.4). At a particular grid point, their values are obtained by interpolation from these tables, using the local value of r. The quantities $\sin\theta$ and $\cos\theta$ in I are computed from $\sin\theta = y/r$ and $\cos\theta = x/r$ at each grid point. All quantities in (5.5) that pertain to the

TABLE 1. *Classification of models and identification symbols.*

	Interaction	Noninteraction
Model A	IA	NIA
Model B	IB	NIB

steering flow, such as the velocity components U, V and gradients of the potential vorticity $\partial\Gamma/\partial x$, $\partial\Gamma/\partial y$ are evaluated from the steering-flow geopotential $\Phi(r, \theta)$ with the usual quasi-geostrophic approximation and by finite differences on the mesh used for numerical integration of (5.5). This mesh consists of a rectangular array of 25×22 squares with mesh length equal to 300 km at the standard latitudes of a 30°/60° Lambert conformal projection.

To obtain some information about the magnitude of the interaction term I, the following quantities were computed for each forecast run:

$$\nu_1 \equiv \{\sum[\mathbf{v} \cdot \nabla(\Gamma + f)]^2\}^{\frac{1}{2}}/D,$$

$$\nu_2 \equiv \{\sum[(\mathbf{V} - \mathbf{C}) \cdot \nabla\gamma]^2\}^{\frac{1}{2}}/D,$$

$$D \equiv \{\sum[\mathbf{V} \cdot \nabla(\Gamma + f)]^2\}^{\frac{1}{2}}.$$

The summation here is taken over all grid points within the vortex domain (about 45 points). The results of computations made with 500-mb data for hurricane Betsy (see section 6) gave $\nu_1 = 1.7$ and $\nu_2 = 2.3$, approximately. Thus, the magnitude of interaction term I of (5.5) for hurricane Betsy was on the average roughly two times larger than that of $\mathbf{V} \cdot \nabla(\Gamma + f)$ in the vortex domain.

6. Results of forecasts

Twenty-four hour and 48-hour forecasts were made with 500-mb data for hurricane Betsy, 13 through 18 August 1956, using four different models to examine the effect of vortex interaction upon the hurricane movement. These models are classified in table 1, where "interaction" and "noninteraction" refer to calculations respectively with and without the interaction term I in prediction equation (5.5). Models A and B correspond to different values of λ, assigned as follows. In the definition of λ, $C_0 \equiv (gD_0)^{\frac{1}{2}}$ is the speed of gravity waves in a homogeneous atmosphere of depth D_0. In order

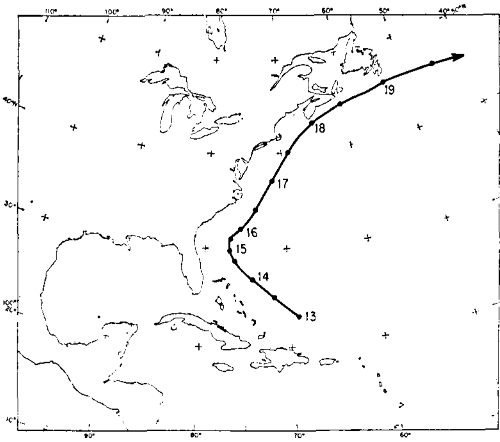

FIG. 2. Observed trajectory of hurricane Betsy, 13–19 August 1956. Twelve-hourly positions (0300 and 1500 GCT) are shown, with dates attached to 0300 positions. Region of integration is slightly larger than region shown here.

to simulate actual atmospheric conditions with the one-layer model, we choose D_0 in such a way that C_0 is approximately equal to the propagation speed of internal gravity waves at the height of the tropopause (≈ 120 m sec^{-1}). From this heuristic reasoning, model A was defined with $D_0 = 5000$ ft $= 1524$ m; this makes $C_0 = 122$ m sec^{-1} and $\lambda^{-1} = C_0/f_0 = 1630$ km (with $f_0 = 0.75 \times 10^{-4}$ sec^{-1}). The value of λ thus obtained is close to that used by the Joint Numerical Weather Prediction Unit in the hemispheric barotropic forecasting model (CRESSMAN, 1958; see also the paper by BOLIN (1956)). To examine the effect of variation in λ upon the hurricane movement, we made forecasts also with $D_0 = 50{,}000$ ft $= 15{,}240$ m (model B); this gives $C_0 = 386$ m sec^{-1} and $\lambda^{-1} = 5150$ km. Note that the increase in D_0 reduces the effect of divergence in the model; in fact, model B is almost nondivergent.[1]

Fig. 2 shows the observed trajectory of hurricane Betsy 1956 at 500 mb in the period 0300 August 13 to 1500 August 19. There are no unusual features of this trajectory. Recurvature began at latitude 27 degrees; by August 18 the vortex was embedded in strong westerly flow. In Fig. 3 are shown the predicted trajectories, using model A (upper part of figure) and model B (lower part). The forecasts initiated

[1] Forecasts were made also with a strictly nondivergent model: the results are very similar to those obtained with model B.

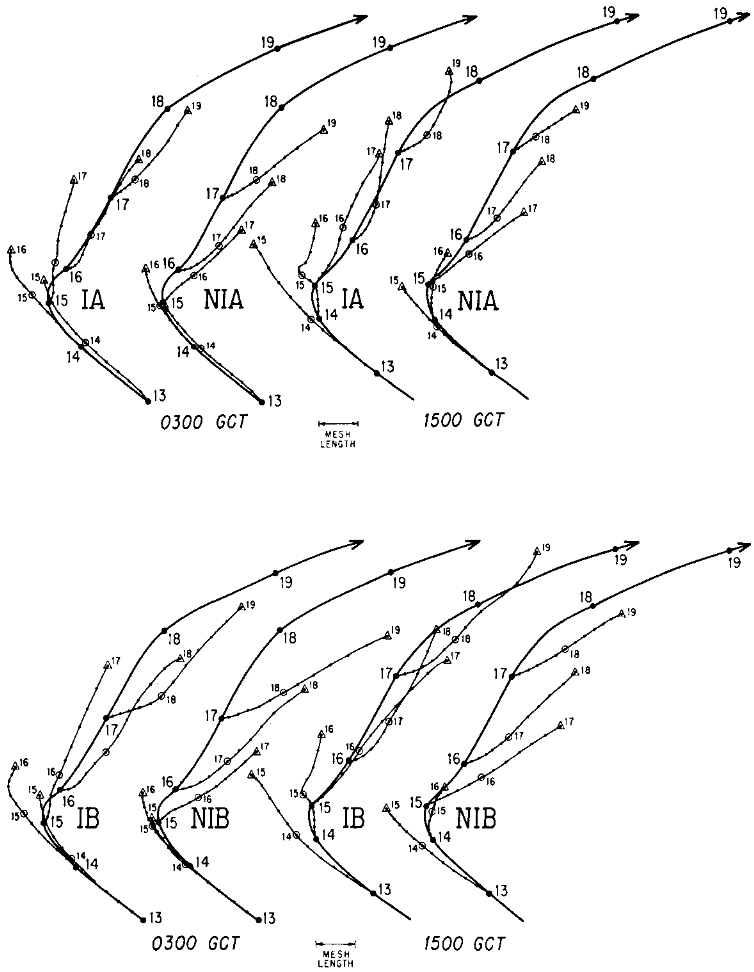

FIG. 3. Predicted trajectories of hurricane Betsy 1956. Upper part, model A; lower part, model B. Left side, forecasts from 0300 GCT data; right side, forecasts from 1500 GCT data. Observed trajectory shown by heavy line, observed 24-hourly positions by large dots. Predicted trajectories shown by light lines, predicted 4-hourly positions by small dots, predicted 24-hour position by circle, predicted 48-hour position by triangle.

from 0300 GCT data are shown on the left, those initiated from 1500 GCT data are on the right.

Qualitative inspection of Fig. 3 confirms that in "noninteraction" forecasts there is a bias that deflects the predicted trajectory to the right of the actual path, as noted in previous work. It is evident also that "interaction" forecasts show a general tendency to correct this bias. This is brought out more clearly in Fig. 4, which shows centroids of "scaled" 24-hour displacement vectors. To obtain these centroids, the length of each 24-hour predicted displacement vector is divided by the length of the corresponding 24-hour observed displacement vector; thus scaled, the predicted vector is plotted relative to the observed, the latter being considered a unit vector, and the mean of these scaled vectors is taken for all forecasts. Both 0300 and 1500 GCT forecasts were used to construct Fig. 4, so that each centroid is determined by ten forecasts. Dispersion of these ten points is represented in Fig. 4 by a circle, the radius of which is equal to the mean of magnitudes of deviations of scaled vectors from their mean.

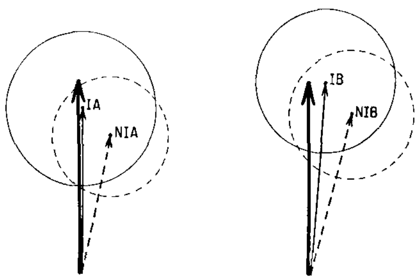

FIG. 4. Heavy arrow represents observed 24-hour "unit" displacement vector of hurricane Betsy 1956. Light arrows point to centroid of predicted 24-hour "scaled" displacement vectors. Broken lines, noninteraction forecasts; solid lines, interaction forecasts. Circles represent dispersion of vectors that determine centroid.

The most interesting aspect of Fig. 4 is the angular deviation of predicted from observed displacement. These angles, as measured from Fig. 4, are listed in Table 2. In model A without interaction, the mean rightward deflection of 13 degrees is virtually eliminated by incorporating interaction terms in prediction of the steering flow.[1] A similar correction takes place in model B, but as would be expected from the analysis of acceleration (section 4), a residual bias remains in this case.

Fig. 4 also shows that in model A, displacement *magnitudes* are too small. The difference in this respect between models A and B is consistent with the fact that divergence contributes dynamically to vortex movement in the direction opposite to that of the steering flow, as noted previously (section 4). This defect in forecasts with model A is brought out more clearly in Table 3, which shows that error-vector magnitudes on the average are smaller in model B (for 48-hour as well as 24-hour forecasts), in spite of the fact that model A greatly reduces the rightward deflection. Also shown in Table 3 (as model C) are results obtained for the same hurricane with a two-level baroclinic

TABLE 2. *Mean angular deviation of predicted from observed 24-hour displacement vector (all are rightward).*

Model	Interaction	Noninteraction
A	0.4°	13°
B	4.8°	15°

TABLE 3. *Mean magnitude of error vector in nautical miles for ten 24-hour and ten 48-hour forecasts of hurricane Betsy 1956.*

	24-hour forecasts[a]		48-hour forecasts[b]	
Model	Interaction	Noninteraction	Interaction	Noninteraction
A	138	135	294	268
B	118	124	211	233
C	—	106	—	256

[a] Magnitude of observed 24-hour displacement: 314 nautical miles.
[b] Magnitude of observed 48-hour displacement: 655 nautical miles.

model (KASAHARA, 1960). This gave better results than model B for 24-hour forecasts, but not for the 48-hour forecasts.

Examination of individual forecasts leads to the conclusion that, particularly on August 16 and 17, prediction of the large-scale flow in middle latitudes was unsatisfactory, especially in model A. Detailed discussion of the possibilities for improvement are beyond the intent of this paper, but it can be said that before any attempt is made to refine these models, the domain of integration should be extended to the complete hemisphere.

7. Conclusions

In the method proposed here for prediction of hurricane movement, the hurricane is defined as an axially symmetric vortex that moves with invariant shape and intensity. A consequence of this definition is that interaction terms appear in the prediction equation for the steering flow (defined as the residual after subtraction of the vortex). In this respect the method can be considered as a generalization of that used by MORIKAWA (1962).

The interaction terms contain the propagation velocity of the vortex—as yet unspecified. The velocity then is defined in such a way as to minimize interactions between vortex and steering flow, and is found to depend upon second-order dynamic effects proportional to vorticity gradients of the steering flow.

[1] Deflections of about 15 degrees were obtained also in "noninteraction" barotropic and baroclinic forecasts of the same hurricane by KASAHARA (1959, 1960).

Some insight is obtained on the consequences of interaction between vortex and steering flow, by an evaluation of vortex propagation acceleration when the steering flow is uniform instantaneously. This acceleration is found to be proportional to the gradient of absolute potential vorticity of the steering flow, with a proportionality factor that depends upon structure of the vortex. For cyclonic vortices the acceleration is in the direction of increasing absolute potential vorticity. The "planetary" contribution to this effect produces the well-known northward acceleration of cyclonic vortices. The "Coriolian" contribution, which comes indirectly through geostrophically induced divergence, is directed at right angles to the steering flow, toward the left for cyclonic vortices.

One of the main conclusions of this investigation is that interaction effects remove the major part of a systematic rightward bias found in earlier hurricane-trajectory forecasts. This conclusion is suggested by the theoretical analysis just described, and is supported by results of numerical computations for hurricane Betsy 1956, which were performed with and without interaction terms in the steering-flow prediction equation.

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