

On representative observations

By DANIEL P. PETERSEN, *Weather System Center, UAC Corporate Systems Center, Farmington, Connecticut*, and DAVID MIDDLETON, *Consulting Physicist, Concord, Massachusetts*

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ABSTRACT

The traditional concept of a "representative observation" is examined and the need for a rigorous and objective definition is discussed in the light of the advancing technology of numerical weather analysis and prediction. Recent advances in the theory of sampling and filtering of multi-dimensional stochastic processes now make meaningful the following definition: a representative observation is a datum pertaining to a particular point and time, which is the result of an optimum filtering operation on the continuous raw data field, under the criterion of minimum average mean square error of reconstruction from the subsequent sampled values on a given space/time lattice.

In many cases of interest, the applicable filter weighting functions can be determined without complete knowledge of the wave-number spectra of the "signal" and "noise" processes. While the required operations can only be approximated in the case of conventional surface and upper air observations, in two important areas—radar and satellite instrumentation—simple and elegant mechanization may be feasible. In addition, the development yields usable insight into such questions as the density and arrangement of observation networks, numerical smoothing of discrete data, and approximations to derivatives used in numerical analysis and prediction.

1. Introduction

One of the earliest and most oft-repeated admonitions given the neophyte meteorologist is, in effect, to "beware the unrepresentative observation". To the information theorist and to the designer of meteorological systems, there is a distinctly disturbing implication that the analyst, by vague appeal to a permanent professional indulgence, is entitled to disregard on any of several grounds the reports delivered in good faith by his observation and data collection network. Moreover, in this era of accelerating mechanization of the meteorological art, rigorous mathematical definitions are needed of concepts that may heretofore have belonged to the professional folklore—definitions that may both be translated into precise computer language and may be subjected to equally rigorous statistical verification. The purpose of this paper is to propose one such rigorous definition of this elusive concept, based on recent advances in the theory of sampling, filtering

and reconstruction of multi-dimensional random processes (PETERSEN & MIDDLETON, 1962).

What then is a "representative observation"? In what sense does a discrete sample measurement convey information about the variation of a continuous field of values within a non-vanishing region of the observation space? Is no more rigorous definition possible than to "...suppose that the station is representative of the area included in the circle ... (having a) 'radius of action' R ..." (WMO Note Technique No. 30, 1960)?

In his well-known textbook, PETERSEN (1940) informs the reader on p. 2 that "... it is necessary to ascertain that the observations which we make use of are *representative*." On pages 445–449, more explicit instructions are given: the analyst's task, for instance, is "to draw the *representative* isobars" by correcting "errors" in the reports; namely, "(1) personal or accidental errors of observations, (2) errors of reduction to sea level, (3) errors caused by the observations being made too early or too late relative to the standard hour of observation, (4) errors of coding, transmission, decoding, or plotting." As a guide to the analyst, the author twice emphasizes that "*simple isobars are much*

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more probable than complicated isobars'. Furthermore, the data should be checked with surrounding reports, with collateral information (e.g., winds) and with previous reports. In other words, actual measurements should be corrected on the basis of preconceived theory and of analyses of the prevailing meteorological situation. Presumably lumped with "errors" here are the bona fide high-frequency components of atmospheric motions which would continue to pose serious "smoothing" problems even if errors of instrumentation, communication, and human frailty were to be completely removed.

In an interesting analysis, THOMPSON (1956) derives the optimum (mean-square) filter for separating additive "noise" from "signal" in a *continuous* two-dimensional field of data. Then, lumping together as "noise" both the short wavelength components of the field and the various sources of data handling errors, he assigns the resultant a Gaussian autocorrelation function with an exponential parameter dependent on the "distance between adjacent observing stations". Using a signal autocorrelation of similar form but different parameters, he computes an approximate filter whose effective "radius of the domain of averaging" he asserts is strongly dependent on the "scale ratio" of the "true" to the "error" field. Closer examination, however, reveals that this "smoothing radius" is *not* a function of the non-dimensional "scale ratio" but is almost proportional to the *signal* correlation distance alone. Its dependence on the *noise* correlation distance (equated by the author with the station-to-station spacing), is, although weak, in the same direction or *inverse* to the change in "scale ratio". The author then reports an empirical investigation using this filter, in which the "noise" added is (apparently) *uncorrelated* from point to point (thus having *zero* correlation distance; a "white noise"), and in which the continuous operations are simulated by discrete ones. The experiment is thus not only performed on an inadequate sample but omits entirely the critical effects of station arrangement and spacing. The concluding recommendation that stations be spaced with a density of 15 over a "characteristic wavelength" of the field (especially in view of the earlier statement that meteorological fields are "not genuinely periodic"), cannot be taken seriously without some basis for assigning a relative value to the *worth* of

a weather information system in relation to its *cost*.

A somewhat different approach has been taken by HOUSE (1960), who obtained an explicit formula for optimum spacing (d) of upper-air stations,

$$d = 2 \left(\frac{3\sigma}{|y''|} \right)^{\frac{1}{3}}, \quad (1)$$

in which σ is the standard deviation of measurement and y'' is the third derivative of the variable function in the region of observation. In obtaining this relation, however, the author assumed that only two sample values are to be used for interpolation, and thus reached the anomalous conclusion that, the less reliable the measurements, the fewer are needed! One would at least intuitively feel that a *dense* network of (rough) observations could somehow be averaged to take advantage of the well-known statistical effect of variance reduction (CRAMÉR, 1946).

On the other hand, faced with the problems of practical implementation of numerical weather analysis and prediction, the various groups associated in the Joint Numerical Weather Prediction Unit have found it necessary to "smooth" the data obtained in discrete form from the weather networks. Several numerical "smoothing operators" have been used, reported for example by SHUMAN (1957). Although these are known to attenuate the shorter wavelengths in the field while leaving virtually unchanged the long-wave components, the elements of the array are generally chosen empirically, based on the reasonableness of the final results. Similar operators have been investigated by FJÖRTOFT (1955) and SIGTRYGGSSON & WIIN-NIELSEN (1957).

In another context the term "representative" has come to connote the effect of individual sensor placement on the usability of its readings. Perhaps a location in a plain meadow, on the windward side of a lake, or at the top of a tower is more "representative" than some other area in the immediate locality. Indeed, the U.S. Air Force is in the process of establishing "rep-ob" observing sites near the intersection of principal runways on its airfields, obviously intending that the measurements obtained therefrom will be "representative" of that immediate vicinity for at least those short time intervals during which critical decisions are made. However, we

shall not use the term in this restricted sense in this paper.

Before proceeding with the mathematical development of our proposed definition of a "representative observation", let us observe here our assumption that the purpose of weather observation in general is to establish the state of the atmosphere; and that, in turn, knowledge of the present state is required in the process of forecasting or prediction (under the tacit assumption that atmospheric dynamics constitute a simple "Markov" process). The forecasting task cannot be considered "distinct from that of weather *analysis*" (THOMPSON, 1961; see also WMO Note Technique No. 30, 1960, and GODSKE *et al.*, 1957), but these must be treated as integral aspects of a single activity, at least for the operational applications of meteorology (as opposed to those of pure scientific research).

2. Definitions and criteria

Let us start by defining more precisely what we mean by an *observation*. We assume first of all the objective existence of a measurable real phenomenon defined over the space under consideration. For example, such a variable might be the northerly component of wind over three physical dimensions and time. We postulate further that the values of this variable at all its space/time arguments are of interest, at least within certain limits of accuracy, and that values at neighboring points are to some extent correlated so that a finite density of sampling points can yield usable information about the function over all space.

The variable of interest is stochastic or random, such that no finite set of real numbers can exactly define its value at all points; it can only with error be interpolated or extrapolated to non-sampling points. It is thus intuitively evident that the accuracy with which such a function can be reconstructed from its sample values is critically dependent on the density of sampling points in the four-dimensional space. Ideally continuous sampling (assuming no errors in instrumentation or transmission) would yield perfect reproduction; but in most situations of meteorological interest continuous space/time measurement is not conceptually possible, and in all cases would be prohibitively costly. Thus, finite sampling plans must be employed to obtain acceptable descriptions of

the process state with reasonable resources of instrumentation, data processing, and communications facilities. This paper will not concern itself with those factors which influence the "reasonableness" of resource allocation, but rather with the techniques for achieving efficient utilization of those resources assumed to have been made available.

At one or more data processing locations, the collected samples may be analyzed by various techniques. In general, the ultimate objective here is to reconstruct the function at specific points not coincident with the arguments of the received data (e.g., displaced in time). The accuracy of such reconstruction is obviously of critical interest, and an objective measure of such accuracy is essential for the comparison of alternative procedures. In what follows, a least-mean-square criterion will be employed, primarily for its mathematical convenience; in specific system design problems, other criteria may prove more significant but may be used only at the cost of mathematical complexity. Furthermore, linear operations on the observed data will be assumed; nonlinear operations would change the detailed derivations but not the general concepts of the presentation.

Accordingly, we define a *simple observation* or *sample value* of the measured variable as a measurement influenced by a negligibly small space/time region of its argument. This measurement operation may be conceived mathematically as modulation of the phenomenological function by a (lattice of) multidimensional unit "delta" or impulse functions, characterizing the "sampling plan".

As we have seen above, it is intuitively evident that some form of weighted average of the observed field surrounding the given sampling point will yield a datum which is more "representative" of the field than a simple observation. That this is indeed the case under rather broad conditions will be demonstrated in the following section. The product of such a procedure, that is, an *optimally prefiltered* datum derived from a random variable *under a given sampling plan*, will be termed a *representative observation*.

3. Mathematical theory

The equations of fluid mechanics, either in their most general form or in the various approximations developed for numerical weather

prediction (THOMPSON, 1961), define the future states of the system in terms of a continuous "synoptic" analysis. The use of such equations for extrapolation of values and patterns implies the tacit assumption both of the Markovian nature of the process (LAX, 1960) and of certain symmetry in the distribution of the various exciting forces. It is, however, manifestly impracticable to obtain at any one time a continuous "initial condition" from which a classical mathematical solution of the problem may proceed. One is thus compelled to reconstruct the field from a finite set of discrete sample values. This procedure is of course not new; "curve fitting" or interpolation is as old as the scientific collection and reduction of data. Nevertheless, the concepts of stationary stochastic processes in the field of communication theory have given new meaning to sampling and interpolation procedures and have introduced useful new techniques. The methods developed for analysis of one-dimensional (time) functions have recently been extended to multi-dimensional (space/time) fields (PETERSEN & MIDDLETON, 1962).

A. ONE-DIMENSIONAL SAMPLING THEORY

Although CAUCHY (1841) stated a form of sampling theorem, WHITTAKER (1915) was apparently the first to observe that there exists a unique "smoothest" curve which can be drawn through an infinite set of ordinate values at equally spaced arguments $t = kT$, $k = 0, \pm 1, \pm 2, \dots$. This curve, containing no Fourier frequency component above half the sampling frequency $\omega_s = 2\pi/T$, may be found from the "cardinal series" expansion

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) g(t - kT), \quad (2)$$

where
$$g(\tau) = \frac{\sin(\pi\tau/T)}{\pi\tau/T} \quad (3)$$

is called the "cardinal" weighting function (Fig. 1).

Applying essentially the inverse of Whittaker's results to communication problems, NYQUIST (1928) showed that signals limited to band B (cps) were effectively describable by real numbers generated at a rate $2B$ per second. KOTEL'NIKOV (1933) and SHANNON (1949), however, enunciated the first usable form of samp-

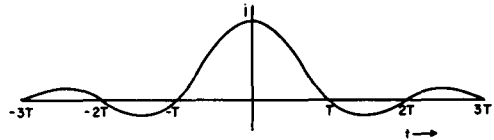


FIG. 1. The "cardinal" weighting function.

ling theorem. They observed that the coefficients of the Fourier expansion of the amplitude spectrum of the given function of time,

$$a_k = -\frac{1}{4\pi B} \int_{-2\pi B}^{2\pi B} F(\omega) \exp(i k \omega / 2B) d\omega, \quad (4)$$

were, except for the factor $2B$, identical with the inverse transform of $F(\omega)$ at sampling instants:

$$f\left(\frac{k}{2B}\right) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} F(\omega) \exp(i k \omega / 2B) d\omega. \quad (5)$$

The inverse term-by-term Fourier transform of the expanded spectrum,

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{2B} f\left(\frac{k}{2B}\right) \\ &\times \left\{ \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \exp[i\omega(t - k/2B)] d\omega \right\} \\ &= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2B}\right) \frac{\sin 2\pi B(t - k/2B)}{2\pi B(t - k/2B)}, \end{aligned} \quad (6)$$

again yields the "cardinal series" (2). The sampling theorem holds for any functions whose spectra are part of a periodic function in frequency with the period $2\pi/T$; however, the reconstruction function $g(\tau)$ depends on the limits of the integral in (6).

In dealing with stochastic (random) processes, the transforms or spectra of individual functions are no longer defined in the usual sense; one must instead consider process-wide statistical averages. It is usually assumed that the process mean value is zero (i.e., that variations are taken with reference to a steady state). Then if one uses a mean-square criterion for evaluating goodness of reconstruction, the only required statistical parameter is the autocovariance

$$K(t, \tau) = E\{f(t)f(t + \tau)\}; \quad E\{f(t)\} \equiv 0. \quad (7)$$

For "stationary" processes, the statistics are independent of the time origin, and the auto-variance is a function of time displacement alone; under these conditions its Fourier transform

$$\Phi(\omega) = \int_{-\infty}^{\infty} K(\tau) e^{-i\omega\tau} d\tau \quad (8)$$

is called the "power spectral density" of the process. BALAKRISHNAN (1957) has shown that, if

$$\Phi(\omega) = 0, \quad |\omega| > 2\pi B, \quad (9)$$

then the "cardinal" series (6) holds also for random processes in the sense of a vanishing mean-square error. A comprehensive treatment of the statistical foundations of communication theory may be found in MIDDLETON (1960).

While the "classical" sampling theory as reviewed above has provided considerable insight into problems of information transmission and data handling, and is especially applicable to communication system design in which, for purposes of frequency multiplexing of signals, one deliberately attempts to construct a rectangular spectral characteristic in each channel, it is basically unsatisfactory for analysis of physical phenomena which do not in fact exhibit "bandlimited" spectra. One must then expect error in reconstructing such functions from their discrete samples. We seek to minimize the error by judicious choice of reconstruction weighting function. One method for determining such an optimum weighting function is the extension of WIENER (1948)-KOLMOGOROV (1934) filter theory to sampled-data systems (see, for instance, RAGAZZINI & FRANKLIN, 1958, and bibliography therein). Several authors (SPILKER, 1960; CHANG, 1961; DERUSSO, 1961; BROWN, 1961) have also considered the prefiltering of data to be sampled. These techniques can all be shown to be special cases of a general multidimensional sampling and filtering theory to be described in the next section.

B. MULTIDIMENSIONAL SAMPLING THEORY

Development of a theory of sampling for functions of multidimensional arguments has

¹ A broad discussion of the role of space/time measurements of Markov stochastic processes governed by known dynamical equations is included in a recent doctoral dissertation (PETERSEN, 1963).

² Not, for the moment, otherwise restricted.

received negligible attention in the literature of information and communication theory or in that of the specific disciplines to which it is applicable. A notable exception is the article by MIYAKAWA (1959), which essentially extended the Kotelnikov-Shannon theorem through the use of multidimensional Fourier series and transformations. The present authors (PETERSEN & MIDDLETON, 1962) have further developed the theory to include the sampling and reconstruction of wave-number-limited stochastic processes, optimum pre- and post-filtering of non-wave-number-limited processes, and efficient sampling of isotropic processes up to eighth-order Euclidean spaces. The present paper concerns the filtering of additive noise from multidimensional sampled data and presents a number of applications and examples of especial interest to meteorological system operations, not treated in the earlier paper.¹

In what follows, the vector argument $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes a set of n independent variables defining the location of a point in a Euclidean "sampling space" (which may be, e.g., three conventional space directions and time). Defined on this space is the (real) function² $f(\mathbf{x})$ which is to be sampled at a lattice of points distributed throughout the space of infinite extent according to a set of n displacement vectors $\{\mathbf{v}_j\} = \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, which form a basis (not necessarily rectangular or unitary) for the space. Thus, the sampling points are the arguments defined by

$$\mathbf{v}_{[m]} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \dots + m_n \mathbf{v}_n,$$

$$m_1, m_2, \dots, m_n = 0, \pm 1, \pm 2, \dots \quad (10)$$

The spectrum of the function $f(\mathbf{x})$ is its Fourier transform (where it exists)

$$F(\omega) = \int_{\mathbf{x}} f(\mathbf{x}) e^{-i\omega \cdot \mathbf{x}} d\mathbf{x} \quad (11)$$

in which the integration extends over the entire space X , and the exponent in the integrand kernel contains the inner or dot product of $\mathbf{x}$ with a multidimensional "wave-number" vector ω :

$$\omega \cdot \mathbf{x} = \omega_1 x_1 + \omega_2 x_2 + \dots + \omega_n x_n. \quad (12)$$

We thus conceive of (11) as effecting a transformation from the "sampling space" X

to the "wave-number" space Ω ; in one dimension ω is the familiar frequency variable, when $x(=t)$ represents time.

The sampling procedure may be represented as a modulation of the phenomenological function by a lattice of multidimensional Dirac "delta" functions:

$$f_s(\mathbf{x}) = f(\mathbf{x}) \sum_{[m]} \delta(\mathbf{x} - \mathbf{v}_{[m]}); \quad (13)$$

which, in turn, may be expanded in a multi-dimensional Fourier series:

$$f_s(\mathbf{x}) = \frac{1}{Q} f(\mathbf{x}) \sum_{[l]} e^{i\mathbf{x} \cdot \mathbf{u}_{[l]}}, \quad (14)$$

which introduces the vector set $\{\mathbf{u}_k\}$ "inverse to" the set $\{\mathbf{v}_j\}$:

$$\begin{aligned} \mathbf{u}_{[l]} &= l_1 \mathbf{u}_1 + l_2 \mathbf{u}_2 + \dots + l_n \mathbf{u}_n, \\ l_1, l_2, \dots, l_n &= 0, \pm 1, \pm 2, \dots; \\ \mathbf{u}_k \cdot \mathbf{v}_j &= 2\pi \delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}. \end{aligned} \quad (15)$$

Q is the hypervolume of the parallelepiped defined by the vectors $\{\mathbf{v}_j\}$. With a change of variable in the integrand, the spectrum of the sampled function becomes

$$F_s(\omega) = \frac{1}{Q} \sum_{[l]} F(\omega - \mathbf{u}_{[l]}); \quad (16)$$

which exhibits, in multidimensional "wave-number" space, the "spectrum repetition" effect of sampling, familiar in the one-dimensional theory. It is clear from (16) that, if $F(\omega)$ is to be recoverable from $F_s(\omega)$ —and thus $f(\mathbf{x})$ from $f_s(\mathbf{x})$ —the successive "images" of $F(\omega)$ on the vector basis $\{\mathbf{u}_k\}$ must not overlap, and a reconstruction filter must be used which passes the spectrum $F(\omega)$ and rejects its images (Fig. 2). This is, in effect, a statement of the multidimensional sampling theorem (PETERSEN & MIDDLETON, 1962).

Let us now turn directly to the general problem of sampling and reconstruction of homogeneous stochastic fields. The function $f(\mathbf{x})$ is now to be considered as a particular member of an ensemble of fields whose statistical properties will be defined as we proceed. We suppose the multidimensional "signal" $s(\mathbf{x})$ to be contaminated with additive noise:

$$f(\mathbf{x}) = s(\mathbf{x}) + n(\mathbf{x}). \quad (17)$$

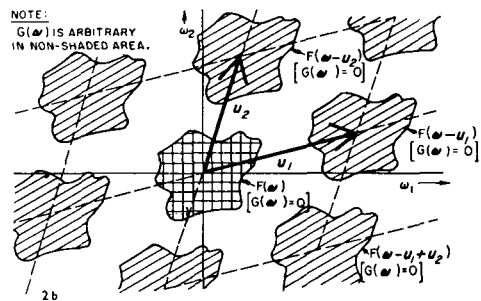
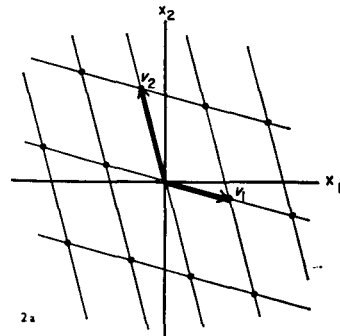


FIG. 2. The multi-dimensional sampling theorem. (a) A lattice of sampling points. (b) Wave-number spectrum repetition and spectrum of reconstruction function. Reprinted by permission of Academic Press, Inc. (Petersen and Middleton, 1962).

We propose to filter (linearly) the raw data field $f(\mathbf{x})$ before sampling on a lattice with the vector basis $\{\mathbf{v}_{jj}\}$:

$$\psi(\mathbf{v}_{[m]}) = \int_{\mathbf{x}} f(\mathbf{x}) \gamma(\mathbf{v}_{[m]} - \mathbf{x}) d\mathbf{x}, \quad (18)$$

where $\gamma(\mathbf{x})$ is the weighting function of the pre-sampling filter. Next, it is desired to reconstruct the original *signal* from the sampled data,

$$\hat{s}(\mathbf{x}) = \sum_{[m]} \psi(\mathbf{v}_{[m]}) g(\mathbf{x} - \mathbf{v}_{[m]}), \quad (19)$$

with minimum mean-square error

$$E = E\{[s(\mathbf{x}) - \hat{s}(\mathbf{x})]^2\}. \quad (20)$$

Expanding the bracket in (20), assuming homogeneous statistics (a "wide-sense stationary" field), and taking $E\{s(\mathbf{x})\} = E\{n(\mathbf{x})\} = 0$ by suitable choice of the origin of measurements, we define a set of covariance functions:

$$\left. \begin{aligned} K_{ss}(\mathbf{x}) &= E\{s(\mathbf{y}) s(\mathbf{y} + \mathbf{x})\}, \\ K_{ff}(\mathbf{x}) &= E\{f(\mathbf{y}) f(\mathbf{y} + \mathbf{x})\}, \\ K_{fs}(\mathbf{x}) &= E\{f(\mathbf{y}) s(\mathbf{y} + \mathbf{x})\}, \\ K_{sf}(\mathbf{x}) &= E\{s(\mathbf{y}) f(\mathbf{y} + \mathbf{x})\} = K_{fs}(-\mathbf{x}) \end{aligned} \right\} \quad (21)$$

and obtain

$$\begin{aligned} \mathbf{E} &= K_{ss}(\mathbf{0}) - 2 \sum_{[m]} g(\mathbf{x} - \mathbf{v}_{[m]}) \int_{\mathbf{x}} \gamma(\mathbf{v}_{[m]} - \mathbf{y}) \\ &\quad \times K_{fs}(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \sum_{[l]} \sum_{[m]} g(\mathbf{x} - \mathbf{v}_{[l]}) g(\mathbf{x} - \mathbf{v}_{[m]}) \\ &\quad \times \int_{\mathbf{x}} \int_{\mathbf{x}} \gamma(\mathbf{v}_{[m]} - \mathbf{y}) \gamma(\mathbf{v}_{[l]} - \mathbf{z}) K_{ff}(\mathbf{z} - \mathbf{y}) d\mathbf{y} d\mathbf{z}. \end{aligned} \quad (22)$$

Next, we hold $\gamma(\mathbf{x})$ temporarily fixed and apply a variational procedure to $g(\mathbf{x})$ to determine the necessary condition for minimum error:

$$\begin{aligned} 2 \int_{\mathbf{x}} \gamma(\mathbf{v}_{[k]} - \mathbf{y}) K_{fs}(\mathbf{x} - \mathbf{y}) d\mathbf{y} &= \sum_{[l]} g(\mathbf{x} - \mathbf{v}_{[l]}) \\ &\quad \times \int_{\mathbf{x}} \int_{\mathbf{x}} \gamma(\mathbf{v}_{[k]} - \mathbf{y}) \gamma(\mathbf{v}_{[l]} - \mathbf{z}) K_{ff}(\mathbf{z} - \mathbf{y}) d\mathbf{y} d\mathbf{z} \\ &\quad + \sum_{[m]} g(\mathbf{x} - \mathbf{v}_{[m]}) \int_{\mathbf{x}} \int_{\mathbf{x}} \gamma(\mathbf{v}_{[m]} - \mathbf{y}) \\ &\quad \times \gamma(\mathbf{v}_{[k]} - \mathbf{z}) K_{ff}(\mathbf{z} - \mathbf{y}) d\mathbf{y} d\mathbf{z}. \end{aligned} \quad (23)$$

[It is not difficult to show that (23) indeed defines a *minimum* error.] Again using a delta-modulation transformation and an n -dimensional version of Parseval's theorem, we may express condition (23) in the wave-number domain:

$$\begin{aligned} 2\Phi_{fs}(\boldsymbol{\omega}) \Gamma(-\boldsymbol{\omega}) &= \frac{G(\boldsymbol{\omega})}{Q} \sum_{[l]} \Gamma(\boldsymbol{\omega} - \mathbf{u}_{[l]}) \Gamma(\mathbf{u}_{[l]} - \boldsymbol{\omega}) \\ &\quad \times [\Phi_{ff}(\boldsymbol{\omega} - \mathbf{u}_{[l]}) + \Phi_{ff}(\mathbf{u}_{[l]} - \boldsymbol{\omega})], \end{aligned} \quad (24)$$

in which $\Gamma(\boldsymbol{\omega})$, $G(\boldsymbol{\omega})$, $\Phi_{fs}(\boldsymbol{\omega})$ and $\Phi_{ff}(\boldsymbol{\omega})$ are Fourier transforms respectively of $\gamma(\mathbf{x})$, $g(\mathbf{x})$, $K_{fs}(\mathbf{x})$ and $K_{ff}(\mathbf{x})$. If there is no prefiltering, $\Gamma(\boldsymbol{\omega}) = 1$ everywhere and (24) becomes

¹ This choice is justified heuristically for the general theory on the basis that accurate reconstruction of the field at all points is equally desirable; were certain points or regions to be preferred over others, these would be measured directly rather than reconstructed from a broad array of observations.

$$2\Phi_{fs}(\boldsymbol{\omega}) = \frac{G(\boldsymbol{\omega})}{Q} \sum_{[l]} [\Phi_{ff}(\boldsymbol{\omega} - \mathbf{u}_{[l]}) + \Phi_{ff}(\mathbf{u}_{[l]} - \boldsymbol{\omega})], \quad (25)$$

which defines the optimum reconstruction weighting function $g(\mathbf{x})$ through its transform $G(\boldsymbol{\omega})$ wherever the summation on the right is nonvanishing. In particular, in the case where noise is absent and the signal is wave-number-limited and sampled on a sufficiently dense lattice, a judicious choice of $G(\boldsymbol{\omega})$ can make (25) an identity, and reconstruction with zero expected error is possible (PETERSEN & MIDDLETON, 1962).

Returning to the general problem (i.e., not assuming wave-number-limited spectra), we next wish to determine the optimum form of $\gamma(\mathbf{x})$ holding $g(\mathbf{x})$ fixed. We observe now that *all* values of $\gamma(\mathbf{x})$, not just a denumerable set, are involved in determining the error at any one point $\mathbf{x}$. Thus our criterion must be to minimize the mean-square error (defined as an *ensemble* average) at some *particular* point $\mathbf{x}$ or to minimize some weighted average of the mean-square error over the space X . Since the error (22) is clearly periodic on the lattice $\{\mathbf{v}_j\}$, we need average only over the parallelepiped b formed by the basis vectors $\{\mathbf{v}_j\}$, and we arbitrarily choose a *uniform* average:¹

$$\begin{aligned} \mathbf{E} &\triangleq \frac{1}{Q} \int_b \mathbf{E}(\mathbf{x}) d\mathbf{x} = K_{ss}(\mathbf{0}) - \frac{2}{Q} \sum_{[m]} \int_b g(\mathbf{x} - \mathbf{v}_{[m]}) \\ &\quad \times \int_{\mathbf{x}} \gamma(\mathbf{v}_{[m]} - \mathbf{y}) K_{fs}(\mathbf{x} - \mathbf{y}) d\mathbf{y} d\mathbf{x} \\ &\quad + \frac{1}{Q} \sum_{[l]} \sum_{[m]} \int_b g(\mathbf{x} - \mathbf{v}_{[l]}) g(\mathbf{x} - \mathbf{v}_{[m]}) \\ &\quad \times \int_{\mathbf{x}} \int_{\mathbf{x}} \gamma(\mathbf{v}_{[m]} - \mathbf{y}) \gamma(\mathbf{v}_{[l]} - \mathbf{z}) K_{ff}(\mathbf{z} - \mathbf{y}) d\mathbf{y} d\mathbf{z} d\mathbf{x}. \end{aligned} \quad (26)$$

Noting that the integrands in (26) can, by a change of variable, be made a function of $(\mathbf{x} - \mathbf{v}_{[m]})$, we may exchange the summation over $[m]$ for integration over the entire space X :

$$\begin{aligned} \mathbf{E} &= K_{ss}(\mathbf{0}) - \frac{2}{Q} \int_{\mathbf{x}} \int_{\mathbf{x}} g(\mathbf{x}) \gamma(-\mathbf{y}) K_{fs}(\mathbf{x} - \mathbf{y}) d\mathbf{y} d\mathbf{x} \\ &\quad + \frac{1}{Q} \sum_{[k]} \int_{\mathbf{x}} \int_{\mathbf{x}} \int_{\mathbf{x}} g(\mathbf{x} - \mathbf{v}_{[k]}) g(\mathbf{x}) \gamma(-\boldsymbol{\sigma}) \gamma(\mathbf{v}_{[k]} - \boldsymbol{\tau}) \\ &\quad \times K_{ff}(\boldsymbol{\tau} - \boldsymbol{\sigma}) d\boldsymbol{\sigma} d\boldsymbol{\tau} d\mathbf{x}. \end{aligned} \quad (27)$$

Again, we convert (27), by using a delta-function convolution and Fourier series, to

$$\begin{aligned} \mathbf{E} = & K_{ss}(0) - \frac{2}{Q} \int_X \int_X g(\mathbf{x}) \gamma(-\mathbf{y}) K_{fs}(\mathbf{x}-\mathbf{y}) d\mathbf{y} d\mathbf{x} \\ & + \frac{1}{Q^2} \sum_{[m]} \int_X \int_X \int_X \int_X g(\mathbf{x}-\boldsymbol{\rho}) g(\mathbf{x}) \gamma(-\boldsymbol{\sigma}) \gamma(\boldsymbol{\rho}-\boldsymbol{\tau}) \\ & \times K_{ff}(\boldsymbol{\tau}-\boldsymbol{\sigma}) e^{i\boldsymbol{\rho} \cdot \mathbf{u}_{[m]}} d\boldsymbol{\sigma} d\boldsymbol{\tau} d\mathbf{x} d\boldsymbol{\rho}. \end{aligned} \quad (28)$$

Now varying $\gamma(\mathbf{x})$, we obtain the necessary condition for a minimum of (26):

$$\begin{aligned} 2 \int_X g(\mathbf{x}) K_{fs}(\mathbf{x}+\mathbf{z}) d\mathbf{x} = & \frac{1}{Q} \sum_{[m]} \int_X \int_X \int_X g(\mathbf{x}-\boldsymbol{\rho}) \\ & \times g(\mathbf{x}) \gamma(\boldsymbol{\rho}-\boldsymbol{\tau}) K_{ff}(\boldsymbol{\tau}+\mathbf{z}) e^{i\boldsymbol{\rho} \cdot \mathbf{u}_{[m]}} d\boldsymbol{\tau} d\mathbf{x} d\boldsymbol{\rho} \\ & + \frac{1}{Q} \sum_{[m]} \int_X \int_X \int_X g(\mathbf{x}-\boldsymbol{\rho}) g(\mathbf{x}) \gamma(-\boldsymbol{\sigma}) \\ & \times K_{ff}(\boldsymbol{\rho}-\boldsymbol{\sigma}-\mathbf{z}) e^{i\boldsymbol{\rho} \cdot \mathbf{u}_{[m]}} d\boldsymbol{\sigma} d\mathbf{x} d\boldsymbol{\rho}; \end{aligned} \quad (29)$$

Finally, Fourier transformation of (29) yields

$$\begin{aligned} 2\Phi_{fs}(\boldsymbol{\omega}) G(-\boldsymbol{\omega}) = & \frac{\Gamma(\boldsymbol{\omega})}{Q} [\Phi_{ff}(\boldsymbol{\omega}) + \Phi_{ff}(-\boldsymbol{\omega})] \\ & \times \sum_{[m]} G(\boldsymbol{\omega}-\mathbf{u}_{[m]}) G(\mathbf{u}_{[m]}-\boldsymbol{\omega}). \end{aligned} \quad (30)$$

We now require the simultaneous solution of (24) and (30) to obtain the optimum *combination* of pre- and post-sampling filters $\Gamma(\boldsymbol{\omega})$ and $G(\boldsymbol{\omega})$. This, however, is not a trivial task; the straightforward approach [solving (30) for $\Gamma(\boldsymbol{\omega})$ and substituting in (24)], leads to the anomalous conclusion that the spectral functions are periodic! However, using an argument essentially the same as that pursued in Appendix E of PETERSEN & MIDDLETON (1962), we conclude that $\Gamma(\boldsymbol{\omega})$ and $G(\boldsymbol{\omega})$ can be nonzero only within a basic cell C of wave-number space (i.e., a region repeatable without overlap on the vector basis $\{\mathbf{u}_k\}$), symmetrical with respect to the origin; and that, within this cell,

$$G(\boldsymbol{\omega}) \Gamma(\boldsymbol{\omega}) = 2Q \frac{\Phi_{fs}(\boldsymbol{\omega})}{\Phi_{ff}(\boldsymbol{\omega}) + \Phi_{ff}(-\boldsymbol{\omega})}. \quad (31)$$

[Note that, if there is no noise, the right side of (31) is simply Q since $\Phi_{ss}(\boldsymbol{\omega})$ must be even.] To find the *shape* of the cell C , we return to (28) and convert it to the wave-number domain:

$$\begin{aligned} \mathbf{E} = & \frac{1}{(2\pi)^n} \int_{\Omega} \Phi_{ss}(\boldsymbol{\omega}) d\boldsymbol{\omega} - \frac{2}{Q} \frac{1}{(2\pi)^n} \\ & \times \int_{\Omega} G(-\boldsymbol{\omega}) \Gamma(-\boldsymbol{\omega}) \Phi_{fs}(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ & + \frac{1}{Q^2} \sum_{[m]} \frac{1}{(2\pi)^n} \int_{\Omega} G(\boldsymbol{\omega}+\mathbf{u}_{[m]}) G(-\boldsymbol{\omega}-\mathbf{u}_{[m]}) \\ & \times \Gamma(\boldsymbol{\omega}) \Gamma(-\boldsymbol{\omega}) \Phi_{ff}(\boldsymbol{\omega}) d\boldsymbol{\omega}. \end{aligned} \quad (32)$$

Then, we substitute (31) and the other constraints on C to obtain

$$\begin{aligned} \mathbf{E} = & \frac{1}{(2\pi)^n} \int_{\Omega} \Phi_{ss}(\boldsymbol{\omega}) d\boldsymbol{\omega} - \frac{4}{(2\pi)^n} \\ & \times \int_C \frac{\Phi_{fs}(\boldsymbol{\omega}) \Phi_{fs}(-\boldsymbol{\omega}) \Phi_{ff}(-\boldsymbol{\omega})}{[\Phi_{ff}(\boldsymbol{\omega}) + \Phi_{ff}(-\boldsymbol{\omega})]^2} d\boldsymbol{\omega}. \end{aligned} \quad (33)$$

Evidently, for minimum average mean-square error, we must choose the shape of the basic wave-number cell such that, of every denumerable set of arguments $\{\boldsymbol{\omega} + \mathbf{u}_{[k]}\}$, that value of $\boldsymbol{\omega}$ lies within the cell for which the second integrand of (33) is greatest, and all others lie outside. Also, since only the product $\Gamma(\boldsymbol{\omega})G(\boldsymbol{\omega})$ is specified by (31), we arbitrarily choose $G(\boldsymbol{\omega})=Q$ within C , as we would require for reconstruction were the signal uncontaminated with noise and ideally wave-number-limited.

We note that, if there is no noise, (33) becomes

$$\mathbf{E} = \frac{1}{(2\pi)^n} \int_{\Omega-C} \Phi_{ss}(\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad (34)$$

so that the error is simply the total spectral intensity *outside* the basic wave-number cell corresponding to the given sampling plan. Evidently, a process which is *both* noise-free and wave-number-limited can be reproduced from its samples on an appropriate lattice with zero average mean-square error.

We now come to the statement of our proposed definition: a *representative observation* is a datum derived from a continuous field of raw data by means of a linear filter whose spectrum $\Gamma(\boldsymbol{\omega})$ has the following properties: for every denumerable set of arguments $\{\boldsymbol{\omega} + \mathbf{u}_{[k]}\}$, we choose

$$\Gamma(\boldsymbol{\omega} + \mathbf{u}_{[k]}) = \frac{2\Phi_{fs}(\boldsymbol{\omega} + \mathbf{u}_{[k]})}{\Phi_{ff}(\boldsymbol{\omega} + \mathbf{u}_{[k]}) + \Phi_{ff}(-\boldsymbol{\omega} - \mathbf{u}_{[k]})} \quad (35)$$

for that value of $[k]$ for which

$$\frac{\Phi_{fs}(\omega + \mathbf{u}_{[k]}) \Phi_{fs}(-\omega - \mathbf{u}_{[k]}) \Phi_{ff}(-\omega - \mathbf{u}_{[k]})}{[\Phi_{ff}(\omega + \mathbf{u}_{[k]}) + \Phi_{ff}(-\omega - \mathbf{u}_{[k]})]^2} \quad (36)$$

is greatest, and $\Gamma(\omega + \mathbf{u}_{[k]}) = 0$ for all other values of $[k]$. The reconstruction weighting function $g(\mathbf{x})$ has a spectrum $G(\omega)$ which equals the constant Q (the hypervolume of the sampling parallelepiped) over the region where $\Gamma(\omega)$ is nonvanishing, and is zero elsewhere. Thus, $g(\mathbf{x})$ is a "canonical" weighting function (PETERSEN & MIDDLETON, 1962); the set $\{g(\mathbf{x} + \mathbf{v}_{[m]})\}$ is orthogonal over the space X , and

$$g(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} = 0 \\ 0, & \mathbf{x} = \mathbf{v}_{[k]} \neq 0 \end{cases} \quad (37)$$

In passing, let us observe that this development justifies in retrospect the study of "wave-number-limited" processes: although not occurring naturally in physical systems, the exigencies of numerical processing of discrete measurement data make desirable the pre-filtering of raw observations to simulate an ideally limited spectral characteristic. It is, of course, understood that such a filter is not strictly realizable because of its demand for an infinite field of raw data.

We note also that the above analysis is based on the assumption that a particular sampling lattice has been specified. We may expect the error (33) to change not only with the *density* of sampling points, but also with their arrangement, i.e., with the relative sizes and orientation of the vectors $\{\mathbf{v}_j\}$. Conversely, given only the required density of observation stations, we may search for the optimum arrangement which minimizes (33). At the present time, this appears to require a trial-and-error procedure. It is, of course, clear that denser sampling, implying a larger wave-number cell C , generally results in smaller error.

4. Applications to meteorological measurement and processing

A. NEED FOR STATISTICAL INFORMATION

In attempting to apply the mathematical development of the preceding section directly to meteorological problems, one becomes painfully aware of the lack of suitable spectral or correlative data on atmospheric motions. The

first point that needs to be made here, therefore, is that a prime objective of climatological research ought to be just such determination of ever more reliable statistical data upon which not only *forecasting* techniques but the *locations* of observing stations and the *analysis* of their reports must be based.

B. DENSITY OF OBSERVATIONS

It was stated earlier that this paper cannot concern itself with the economic worth of weather information and the resultant decisions by cognizant agencies on the overall density of observations in space and time. However, it is axiomatic that the value of weather information, measured as it must be by the losses that would occur in its absence as a result of incorrect operational decisions, must be a function of its accuracy in the pertinent situations. Such an evaluation can only be performed through statistical analyses of all operations sensitive to meteorological conditions, in connection with estimates of the (maximum achievable) accuracy of forecasting as a function of the density of observing networks. One cannot, by examining a curve of accuracy of interpolation vs. density of observations (THOMPSON, II, 1957), select a point of "diminishing return" without regard to the ultimate operational worth of meteorological information.

We may comment also on the conclusion of the WMO Working Group on Networks (WMO Note Technique No. 30, 1960) that the density of stations should not be so great as to make *instrumentation* (rather than *interpolation*) the dominant source of error. The above development shows that the effect of instrumentation "noise" on interpolation error may be reduced by filtering a dense (or continuous) field of raw data. Whether a given investment may more profitably be applied toward instrumentation improvement or toward increasing the density of observation depends on the particular situation and is not at all a foregone conclusion.

Some analyses of atmospheric motions have disclosed a tendency for disturbances of wavelength larger than a critical value to amplify spontaneously while smaller wavelengths decay (LORENZ, 1953; THOMPSON, I, 1957). Even apart from this phenomenon of nonlinear instability, however, it can be shown, at least in simple situations (e.g., thermal diffusion), that there is

a natural tendency for *selective* decay of wavelengths in an initial field of values. If the process is stationary, the wavelengths which decay must be replaced by fluctuations generated by "noise" excitation, which are of course wholly unpredictable if we are dealing with a genuine Markov process. Thus, one concludes that the *longer* the forecast period, the *longer* the wavelengths that are of interest in the analysis. This does *not*, however, justify a sparse observing network for these fields since the wavelengths used must be known with high precision. In other words, what we require for long-range prediction is still a *dense* network of observations which can be *prefiltered* to a small wave-number passband, eliminating to the greatest possible extent the disturbing noise and short-wavelength "aliasing" errors.

C. ARRANGEMENTS OF SAMPLING LATTICES

Although in the general case the selection of an optimum sampling lattice of given density would appear to be a trial-and-error procedure (although one which could, given the necessary statistical data, be readily programmed for digital computer exploration), further assumptions and constraints may allow more definitive statements to be made. Let us, for example, suppose that expression (36) is *isotropic*—that is, a function of wave-number magnitude $|\omega|$ alone—and *monotonically decreasing* with increasing wave-number magnitude. The region over which $\Gamma(\omega)$ is nonvanishing is then evidently a repeatable cell of given hypervolume which is most nearly hyperspherical in shape. But this is precisely the classical geometrical problem of close-packing of hyperspheres, which has been rigorously solved for the first eight Euclidean spaces (COXETER, 1951), and for

which the optimum sampling lattices may be immediately specified (PETERSEN & MIDDLETON, 1962). In one dimension, $G(\omega)$ becomes, of course, the ideal low-pass filter (Fig. 3). In two dimensions, the closest packing of circles in the wave-number plane occurs with their centers on a 60° rhombic lattice; its inverse, the sampling lattice itself, is also a 60° rhombic (sometimes, incorrectly, referred to as "equilateral triangular"), (Fig. 4). Close packing of spheres in three-dimensional space occurs with their centers in a "face-centered cubic" lattice, whose inverse, the optimum sampling lattice, is "body-centered cubic" (Fig. 5).

We remark here that others (WMO Note Technique No. 30, 1960) have also concluded that a "triangular" network is more "efficient" than a square grid for sampling two-dimensional processes. This result, however, was apparently based on the concept of a "radius of influence"

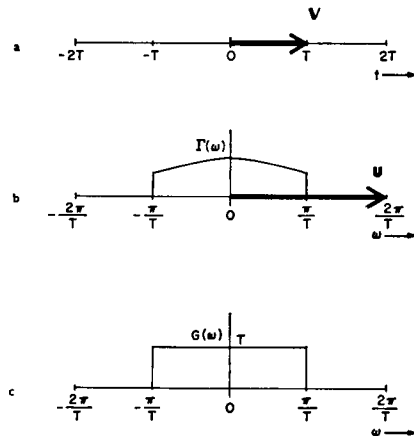


FIG. 3. Sampling a one-dimensional isotropic process. (a) Sampling lattice in time domain. (b) Presampling filter spectrum. (c) Reconstruction filter spectrum.

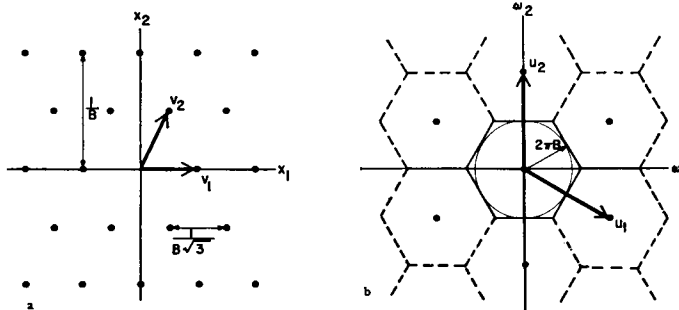


FIG. 4. Sampling a two-dimensional isotropic function. (a) Minimum density sampling lattice. (b) Closest packing of repetitive spectra.

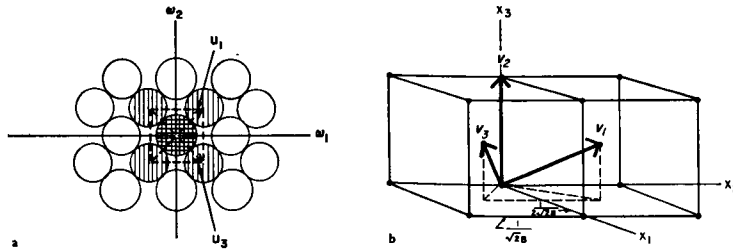


FIG. 5. Sampling a three-dimensional isotropic function. (a) Closest packing of repetitive spectra per layer (face-centered cubic lattice). (b) minimum density sampling lattice (body-centered cubic lattice). Reprinted by permission of Academic Press, Inc. (Petersen and Middleton, 1962).

of each observing station. We have shown that optimum reconstruction or interpolation of sampled data is governed by the spectral function in wave-number space. The fact that, for two-dimensional isotropic processes the identical result is obtained may be attributed to geometrical coincidence.

The concept of *efficiency* of various sampling plans must be carefully defined and applied with caution. Even under the assumptions of isotropy and monotonicity, a reasonable definition must depend on the cross-sectional shape of the spectra of signal and noise. MIYAKAWA (1959) proposed as a definition of sampling efficiency for wave-number-limited processes the ratio of the actual wave-number "coverage" of the process spectrum to the hypervolume of the smallest repeatable cell corresponding to the given sampling lattice and fully enclosing the process spectrum. Thus, for example, the efficiency of a square lattice for sampling a two-dimensional isotropic function was found to be 78.5 %; while that of a 60° rhombic lattice was 90.8 %. It is worth noting, however, that in the extreme case (when the spectrum is flat out to its wave-number cutoff), an increase in the size of the square lattice to equate its

density with that of the rhombic lattice entails an error of only 4.9 % (Fig. 6). Under more realistic circumstances the difference would be considerably less.

An "optimum" arrangement of sampling points may not always be permitted in system design. For instance, stations in the surface meteorological network are constrained to be fixed in position; thus one of the sampling lattice vectors must be parallel to the "time" axis of coordinates in the four-dimensional sampling space. On the other hand, observations from aircraft, ships or balloons are constrained by the speed of the transporting vehicle, thus fixing the slope of one of the sampling lattice vectors. Finally, it may be desired to scan the various stations on a network in sequential order from a central computer control complex, the period between calls being equivalent to the length of data messages at the prescribed signalling speeds. All of these constraints restrict in some respect the ability of the system designer to optimize the size and shape of the sampling lattice, while at the same time simplifying that task by reducing the number of variables under his control.

Finally, we observe that, in a time/space measurement problem, absolute minimum sampling density may not necessarily be desirable. It is characteristic of most measuring instruments that the cost of the instrument and its placement, not the cost per measurement, is significant. Thus, one may often wish to "trade off" time vs. space density, even to the extent of going to essentially continuous measurement in time. The analysis above allows the system designer to evaluate, for example, the possible combinations of time and space density which yield a specified mean-square error in conjunction with their associated optimum pre- and post-sampling filter functions.

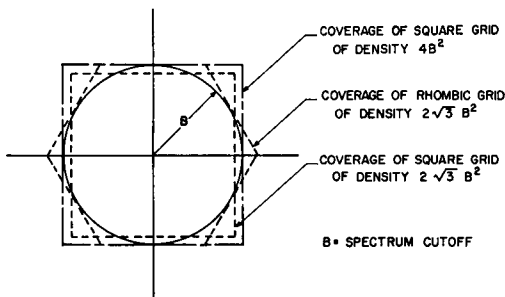


FIG. 6. Coverage of bounded isotropic spectrum.

D. FILTER WEIGHTING FUNCTIONS

The filter weighting functions $\nu(\mathbf{x})$ and $g(\mathbf{x})$ discussed in Section 3 are defined as the inverse transforms respectively of $\Gamma(\omega)$ and $G(\omega)$, which in general (for homogeneous processes with uncorrelated noise) are real quantities defined over a basic cell of wave-number space. The inverse Fourier transforms will thus depend both on the *shape* of the wave-number cell and on the magnitude of the filter spectrum across the cell. In the general case, of course, this integration may be difficult or impossible to perform analytically.

Having arbitrarily absorbed in $\Gamma(\omega)$ the necessary noise-suppression characteristics, we find the function $G(\omega)$ to be simply flat over a basic cell, and thus $g(\mathbf{x})$ to have the sometimes desirable quality of orthogonality. Where the shape of the cell is simple, $g(\mathbf{x})$ may be readily calculated, and it is instructive to observe its behavior in selected cases.

For one-dimensional processes with monotonic spectra, we obtain the familiar $(\sin 2\pi Bt)/2\pi Bt$ "cardinal" weighting function illustrated in Fig. 1. In two dimensions, using a square wave-number cell, we have

$$\frac{\sin 2\pi Bx_1}{2\pi Bx_1} \frac{\sin 2\pi Bx_2}{2\pi Bx_2},$$

however, using a rhombic sampling lattice and a hexagonal wave-number cell (Fig. 4), we arrive at the more complicated expression

$$g(x_1, x_2) = \frac{2x_1 \cos \frac{2\pi Bx_1}{\sqrt{3}} \cos 2\pi Bx_2 - 2x_1 \cos \frac{4\pi Bx_1}{\sqrt{3}} - 2\sqrt{3}x_2 \sin \frac{2\pi Bx_1}{\sqrt{3}} \sin 2\pi Bx_2}{(2\pi B)^2 x_1(x_1^2 - 3x_2^2)}. \quad (38)$$

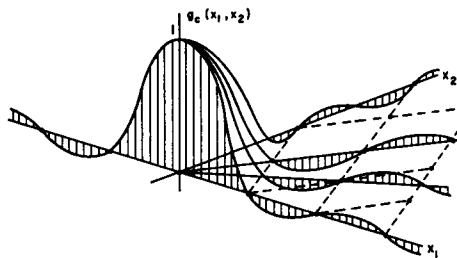


FIG. 7. A canonical weighting function.

Fig. 7 shows the variation of (38) over the (x_1, x_2) plane. We note that it vanishes at all sampling points, equals unity at the origin, and exerts its heaviest weighting within the area defined by the "ring" of sampling points immediately surrounding the origin.

In three dimensions, the "canonical" isotropic weighting function is found by integrating a Fourier kernel over a regular rhombic dodecahedron, with the result

$$g(x_1, x_2, x_3) = \frac{1}{2(\sqrt{2\pi B})^3 [x_1^4 + x_2^4 + x_3^4 - 2(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2)]} \times \{ -2x_1 \sin 2\pi\sqrt{2}Bx_1 - 2x_2 \sin 2\pi\sqrt{2}Bx_2 - 2x_3 \sin 2\pi\sqrt{2}Bx_3 + (x_1 + x_2 + x_3) \sin \pi\sqrt{2}B(x_1 + x_2 + x_3) + (x_1 - x_2 + x_3) \sin \pi\sqrt{2}B(x_1 - x_2 + x_3) + (x_1 - x_3 - x_2) \sin \pi\sqrt{2}B(x_1 - x_3 - x_2) + (x_1 + x_2 - x_3) \sin \pi\sqrt{2}B(x_1 + x_2 - x_3) \}. \quad (39)$$

Variation of (39) along several lines in the three-dimensional space is shown in Fig. 8. The remarks made above with reference to the two-dimensional function also hold here.

For isotropic spectra, one may at the expense of a few percent of reconstruction accuracy use a weighting function derived by Fourier integration over a circular or spherical region in wave-number space instead of over a complete basic cell. In two dimensions this yields

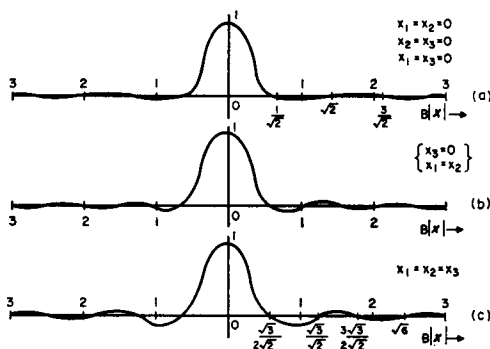


FIG. 8. Variation of three-dimensional canonical weighting function along radial lines. Reprinted by permission of Academic Press, Inc. (Petersen and Middleton, 1962).

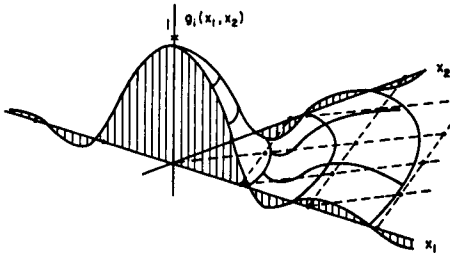


FIG. 9. An isotropic weighting function.

$$g(r) = \frac{J_1(2\pi Br)}{2\sqrt{3}Br}, \quad r = \sqrt{x_1^2 + x_2^2}; \quad (40)$$

and, in three dimensions,

$$g(r) = \left(\frac{B}{r}\right)^{\frac{3}{2}} \frac{1}{4\sqrt{2}B^3} J_{\frac{3}{2}}(2\pi Br) = \frac{\pi}{\sqrt{2}(2\pi Br)^3} \times [\sin 2\pi Br - 2\pi Br \cos 2\pi Br], \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \quad (41)$$

The variation of weighting functions (40) and (41) with distance in sampling space is shown in Figs. 9 and 10.

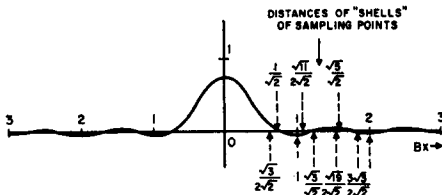


FIG. 10. A three-dimensional isotropic weighting function. Reprinted by permission of Academic Press, Inc. (Petersen and Middleton, 1962).

E. IMPLEMENTATION

One may well ask whether we have not constructed a rather useless concept with the above definition of a "representative observation". Obviously, it is impossible at each station to measure atmospheric phenomena at a continuum of points throughout the entire atmosphere or even a limited local region. How then can we apply this idealized weighting or prefilter function?

First of all, it is clear that the appropriate weighting may be *approximated* by use of the discrete readings of stations surrounding the lattice point in question. For example, in the United States, one may use the larger number

of hourly "airways" reports to enhance the "representativeness" of the fewer "synoptic" observations which are incorporated into broad-scale analyses and exchanged internationally. Where eventually local "mesonets" are established, their observations could also be incorporated into a refined lattice datum. This approach, of course, would logically culminate in the abandonment of actual "synoptic" observation stations as such, since a central computer could "smooth" the fine-grid data into a coarser set of pseudo-observations on an idealized lattice for international exchange, broadscale analyses, and climatological archives. We will discuss in the next section the smoothing techniques which are consistent with this concept.

However, in the special but important case of two-dimensional fields whose magnitudes may be converted into optical intensities, we have an opportunity to implement directly the techniques developed here. Such a situation arises in the meteorological domain in the encoding of radar and satellite reconnaissance presentations. In both cases, digital codes are sought which quantize the received signals in both location and amplitude for discrete transmission. In the former case, the code presently used is essentially a verbal description of the appearance of the primary display; in the latter, a line-by-line scan is used with subsequent sampling and digitization.

It is well known (CUTRONA, LEITH, PALERMO & PORCELLO, 1960) that a coherent optical system performs successive two-dimensional Fourier transformations from one focal plane to the next. [A "coherent" system is one in which the relative phase of light radiation at different points is constant; this can be achieved, for instance, by using for excitation a point source of monochromatic light.] This affords the interesting possibility of performing a prefiltering operation on the raw pictorial data directly in a wave-number plane, and thus obviating a complicated two-dimensional convolution. We have shown that the optimum operation is one of ideal wave-number limiting; this becomes, optically, simply an *aperture* in the wave-number focal plane with a variation of transparency across the opening as required by relationship (35). Figure 11 shows an arrangement applicable to an isotropic spectrum; note that it is simple to synthesize the exact

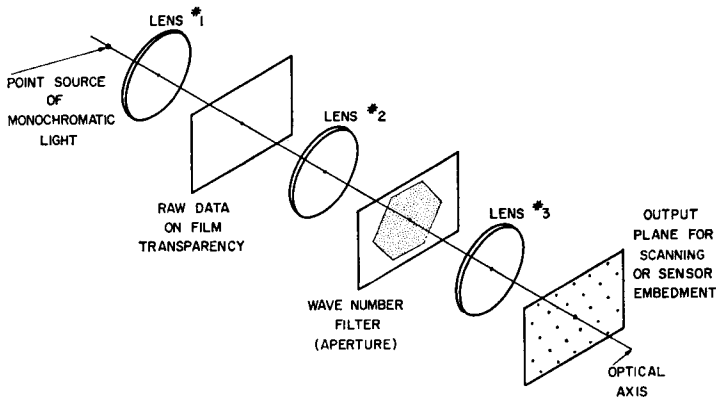


FIG. 11. An optical filter for pre-sample processing of two-dimensional data.

filter desired even when the weighting function cannot be obtained in closed form. At the output focal plane, photocells may be embedded in the desired sampling pattern, their outputs digitized and transmitted; or the resulting display may be scanned and sampled at the appropriate points. Note that if *continuous* scanning is desired, the proper wave-number filter becomes a *slit* of width inverse to the scan line spacing.

In many systems, the received information will be used directly in digital manipulations. Where reconstruction is desired for manual viewing, the optical system may be reversed by exciting (coherent) light sources in the prescribed sampling pattern with an intensity proportional to the received signal distribution. The identical wave-number aperture is used in the transform focal plane (with the variation in transparency removed), and an image is recorded on a sensitive plate exposed at the inverse transform plane. This image will not, of course, coincide with the original picture but will be an *ideally smoothed* representation corresponding to the given sampling density.

We observe in passing that the application of coherent (laser) amplification to the originally received field of light intensity may obviate the requirement of conventional optical technique for an intermediate step of photographic recording, thus making such a system practical for unattended (e.g., satellite surveillance) operation.

F. SMOOTHING OPERATORS FOR NUMERICAL DATA ANALYSIS

A need for smoothing or filtering of raw data before the application of numerical analysis

and prediction techniques has become manifest in practical weather data processing systems. "Smoothing operators" are used today for meteorological analysis at JNWPU (Joint Numerical Weather Prediction Unit, Suitland, Maryland) (SHUMAN, 1957). These consist of an array of numbers corresponding to the lattice of available reports, signifying the weighting to be applied to measurements at each point to obtain a smoothed datum at a specified point in the array (generally the center lattice point). The sampled field is smoothed by successively displacing the "operator" array to each lattice point of interest. Generally, only the five or nine adjacent points in a square lattice are used and the weighting coefficients are chosen empirically on the basis of end (forecast) results. However, some attempt has been made to analyze the effect of relative weighting coefficients and of repeated smoothing operations on certain idealized (two-dimensional) wave signals.

The theory of multidimensional sampling and reconstruction allows considerably deeper insight into the discrete smoothing problem. It is possible to design smoothing operators to specification without tedious empirical search and without restriction to two-dimensional fields, square lattices, locations of final data points, etc.

First of all, we remark that the operation of *smoothing* implies the elimination of high wave numbers from the received data. [Narrowband filtering is indeed feasible and does not affect the subsequent argument.] Since, as we have seen, the bona fide information content of sampled data is strictly limited to a band of wave numbers determined by the sampling

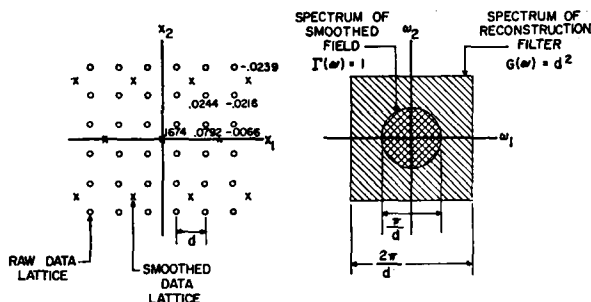


Fig. 12. Smoothing-calculation of an array.

lattice, the lattice of smoothed data must be less dense than the raw data lattice. Otherwise, the resulting samples become highly redundant.

Next, we define the concept of discrete data smoothing as follows: from the original (raw data) samples we reconstruct in an optimal manner the continuous field from which they were drawn. For this purpose, a reconstruction function $G(\omega)$ defined by (25) is used. Then, before this continuous field is sampled, a *prefiltering* operation $\Gamma(\omega)$ is to be performed which, as discussed above, consists of ideal wave-number limiting over a cell corresponding to the selected (smoothed data) lattice. If it is desired to reconstruct the field from the samples of this resulting smoothed field, an ideal filter of identical cell shape must be used.

Let us now suppose that our desired smoothed datum is located at the origin of sampling space, and that we wish to compute the weighting to be applied to a measurement at a raw sampling point $\mathbf{x}_{[k]}$. The contribution to reconstruction of the continuous field at a point $\mathbf{x}$ is defined by $g(\mathbf{x} - \mathbf{x}_{[k]})$, while the contribution to the smoothed sample at $\mathbf{0}$ by the field value at $\mathbf{x}$ is $\gamma(\mathbf{0} - \mathbf{x})$. Thus the total contribution of the sample at $\mathbf{x}_{[k]}$ to the datum at $\mathbf{0}$ is

$$\hat{f}_{[k]}(\mathbf{0}) = f(\mathbf{x}_{[k]}) \int_{\mathbf{x}} \gamma(\mathbf{0} - \mathbf{x}) g(\mathbf{x} - \mathbf{x}_{[k]}) d\mathbf{x}. \quad (42)$$

But the integral in (42) is a convolution which may be transformed by Parseval's theorem to

$$\hat{f}_{[k]}(\mathbf{0}) = \frac{f(\mathbf{x}_{[k]})}{(2\pi)^n} \int_{\Omega} \Gamma(\omega) G(\omega) e^{-i\omega \cdot \mathbf{x}_{[k]}} d\omega. \quad (43)$$

Thus, the weighting to be applied to the measured value at $\mathbf{x}_{[k]}$ is the inverse transformation

of the product $\Gamma(\omega)G(\omega)$, evaluated at $(-\mathbf{x}_{[k]})$. But since $\Gamma(\omega)$ is an ideal wave-number-limiting filter, the product $\Gamma(\omega)G(\omega)$ is equivalent to viewing $G(\omega)$ through a *window* whose shape is that of the desired spectral cell of the smoothed samples.

We note that the identical result is obtained if we simply demand that the desired end is not reconstruction of the original data field but of that field ideally smoothed by a filter $\Gamma(\omega)$.

As an example, let us consider the situation shown in Fig. 12. Here, the original field is assumed sampled on a square lattice of side d and (for lack of more definitive information) it is to be reconstructed with an ideal (square) wave-number filter having a uniform spectral weighting d^2 . The smoothed data are to be isotropically limited to wavelengths less than $1/4d$. The minimum lattice for this purpose would be a 60° rhombic with side $4d/\sqrt{3}$; however, to achieve a *stationary* smoothing operator (requiring only one array of numbers applicable to all points of the final lattice), the sampling cell size must be decreased to an area of $4d^2$. It then makes no difference whether adjacent rows are staggered as shown in Fig. 12 or not.

Performing the indicated integration (43), we obtain simply

$$\begin{aligned} \hat{f}_{[k]}(\mathbf{0}) &= \frac{f(\mathbf{x}_{[k]})}{(2\pi)^2} \int_{|\omega| < \pi/2d} d^2 e^{-i\omega \cdot \mathbf{x}_{[k]}} d\omega \\ &= f(\mathbf{x}_{[k]}) \frac{J_1(\pi x/2d)}{4(x/d)}, \quad x = |\mathbf{x}_{[k]}|. \end{aligned} \quad (44)$$

Because of symmetry, only one octant of values need be calculated from (44); the result, for the innermost six rings of points, are as shown in Fig. 12. The weighting has a finite value over

the entire plane and, for practical application, must be truncated (with appropriate adjustments to leave the mean value of the sampled function unchanged) to some reasonable number of points. (Note that the properties of Fourier approximation ensure that such a truncated set defines a least-mean-square approximation to the desired filter spectrum in the wave-number domain.) If the smoothed data are later to be reconstructed, the same Bessel function must be used for interpolation.

G. NUMERICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

THOMPSON (1961) has stated that the successful use of high-speed digital data processing equipment for meteorological prediction depends on two factors: (1) appropriate simplification of the equations of motion to eliminate solutions (e.g., sound and gravity waves) which are not of meteorological significance, and (2) development of a suitable and computationally stable numerical method of solution of linear differential equations.

Since this paper is not addressed to the *physics* of random processes, i.e., the development of dynamic equations of motion to physical systems, we shall not concern ourselves with item (1) above. With regard to item (2), we limit our remarks to a discussion of the application of multidimensional sampling theory to the design of discrete differential operators.

Let us suppose, as does Thompson, that the essential step in a numerical procedure for the solution of the atmospheric equations of motion involves the solution of an equation of the form

$$\nabla^2 \frac{\partial z}{\partial t} + \mu \frac{\partial z}{\partial t} = F, \quad (45)$$

in which μ and F depend on z and are known on a discrete lattice of points. With the solution of (45), z may be extrapolated over a suitably small time interval, μ and F may be recalculated, and the procedure repeated.

Now it is evident that since we have, for N data points, only N equations of the form (45), we must ensure that the number of unknown quantities is no greater: namely, the N values of $\partial z/\partial t$. Thus, the Laplacian derivative must be expressed or approximated as a function of the unknown values of $\partial z/\partial t$ at the sampling points. For this purpose, the standard technique is a "finite-difference" approximation: since

$$\begin{aligned} \nabla^2 \phi &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lim_{\Delta x \rightarrow 0} \\ &\quad \frac{[\phi(x + \Delta x, y) - \phi(x, y)]/\Delta x}{\Delta x} \\ &\quad \times \frac{-[\phi(x, y) - \phi(x - \Delta x, y)]/\Delta x}{\Delta x} \\ &\quad + \lim_{\Delta y \rightarrow 0} \frac{[\phi(x, y + \Delta y) - \phi(x, y)]/\Delta y}{\Delta y} \\ &\quad \times \frac{-[\phi(x, y) - \phi(x, y - \Delta y)]/\Delta y}{\Delta y} \end{aligned} \quad (46)$$

we may approximate $\nabla^2 \phi$ (for a square lattice of side d) by simply letting $\Delta x = \Delta y = d$ and omitting the limit operation:

$$\begin{aligned} \nabla_d^2 \phi \approx & \frac{1}{d^2} [\phi(x + d, y) + \phi(x - d, y) \\ & + \phi(x, y + d) + \phi(x, y - d) - 4\phi(x, y)]. \end{aligned} \quad (47)$$

We call (47) a "finite-difference differential operator". Substituting in (45), with $\phi = \partial z/\partial t$, we obtain a determinate set of linear equations for the N values of $\partial z/\partial t$. Because N may be very large, the solution is performed by some type of "relaxation" procedure rather than a matrix inversion, but this is immaterial to our present discussion.

We note next that (47) is not a unique approximation to the Laplacian (46). In fact (45), being defined only at discrete points, does not have a unique solution without some specification of the "curvature" $\nabla^2(\partial z/\partial t)$ in terms of the functional values of $\partial z/\partial t$. We could, *reductio ad absurdum*, choose $\nabla^2(\partial z/\partial t) = 0$ at every lattice point and still have the solution represent samples of a *continuous* and differentiable function!

To investigate the implications of the approximation (47), let us define a quadratic function

$$\psi(x, y) = ax^2 + by^2 + cxy + ex + fy + g, \quad (48)$$

which passes through the five points $\phi(x, y)$, $\phi(x + d, y)$, $\phi(x - d, y)$, $\phi(x, y + d)$, $\phi(x, y - d)$. By direct substitution it is easy to see that

$$\begin{aligned} \nabla^2 \psi(x, y) &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2a + 2b \\ &= \frac{1}{d^2} [\phi(x + d, y) + \phi(x - d, y) + \phi(x, y + d) \\ &\quad + \phi(x, y - d) - 4\phi(x, y)]. \end{aligned} \quad (49)$$

Thus, the finite-difference approximation is in effect a requirement that the Laplacian of the solution at each point be identical to that of a *quadratic surface* passing through that point and the four surrounding points. One might intuitively suspect that the function $\partial z/\partial t$ constrained to such quadratic "curvature" at the lattice points might of necessity possess Fourier components of relatively high wave numbers; and that these might, as the iterated computation proceeds, generate "aliasing" errors in the extrapolated field $z(x, y)$.

Since, as we have seen, the representation of a field of values in sampled form implies the abandonment of wave-numbers outside a cell of definitely limited extent in wave-number space [indeed, the data entering into (45) are often discretely "smoothed" before further processing], it would appear advisable to restrict the solution of (45) to functions having the same limited spectrum. In effect, this restriction allows the reconstruction of the solution from its samples, and in turn the calculation of the Laplacian of the resulting *continuous* field. But since the reconstruction is effected by means of a linear combination of weighting functions, the contribution of each value of $\partial z/\partial t$ to the Laplacian at a given point is simply *the Laplacian of the weighting function* evaluated at that displacement of its argument. We thus obtain a discrete array of numbers which we may call a *smooth differential operator*. Note that, since we have developed a technique for finding reconstruction functions associated with *any* periodic lattice, we are no longer restricted in this numerical procedure to a square or cubic grid if another (e.g., the optimum "isotropic" lattice) should prove more efficient for *measurement* purposes. Moreover, *every* point in the array will have a nonzero weighting on the Laplacian at a given location (not only the five adjacent points), and thus more computing capacity will be needed to solve the resulting set of equations. It is not known at this time what improvement in results may be expected using this technique, and therefore what additional investment in equipment may be justified for a particular system design.

Let us consider two examples of the above, applying to one- and two-dimensional processes. In the first case, we define a function $\phi(t)$ known [or to be determined by an equation such as (45)] at discrete points on the real line spaced T

units apart. We wish the solution to be limited to the band $-\pi/T < \omega < \pi/T$; thus the function may be expanded in a Kotel'nikov-Shannon series

$$\phi(t) = \sum_{-\infty}^{\infty} \phi(kT) \frac{\sin(\pi/T)(t - kT)}{(\pi/T)(t - kT)}. \quad (50)$$

The second derivative at a point $t = nT$ then becomes

$$\begin{aligned} \phi''(nT) = & -\frac{\pi^2}{3T^2} \phi(nT) + \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} (-1)^{k+1} \\ & \times \phi[(n-k)T] \left[\frac{2}{(kT)^3} \right]. \end{aligned} \quad (51)$$

The coefficients of $\phi(kT)$ in (51) define the smooth differential operator for this case. The applicable finite-difference operator is the familiar expression

$$\phi''(nT) \approx \frac{1}{T^2} \{ \phi[(n+1)T] + \phi[(n-1)T] - 2\phi(nT) \} \quad (52)$$

The differential operators (51) and (52) are shown in Fig. 13.

As an example of a two-dimensional operator, let us consider a function isotropically limited to wavelengths $d/\sqrt{3}$ and sampled on the corresponding 60° rhombic lattice of typical spacing d . This field can be reconstructed from its samples by means of the weighting function [cf. (40)]:

$$g(\mathbf{x}) = \frac{J_1(2\pi Bx)}{2\sqrt{3}Bx}, \quad x = |\mathbf{x}|. \quad (53)$$

We now find the Laplacian of (53) to be

$$\begin{aligned} \nabla^2 g(\mathbf{x}) = & -\frac{2}{\sqrt{3}Bx^3} \{ [(\pi Bx)^2 - 1] J_1(2\pi Bx) \\ & + \pi Bx J_0(2\pi Bx) \}, \end{aligned} \quad (54)$$

which must be evaluated at each sample point.

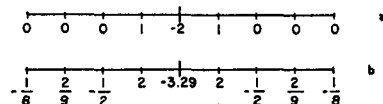


FIG. 13. One-dimensional second-differential operators. (a) Finite-difference operator. (b) Smooth operator.

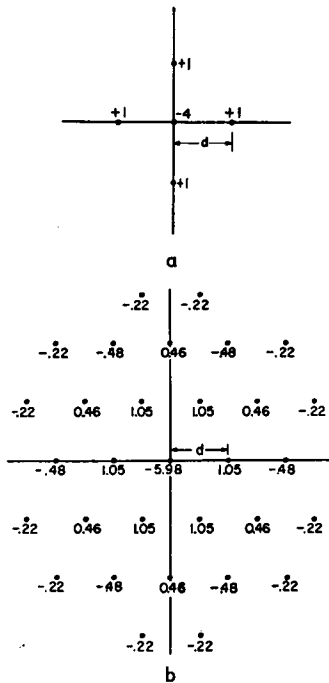


FIG. 14. Two-dimensional isotropic Laplace operators. (a) Finite-difference operator. (b) Smooth operator.

The central portion of the resulting array is shown as Fig. 14b; this is to be compared with the two-dimensional finite-difference operator (47), shown as Fig. 14a.

H. LIMITATIONS

The principal limitation of the analysis presented in Section 3 is its applicability only to data spaces of infinite extent. Although the analysis is unaffected by a requirement that one of the coordinates of the sampling space be time (provided the appropriate statistical relationships are known), practical "real-time" systems are constrained by "realizability" conditions which make available only half the total data space (i.e., "past" time). Techniques have been developed in the one-dimensional theory to deal with this problem; their extension to multidimensional spaces has not yet been wholly successful.

The average mean-square criterion for reconstruction accuracy may not be applicable to many users of meteorological information; however, it is probably not feasible to find an alternate criterion appropriate to the complete *ensemble* of users.

The assumption of statistical homogeneity ("stationarity") in atmospheric processes is open to serious question. The extension to processes with homogeneous increments (Tatarski, 1961) may be fruitful in this regard.

Only periodic sampling lattices are considered in order to retain the advantages of manipulations in "wave-number" space.

The effect of measurement quantization is not specifically included; however, this may usually be treated as additive "noise" (Widrow, 1961).

While the above limitations of the theory presented in this paper strongly point up the need for further work, they are by no means peculiar to this analysis. It is felt that, if applied with due caution, the results obtained can lend considerable insight and some practical methods of handling discrete data measurement and processing problems.

5. Summary

This paper has examined the well-known concept in meteorological science of the "representative observation" and has proposed the establishment of a rigorous definition of this term whose current usage betrays an anarchical attitude toward reported meteorological measurements. The suggestion is made to define a representative observation as a datum which has been so modified as to eliminate or reduce to a minimum its contamination with additive disturbing noise as well as bona fide atmospheric fluctuations of a scale smaller than that observable with the given measurement network. The mathematical basis for such a definition involves the determination of optimum pre- and post-sampling filter functions based on a criterion of minimum mean-square error, averaged over the observation space. The implications of spectral intensity distributions in the multidimensional "wave-number" space are emphasized in the analysis.

Although for the usual surface and upper-air observations the required continuous prefiltering operation is unrealizable, approximations are always possible and in some cases—notably radar and satellite data collection—the idealized operation may be easily mechanized. The concept furthermore affords new insight into the application of smoothing techniques and the numerical solution of dynamic prediction models.

Ostensibly directed toward "deterministic" rather than "statistical" approaches to weather prediction, this work has served to emphasize the importance of adequate statistical data on atmospheric motions, at least up to second-

moment functions, which have a profound influence even on such deceptively straightforward tasks as the determination of the spatial density and lattice arrangements and the time periods of meteorological measurements.

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