# Currents induced by tides and gravity waves

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ABSTRACT

Two-dimensional oscillatory potential flow was shown by Rayleigh and Schlichting to give rise to a steady second-order streaming at the outer edge of a laminar boundary layer. Their method is here extended to a three-dimensional oscillatory flow which may be represented at a plane boundary by two harmonic oscillations ($U$, $V$) of frequency $\sigma/2\pi$ in perpendicular horizontal directions.

The resulting second-order mean velocity components are interpreted in terms of bottom currents induced by tidal motion in shallow seas, or by doubly modulated gravity waves. These currents depend critically on the relative magnitudes of $U$, $V$ and $f/\sigma$, where $f$ is the Coriolis parameter.

It is suggested that on the tidal scale, such currents may contribute to the circulation within the North Sea, while on the gravity wave scale they contribute to the formation of offshore sandbars around the coasts.

## 1. Introduction

Previous investigations of non-linear effects in oscillatory boundary layers have been concerned with two-dimensional motion. RAYLEIGH (1883) and SCHLICHTING (1932) established the existence of a steady second-order streaming at the outer edge of a laminar boundary layer induced by a first order oscillatory potential flow. LONGUET-HIGGINS (1953) calculated the corresponding Lagrangian particle drift velocities produced by first-order progressive and standing waves, and indicated the physical importance of the mechanism in the transport of sediment beneath gravity waves in shallow water. Schlichting's result was used by ABBOTT (1960) to discuss bottom currents induced by tidal oscillations in slowly converging estuaries, and was further generalised by HUNT (1961) to a two-layer system, again in two-dimensional motion, with reference to sediment transport in estuaries.

It is natural to enquire whether the oscillatory tidal motion in large seas and oceans can induce steady bottom currents which might significantly contribute to the general circulation. In addition, the movement of loose bed material by steady currents in conjunction with tidal oscillations can lead to important changes in bottom topography. The movements of the Goodwin Sands and the Brake Bank provide well documented examples, CLOET (1962).

Away from the constraining influence of narrow channels and estuaries, tidal motion is frequently three-dimensional in character, and near the sea bed a tidal constituent can be represented by the sum of two harmonic oscillations of frequency $\sigma/2\pi$ in perpendicular horizontal directions. In the following sections a solution to the boundary layer problem is sought subject to these prescribed horizontal oscillatory components at its outer edge. Variation of $\sigma$ permits the inclusion of three-dimensional gravity waves as well as tidal oscillations. An interesting feature of the problem is due to the occurrence of the Corioli parameter $f$ in a linear term in the boundary layer equations. As $f\to 0$ approaching the equator, or as $\sigma\to\infty$ for high frequency waves, this linear term becomes smaller than the non-linear terms in the equations. A convenient method of treatment is to consider separate cases in which $f/\sigma$ is of the order $\varepsilon^n$, where $n$ is an integer and the parameter $\varepsilon$ is a measure of the wave slope.

It is found that the second order Eulerian mean velocity at the outer edge of the boundary layer vanishes for tidal oscillations in midlatitudes where $2f/\sigma = 0(1)$, but is non-zero for gravity waves where $2f/\sigma = 0(\varepsilon)$. The Lagrangian particle drift velocity is non-vanishing in both

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cases. The solutions for the particle drift velocity in its two horizontal components, again at the outer edge of the boundary layer, are applied to some examples of wave motions in a homogeneous fluid of constant depth.

The solutions for $2f/\sigma = 0(1)$ are applied to progressive and reflected Kelvin and Sverdrup waves. It seems possible that the currents induced by these tidal waves may account in part for the observed bottom currents in shallow seas, and contribute to their general circulation.

Solutions for $2f/\sigma = 0(\varepsilon)$ are applied to the case of long crested irrotational waves obliquely incident on a straight coast. The computed bottom currents provide a three-dimensional extension of the solutions given by Longuet-Higgins for two-dimensional waves. The results are consistent with the formation of a series of sand-bars or long sand ripples offshore, dependent on the wavelength and orientation of the incident waves.

No account has been taken here of the effects of a variable depth. The distribution of velocity gradients beneath tidal and gravity waves in such conditions must be expected to lead to significant differences in the magnitudes of the bottom currents.

## 2. Boundary layer equations

We employ Cartesian coordinates $(x, y, z)$ fixed with respect to a plane boundary $z = 0$, with the $z$-axis vertically upwards, the $x$-axis to the east, and the $y$-axis to the north. With velocity components $\mathbf{u} = (u, v, w)$ the equations of motion are

$$\left.\begin{aligned} \frac{du}{dt} - 2v\Omega\sin\phi + 2w\Omega\cos\phi &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\ \frac{dv}{dt} + 2u\Omega\sin\phi &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\ \frac{dw}{dt} + 2u\Omega\cos\phi + g &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \end{aligned}\right\} \quad (1)$$

where $\phi$ is the latitude, $\Omega$ the earth's angular velocity ($7.29 \times 10^{-5}$ sec$^{-1}$) and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

A boundary layer approximation near $z = 0$ reduces equations (1) to the form

$$\frac{du}{dt} - 2fv = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \quad (2)$$

$$\frac{dv}{dt} + 2fu = -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \quad (3)$$

for large values of the Reynolds number $R = c\lambda/\nu$ where $c$ denotes wave speed, $\lambda$ the wavelength, and $f = \Omega\sin\phi$. The equation of motion in the $z$-direction becomes

$$g + \frac{1}{\varrho}\frac{\partial p}{\partial z} = 0(1), \quad \Omega\cos\phi \neq 0,$$

or

$$g + \frac{1}{\varrho}\frac{\partial p}{\partial z} = 0(\delta), \quad \Omega\cos\phi = 0,$$

where $\delta$ denotes the boundary layer thickness. It follows that the pressure change across the layer is of order $\delta$ at most, so that the external flow

$$u = U(x,y,t), \quad v = V(x,y,t), \quad w = 0, \quad (4)$$

gives rise to the pressure gradients

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = U_t + UU_x + VU_y - 2fV \quad (5)$$

and

$$-\frac{1}{\varrho}\frac{\partial p}{\partial y} = V_t + UV_x + VV_y + 2fU, \quad (6)$$

where suffixes denote partial derivatives. Equations (2) and (3), with pressure gradients given by (5) and (6), are to be solved together with the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (7)$$

subject to the condition (4) at $z = \delta$, and

$$u = v = w = 0 \quad \text{on} \quad z = 0. \quad (8)$$

## 3. Method of solution

Following Schlichting, a solution to equations (2) and (3) will be obtained by successive approximation supposing the first-order velocity components to be small. The motion within the boundary layer is induced by the oscillatory external flow $(U, V, 0)$ caused by tidal or gravity waves. Accordingly, we express $(u, v, w)$ as power series in a small parameter $\varepsilon$ which is of the order of a wave amplitude to wavelength ratio. Where the tidal velocity is rotatory, both $U$ and $V$ are of the same order of magnitude $\approx 0(\varepsilon)$. Approaching a meridional boundary though, $V = 0(\varepsilon)$ and $U \to 0$ passing through values $U = 0(\varepsilon^m)$, where $m > 1$. Similarly approaching a zonal boundary $U = 0(\varepsilon)$, and $V \to 0$ passing through values $V = 0(\varepsilon^m)$, $m > 1$. Both limiting conditions are readily obtained from the solutions of sections 4 and 5 by assigning the appropriate magnitudes to $U$ and $V$. Such limiting conditions are not confined to the vicinity of fixed boundaries, however. Examples readily quoted include the reflected Kelvin wave model for the semi-diurnal tide in the northerly regions of the North Sea, TAYLOR (1920), in which the currents are rectilinear not merely adjacent to the coastline but across the full width of the region. Here, then, we may write $V = 0(\varepsilon)$, $U = 0$.

A different situation arises with regard to the parameter $f/\sigma$, where $\sigma/2\pi$ is the wave frequency. Once the method of solution by successive approximation has been adopted, $f/\sigma$ can only assume magnitudes corresponding to integral powers of $\varepsilon$. Furthermore, if $f/\sigma = 0(\varepsilon^n)$, each integral value of $n$ requires separate consideration. We cannot allow $f/\sigma$ to tend to zero in a solution obtained for $f/\sigma = 0(1)$ say, since for sufficiently small values the Coriolis term, linear in $(u, v, w)$, must be discarded from the linearised first approximation but retained in the second or some higher order approximation. For a discussion of steady second-order currents it will be necessary to consider only two cases, namely $f/\sigma = 0(1)$, and $f/\sigma = 0(\varepsilon^n)$, $n \geq 1$.

## 4. Low frequency oscillations

For diurnal and semi-diurnal tides in mid-latitudes and for long period tidal constituents in low latitudes we have

$$2f/\sigma = 0(1). \tag{9}$$

For simplicity we may suppose the horizontal components of tidal velocity to be of the same order of magnitude, i.e.

$$u = 0(\varepsilon), \quad v = 0(\varepsilon). \tag{10}$$

We therefore have

$$\mathbf{u} = \varepsilon \mathbf{u}_0 + \varepsilon^2 \mathbf{u}_1 + \dots \tag{11}$$

where $\mathbf{u} = (u, v, w)$, $\mathbf{u}_0 = (u_0, v_0, w_0)$ etc. The first-order boundary layer equations, obtained by substituting (11) in (2) and (3) retaining terms of order $\varepsilon$, are

$$\frac{\partial u_0}{\partial t} - 2fv_0 = \frac{\partial U}{\partial t} - 2fV + \nu \frac{\partial^2 u_0}{\partial z^2} \tag{12}$$

and

$$\frac{\partial v_0}{\partial t} + 2fu_0 = \frac{\partial V}{\partial t} + 2fU + \nu \frac{\partial^2 v_0}{\partial z^2} \tag{13}$$

determining $u_0$ and $v_0$, while $w_0$ follows immediately on retaining terms of order $\varepsilon$ in the continuity equation (7). The boundary conditions are

$$\left.\begin{aligned} u_0 = v_0 = w_0 = 0 \quad &\text{on} \quad z = 0, \\ u_0 = U,\, v_0 = V,\, w_0 = 0 \quad &\text{at} \quad z = \infty. \end{aligned}\right\} \tag{14}$$

For an external flow given by the real part of

$$U = U_0(x, y)\, e^{i\sigma t}, \quad V = V_0(x, y)\, e^{i\sigma t}, \tag{15}$$

equations (12) and (13) possess a solution of the form

$$\left.\begin{aligned} u_0 &= \{U_0(x, y) F_1(\eta) + V_0(x, y) F_2(\eta)\} e^{i\sigma t}, \\ v_0 &= \{V_0(x, y) F_1(\eta) - U_0(x, y) F_2(\eta)\} e^{i\sigma t}, \end{aligned}\right\} \tag{16}$$

where

$$\eta = z(\sigma/2\nu)^{\frac{1}{2}} \tag{17}$$

and, from (7),

$$w_0 = -\left(\frac{2\nu}{\sigma}\right)^{\frac{1}{2}} \left\{ \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) \int_0^\eta F_1(\eta)\, d\eta + \left(\frac{\partial V_0}{\partial x} - \frac{\partial U_0}{\partial y}\right) \int_0^\eta F_2(\eta)\, d\eta \right\} e^{i\sigma t}. \tag{18}$$

The resulting separation of variables leads to equations for $F_1$, $F_2$:

$$\frac{1}{4}\frac{d^4 F_1}{d\eta^4} - i\frac{d^2 F_1}{d\eta^2} + \left(\frac{4f^2}{\sigma^2} - 1\right) F_1 = \frac{4f^2}{\sigma^2} - 1, \tag{19}$$

$$\frac{1}{4}\frac{d^4 F_2}{d\eta^4} - i\frac{d^2 F_2}{d\eta^2} + \left(\frac{4f^2}{\sigma^2} - 1\right) F_2 = 0. \tag{20}$$

The boundary conditions are, from (14).

$$F_1(0) = F_2(0) = F_2(\infty) = 0, \quad F_1(\infty) = 1, \tag{21}$$

and from (12), (13),

$$\left.\begin{aligned} F_1''(0) &= -2i, & F_1''(\infty) &= 0, \\ F_2''(0) &= 4f/\sigma, & F_2''(\infty) &= 0. \end{aligned}\right\} \tag{22}$$

For $2f/\sigma < 1$, the solutions are

$$F_1(\eta) = 1 - \tfrac{1}{2}\{\exp[-(1+i)(1+2f/\sigma)^{\frac{1}{2}}\eta] + \exp[-(1+i)(1-2f/\sigma)^{\frac{1}{2}}\eta]\} \tag{23}$$

and $$F_2(\eta) = -\frac{i}{2}\left\{\exp[-(1+i)(1+2f/\sigma)^{\frac{1}{2}}\eta) - \exp[-(1+i)(1-2f/\sigma)^{\frac{1}{2}}\eta]\right\}. \tag{24}$$

For $2f/\sigma > 1$, it is only necessary to change the sign of the coefficient of $\eta$ in the second exponential term in each of (23) and (24). For latitudes very close to that at which $1 - 2f/\sigma = 0$, the preceding simple boundary layer formulation fails, and no separable solution of the form (16) can be obtained. The solutions for the steady currents outside the boundary layer for $1 - 2f/\sigma > 0$ and $1 - 2f/\sigma < 0$ are of the same form, and being independent of this parameter suggest that they may remain valid through the narrow zonal belt in which the present analysis breaks down, although this has not been established. The parameter $1 - 2f/\sigma$ does not vanish at all in mid-latitudes for tides of the semi-diurnal species, and for diurnal tides only at around 30° latitude.

The equations for the second-order terms are

$$\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z} - 2fv_1 = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \nu\frac{\partial^2 u_1}{\partial z^2}, \tag{25}$$

$$\frac{\partial v_1}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z} + 2fu_1 = U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \nu\frac{\partial^2 v_1}{\partial z^2}. \tag{26}$$

Possible second-order terms in $\partial U/\partial t$, $2fV$ etc., have been omitted from (25) and (26) since they do not contribute to the steady part of $u_1$ and $v_1$, the external tidal flow being harmonic in the time by hypothesis. Substituting equations (15)–(18) in (25) and (26) leads to velocity components of the form

$$(u_1, v_1) = (\bar{u}_1, \bar{v}_1) + (u_1^{(2)}, v_1^{(2)})e^{2i\sigma t}. \tag{27}$$

The steady components $\bar{u}_1$, $\bar{v}_1$ satisfy equations which are very lengthy, but which are of the form

$$\frac{\partial^2 \bar{u}_1}{\partial \eta^2} + \frac{4f}{\sigma}\bar{v}_1 = G_1(x, y, z),$$

$$\frac{\partial^2 \bar{v}_1}{\partial \eta^2} - \frac{4f}{\sigma}\bar{u}_1 = G_2(x, y, z),$$

where $G_1(x,y,z)$ and $G_2(x,y,z)$ tend to zero exponentially as $z \to \infty$, for both $2f/\sigma > 1$ and $2f/\sigma < 1$. It is readily shown that there are no solutions for $\bar{u}_1$, $\bar{v}_1$, bounded as $z \to \infty$, which do not tend to zero. Since we are principally concerned with the values of $\bar{u}_1$, $\bar{v}_1$ at the outer edge of the boundary layer, it is sufficient to note then that for $2f/\sigma = 0(1)$,

$$\bar{u}_1(x, y, \infty) = \bar{v}_1(x, y, \infty) = 0. \tag{28}$$

Although the mean Eulerian velocity $\bar{\mathbf{u}}$ beyond the boundary layer therefore vanishes in this case to the second order, the Lagrangian particle velocity or drift velocity $\bar{\mathbf{U}}$ does not vanish. Using the relation given by LONGUET-HIGGINS (1953),

$$\bar{\mathbf{U}}_1 = \bar{\mathbf{u}}_1 + \overline{\left(\int \mathbf{u}_0\, dt \cdot \nabla\right)\mathbf{u}_0} \tag{29}$$

where, from (11), $\mathbf{u}_0 = (u_0, v_0, w_0)$ and $\mathbf{u}_1 = (u_1, v_1, w_1)$. Denoting the components of the second-order drift velocity $\bar{\mathbf{U}}_1$ by $(\bar{U}_1, \bar{V}_1)$, we have from (16), (28) and (29) the limiting values at the outer edge of the boundary layer

$$\bar{U}_1(x, y, \infty) = -\frac{i}{2\sigma}\left(U_0\frac{\partial U_0^*}{\partial x} + V_0\frac{\partial U_0^*}{\partial y}\right) \tag{30}$$

and $$\bar{V}_1(x, y, \infty) = -\frac{i}{2\sigma}\left(U_0\frac{\partial V_0^*}{\partial x} + V_0\frac{\partial V_0^*}{\partial y}\right), \tag{31}$$

where the real parts of

$$U = U_0(x, y)e^{i\sigma t}, \quad V = V_0(x, y)e^{i\sigma t},$$

are the components of the external tidal motion, (*) denoting the complex conjugate.

Considering now the situation in which the tidal currents are elliptic harmonic with large eccentricity, we write

$$U = 0(\varepsilon), \quad V = 0(\varepsilon^m) \quad m = 2, 3 \ldots, \tag{32}$$

where $\mathbf{u}$ is again given by the series expansion (11). It is easily seen that the solution is now obtained by setting $V = 0$ in the preceding results, (30), (31). The boundary conditions on $\mathbf{u}_0$ are

$$\left.\begin{array}{lll} u_0 = v_0 = w_0 = 0 & \text{on} & z = 0, \\ u_0 = U, \quad v_0 = w_0 = 0 & \text{at} & z = \infty \end{array}\right\}$$

and we have

$$u_0 = U_0(x, y) F_1(\eta) e^{i\sigma t}, \quad v_0 = -U_0(x, y) F_2(\eta) e^{i\sigma t}, \tag{33}$$

where $F_1$ and $F_2$ are again given by (23) and (24). The equations for $u_1$, $v_1$ are (25) and (26) with $V = 0$ so that again

$$\bar{u}_1(x, y, \infty) = \bar{v}_1(x, y, \infty) = 0,$$

and, from (30), (31)

$$\bar{U}_1(x, y, \infty) = -\frac{i}{2\sigma} U_0 \frac{\partial U_0^*}{\partial x}, \quad \bar{V}_1(x, y, \infty) = 0. \tag{34}$$

The corresponding results for the case in which

$$U = 0(\varepsilon^m) \quad m = 2, 3 \ldots, \quad V = 0(\varepsilon),$$

are readily obtained by interchanging the roles of $U_0$ and $V_0$ so that

$$u_0 = V_0(x, y) F_2(\eta) e^{i\sigma t}, \quad v_0 = V_0(x, y) F_1(\eta) e^{i\sigma t}, \tag{35}$$

$$\bar{U}_1(x, y, \infty) = 0, \quad \bar{V}_1(x, y, \infty) = -\frac{i}{2\sigma} V_0 \frac{\partial V_0^*}{\partial y}. \tag{36}$$

In each case the real part is to be taken.

It follows from these solutions that rectilinear tidal oscillations induce steady currents near the bed along the direction in which the tidal flow occurs. The magnitude of these steady currents just *outside* the viscous boundary layer, at a height above the bed of order $\delta = (2\nu/\sigma)^{\frac{1}{2}}$, are given by (34) or (36).

An equally important feature of these solutions is the predicted existence of an oscillatory transverse flow within the boundary layer which vanishes at $z = 0$ and for $z \geq \delta$. The magnitude of this oscillatory flow varies within $0 < z < \delta$ and varies with time according to the real part of (33) or (35) where $F_2(\eta)$ is given by equation (24). For example, at the latitude 52° N a rectilinear semidiurnal tidal flow $V = V_0(x, y) e^{i\sigma t}$, $U = 0$, induces a transverse oscillatory flow within $0 < z < \delta$ which takes the value

$$u_0 = -0 \cdot 198\, V_0(x, y) e^{i(\sigma t - 1.22)}$$

at a height $z = \delta/2$ above the sea bed.

It is evident that close to a vertical barrier or coastline or within a narrow estuary where transverse flow is inhibited or is completely absent, both in the main body of the fluid and in the boundary layer, then the present method of solution fails. The transverse oscillatory pressure gradient, necessary to balance the Coriolis force, becomes inconsistent with the boundary layer formulation of § 2 for purely two-dimensional motion. From equation (3) it is evident that with $v \equiv 0$, $\partial p/\partial y$ changes through the viscous layer by an amount of order $\varepsilon$ which is generally greater than $\delta/h$, (where $h$ is the depth) unless the tidal amplitude is extremely small.

## 5. High frequency oscillations

For high frequency oscillatory flows, generally classified as gravity waves, $f/\sigma$ is small and may be represented in magnitude by

$$2f/\sigma = 0(\varepsilon^n), \quad n = 1, 2 \ldots \tag{37}$$

for some integral value of $n$. Also included in this category are flows of tidal periods in the immediate vicinity of the equator.

Writing $U = 0(\varepsilon)$, $V = 0(\varepsilon)$, and using the expansion (11), the first-order boundary layer equations become

$$\frac{\partial u_0}{\partial t} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 u_0}{\partial z^2} \tag{38}$$

and

$$\frac{\partial v_0}{\partial t} = \frac{\partial V}{\partial t} + \nu \frac{\partial^2 v_0}{\partial z^2} \tag{39}$$

subject to the boundary conditions (14) as before. To this order, the equations of motion (38) and (39) are of the same form as in the corresponding two-dimensional flow. Accordingly, the substitutions

$$\left.\begin{aligned} u_0 &= U_0(x,y)F_3(\eta)e^{i\sigma t},\\ v_0 &= V_0(x,y)F_3(\eta)e^{i\sigma t}\end{aligned}\right\} \tag{40}$$

reduce (38) and (39) to the form

$$\tfrac{1}{2}i\frac{d^2F_3}{d\eta^2}+F_3=1, \tag{41}$$

where $F_3(0)=0$, $F_3(\infty)=1$, and the solution given by SCHLICHTING (1932) is

$$F_3(\eta)=1-e^{-(1+i)\eta}. \tag{42}$$

From the continuity equation, the component of velocity in the $z$-direction is

$$w_0=-\left(\frac{2\nu}{\sigma}\right)^{\frac{1}{2}}e^{i\sigma t}\int_0^\eta\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)F_3(\eta)\,d\eta. \tag{43}$$

When $n=1$ in (37) and $2f/\sigma=0(\varepsilon)$, the second-order velocity components $(u_1, v_1)$, determined by terms of order $\varepsilon^2$ in (2) and (3), contain steady terms and terms proportional to $e^{i\sigma t}$ and $e^{2i\sigma t}$. When $n\geqslant 2$ however, only steady terms and those proportional to $e^{2i\sigma t}$ occur. The time-average second-order boundary layer equations, which determine the steady part of $(u_1, v_1)$ namely $(\bar{u}_1, \bar{v}_1)$, are the same for all $n\geqslant 1$,

$$\overline{u_0\frac{\partial u_0}{\partial x}}+\overline{v_0\frac{\partial u_0}{\partial y}}+\overline{w_0\frac{\partial u_0}{\partial z}}=\overline{U\frac{\partial U}{\partial x}}+\overline{V\frac{\partial U}{\partial y}}+\nu\frac{\partial^2\bar{u}_1}{\partial z^2}, \tag{44}$$

$$\overline{u_0\frac{\partial v_0}{\partial x}}+\overline{v_0\frac{\partial v_0}{\partial y}}+\overline{w_0\frac{\partial v_0}{\partial z}}=\overline{U\frac{\partial V}{\partial x}}+\overline{V\frac{\partial V}{\partial y}}+\nu\frac{\partial^2\bar{v}_1}{\partial z^2}. \tag{45}$$

The boundary conditions are

$$\left.\begin{aligned} &\bar{u}_1=\bar{v}_1=0 \quad \text{on} \quad z=0,\\ &\bar{u}_1, \bar{v}_1 \text{ finite} \quad \text{at} \quad z=\infty.\end{aligned}\right\} \tag{46}$$

By means of the substitutions

$$-\sigma\bar{u}_1=F_4(\eta)\,U_0\frac{\partial U_0^*}{\partial x}+F_5(\eta)\,V_0\frac{\partial U_0^*}{\partial y}+F_6(\eta)\,U_0\frac{\partial V_0^*}{\partial y}$$

and

$$-\sigma\bar{v}_1=F_4(\eta)\,V_0\frac{\partial V_0^*}{\partial y}+F_5(\eta)\,U_0\frac{\partial V_0^*}{\partial x}+F_6(\eta)\,V_0\frac{\partial U_0^*}{\partial x},$$

equations (44) and (45) lead to equations for $F_4$, $F_5$, $F_6$ independent of $U_0$, $V_0$, namely

$$\frac{d^2F_4}{d\eta^2}=1-|F_3|^2+F_3'\int_0^\eta F_3^*\,d\eta,$$

$$\frac{d^2F_5}{d\eta^2}=1-|F_3|^2,$$

$$\frac{d^2F_6}{d\eta^2}=F_3'\int_0^\eta F_3^*\,d\eta,$$

where $F_3$ is given by equation (42). Solutions satisfying the boundary conditions (46) are

$$F_4=-\tfrac{1}{2}(1+3i)e^{-(1+i)\eta}+\tfrac{1}{2}ie^{-(1-i)\eta}-\tfrac{1}{4}(1-i)e^{-2\eta}+\tfrac{1}{2}(1-i)\eta e^{-(1+i)\eta}+\tfrac{3}{4}(1+i), \tag{47}$$

$$F_5=-\tfrac{1}{2}ie^{-(1+i)\eta}+\tfrac{1}{2}ie^{-(1-i)\eta}-\tfrac{1}{4}e^{-2\eta}+\tfrac{1}{4}, \tag{48}$$

$$F_6=-\tfrac{1}{2}(1+2i)e^{-(1+i)\eta}+\tfrac{1}{2}(1-i)\eta e^{-(1+i)\eta}+\tfrac{1}{4}ie^{-2\eta}+\tfrac{1}{4}(2+3i). \tag{49}$$

At the outer edge of the boundary layer, as $z\to\infty$,

$$F_4\to\tfrac{3}{4}(1+i), \quad F_5\to\tfrac{1}{4}, \quad F_6\to\tfrac{1}{4}(2+3i)$$

so that, (retaining only the real parts)

$$\bar{u}_1(x,y,\infty)=-\frac{1}{4\sigma}\left\{3U_0\frac{\partial U_0^*}{\partial x}+V_0\frac{\partial U_0^*}{\partial y}+2U_0\frac{\partial V_0^*}{\partial y}+3iU_0\left(\frac{\partial U_0^*}{\partial x}+\frac{\partial V_0^*}{\partial y}\right)\right\} \tag{50}$$

and

$$\bar{v}_1(x,y,\infty) = -\frac{1}{4\sigma}\left\{3V_0\frac{\partial V_0^*}{\partial y} + U_0\frac{\partial V_0^*}{\partial x} + 2V_0\frac{\partial U_0^*}{\partial x} + 3iV_0\left(\frac{\partial U_0^*}{\partial x} + \frac{\partial V_0^*}{\partial y}\right)\right\}. \quad (51)$$

The corresponding Lagrangian drift velocity components, obtained from (29) using (40), (50) and (51), are

$$\bar{U}_1(x,y,\infty) = -\frac{1}{4\sigma}\left\{3U_0\frac{\partial U_0^*}{\partial x} + V_0\frac{\partial U_0^*}{\partial y} + 2U_0\frac{\partial V_0^*}{\partial y} + i\left(5U_0\frac{\partial U_0^*}{\partial x} + 3U_0\frac{\partial V_0^*}{\partial y} + 2V_0\frac{\partial U_0^*}{\partial y}\right)\right\} \quad (52)$$

and

$$\bar{V}_1(x,y,\infty) = -\frac{1}{4\sigma}\left\{3V_0\frac{\partial V_0^*}{\partial y} + U_0\frac{\partial V_0^*}{\partial x} + 2V_0\frac{\partial U_0^*}{\partial x} + i\left(5V_0\frac{\partial V_0^*}{\partial y} + 3V_0\frac{\partial U_0^*}{\partial x} + 2U_0\frac{\partial V_0^*}{\partial x}\right)\right\}. \quad (53)$$

It is necessary of course to take real parts of equations (52) and (53) to obtain the predicted drift velocities.

On setting $V_0 = 0$ in equation (52) we recover the solution given by LONGUET-HIGGINS (1953) for two-dimensional gravity waves. The drift velocity just above the viscous layer at the bed is then given by the real part of the expression

$$-\frac{1}{4\sigma}(3+5i)\,U_0\frac{\partial U_0^*}{\partial x},$$

so that

$$\bar{U}_1 = -\frac{1}{8\sigma}\left\{3\left(U_0\frac{\partial U_0^*}{\partial x} + U_0^*\frac{\partial U_0}{\partial x}\right) + 5i\left(U_0\frac{\partial U_0^*}{\partial x} - U_0^*\frac{\partial U_0}{\partial x}\right)\right\}. \quad (54)$$

## 6. Application to tidal co-oscillations

The results derived in § 4 for low frequency oscillations may be applied to semi-diurnal co-oscillations. Sufficient illustration of the method will be provided by two simple examples from linearised long wave theory.

We consider first a southward progressing Kelvin wave of elevation $z = \zeta_1(x,y,t)$ given by

$$U = 0, \quad V = -A\frac{c}{h}e^{-2fx/c}\cos(ky+\sigma t),$$

$$\zeta_1 = h + Ae^{-2fx/c}\cos(ky+\sigma t), \quad (55)$$

where $c^2 = (\sigma/k)^2 = gh$, and $A$ is a constant. Substituting for $V_0$ in equation (36) we find

$$\bar{U}_1 = 0, \quad \bar{V}_1 = -\tfrac{1}{2}c\left(\frac{A}{h}\right)^2 e^{-4fx/c}. \quad (56)$$

The addition of a similar northward progressing Kelvin wave $\zeta_2$, where

$$\zeta_2 = h + Ae^{2fx/c}\cos(ky-\sigma t), \quad (57)$$

leads to a resultant drift velocity near the bed,

$$\bar{U}_1 = 0, \quad \bar{V}_1 = c\left(\frac{A}{h}\right)^2 \sinh\frac{4fx}{c}. \quad (58)$$

The application of this model to the northerly regions of the North Sea, TAYLOR (1920), suggests the existence of a southward bottom current along its western side together with a northward bottom current along its eastern side. These currents vanish along the line of amphidromic points $x = 0$. With $A = 50$ cm, $h = 4000$ cm, $\phi = 58°$ N, then

$$\bar{V}_1 \approx 1\cdot 0 \text{ cm/sec}$$

at $x = 160$ km.

The second example is a southward progressing Sverdrup wave of elevation $\zeta_3(x,y,t)$ given by

$$U = \frac{2fA}{k_1 h}\cos(k_1 y + \sigma t),$$

$$V = -\frac{\sigma A}{k_1 h}\sin(k_1 y + \sigma t),$$

$$\zeta_3 = h + A\sin(k_1 y + \sigma t), \quad (59)$$

where $k_1^2 = (\sigma^2 - 4f^2)/gh$. Substituting for $U_0$ and $V_0$ in equations (30) and (31),

$$\bar{U}_1 = 0, \quad \bar{V}_1 = -\tfrac{1}{2}\left\{\frac{gh}{1-4f^2/\sigma^2}\right\}^{\frac{1}{2}}\left(\frac{A}{h}\right)^2. \quad (60)$$

The addition of similar northward progressing Sverdrup wave $\zeta_4$ given by

$$\zeta_4 = h + A \sin (k_1 y - \sigma t) \tag{61}$$

leads to the resulting drift velocity near the bed

$$\bar{U}_1 = -\frac{2f}{k_1}\left(\frac{A}{h}\right)^2 \sin 2k_1 y, \quad \bar{V}_1 = 0. \tag{62}$$

This result is quite different from that corresponding to the superposition of Kelvin waves (58), where the drift was in the direction of the individual wave propagation. For Sverdrup waves, (62) indicates a transverse drift vanishing at $2k_1 y = n\pi$. Its maximum value, occurring at $2k_1 y = (n + \frac{1}{2})\pi$, takes the value

$$(\bar{U}_1)_{\max} = 0 \cdot 5 \text{ cm/sec}$$

for $A = 50$ cm, $h = 4000$ cm, $\phi = 58°$ N.

## 7. Application to doubly-modulated gravity waves

The results of § 5 concerning high frequency oscillations are readily applied to the case of doubly modulated gravity waves. Such motions are caused for example by the oblique incidence of long crested waves against a coastline. If small-amplitude irrotational waves propagate in a direction which makes an angle $\tan^{-1}(q/p)$ with a coastline $y = 0$, the resulting motion may be represented by the velocity potential

$$\Phi = \frac{Ag}{\sigma}\frac{\cosh rz}{\cosh rh} \cos qy \sin (px + \sigma t), \tag{63}$$

where $\quad \sigma^2 = gr \tanh rh, \quad r^2 = p^2 + q^2$

and $\quad \mathbf{u} = -\operatorname{grad} \Phi. \quad (64)$

The surface elevation is given by

$$\zeta = h + A \cos qy \cos (px + \sigma t) \tag{65}$$

which indicates a wave having the characteristics of a progressive wave in the longshore (negative $x$) direction, and a standing wave in the offshore direction. From equations (63) and (64) may be derived the oscillatory potential flow ($U_0$, $V_0$) at the outer edge of the boundary layer, and taking the real parts of equations (52) and (53) we have the drift components

$$\bar{U}_1 = -\frac{pgA^2}{4\sigma r \sinh 2rh}\{5r^2 + (r^2 + 4p^2)\cos 2qy\} \tag{66}$$

and $$\bar{V}_1 = -\frac{qgA^2}{4\sigma r \sinh 2rh}\{r^2 + 2q^2\} \sin 2qy. \tag{67}$$

These solutions reduce to those of LONGUET-HIGGINS (1953) for the two-dimensional standing wave when $p = 0$, and for the two-dimensional progressive wave when $q = 0$.

From (66) we see that $\bar{U}_1 < 0$ everywhere, that is the longshore component of bottom drift is everywhere in the direction of propagation of the progressive component of the wave motion. Moreover, $|\bar{U}_1|$ is a maximum at $y = n\pi/q$, that is beneath antinodes of the offshore standing wave pattern. Similarly $|\bar{U}_1|$ is a minimum at $y = (n + \frac{1}{2})\pi/q$ where the nodes of the standing wave pattern occur.

The offshore component $\bar{V}_1$ of the bottom drift at the outer edge of the boundary layer is directed towards the shore for

$$2l\pi < 2qy < (2l + 1)\pi, \quad l = 0, 1, 2 \ldots$$

and is directed seawards for

$$(2l + 1)\pi < 2qy < 2(l + 1)\pi, \quad l = 0, 1, 2 \ldots.$$

In so far as these currents are capable of moving sediment, in conjunction with the oscillatory wave motion, it follows that erosion is to be expected in the vicinity of $2qy = (2l + 1)\pi$, $l = 0, 1, 2 \ldots$, and deposition to occur where $2qy = 2l\pi$, $l = 0, 1, 2 \ldots$. A series of sand-bars parallel to the coast would be consistent with such a process.

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