The mathematical equivalence of multi-level and vertically integrated numerical forecasting models[1]

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ABSTRACT

It is shown that, if the equations for an n-level model are averaged, the resulting equations have mathematical forms which, under certain conditions, are identical with those of the vertically integrated n-parameter model. The implications for the temperature and ω-profiles are discussed in order to achieve exact mathematical equivalence. A physically consistent vertically integrated model yields temperature and ω-profiles which depart from those required for mathematical exactness, yet appears to lead to more realistic results.

1. Introduction

During the late 1940's and early 1950's, there were two approaches to the development of numerical forecasting models. One was to divide the atmosphere into layers, and appropriate equations governing the motions within each layer were derived. The other was to assume that the variation with height of quantities was known, and these relations were substituted into the vertically averaged equations. Thus an n-level model was one in which one had to solve a system of $2n-1$ equations in $2n-1$ unknowns. An n-parameter model was one in which one had to solve $n+1$ equations in $n+1$ unknowns. There may be reasons for preferring one approach to the other. For example, the vertically integrated approach is useful in formulating certain problems in which resolution along the vertical is not critical. There may also be computational reasons for selecting one of the approaches.

In order to be able to make a scientifically legitimate choice between these two approaches, it must be known whether they are equivalent, i.e., if one chooses one approach over the other, is one then solving the same meterological problem? The vertically integrated approach seems rather artificial, and its use may lead one to the uneasy feeling that his problem is mathematically neat, but physically confusing. In the early days of model development, the statement was often made that these two approaches are mathematically equivalent. Indeed, ELIASSEN & KLEINSCHMIDT (1957) indicated the equivalence for $n=2$. But a proof of the mathematical equivalence of the two approaches for any n does not seem to exist in the literature. It is the purpose of the present investigation to demonstrate the degree of mathematical equivalence between the two formulations.

2. The multi-level equations

The general quasi-geostrophic equations for an n-level model have been derived by CHARNEY & PHILLIPS (1953). In what follows we shall not make the quasi-geostrophic assumption for the wind, but shall assume that the wind shear is given by the thermal wind.

We divide the atmosphere into n layers of equal thickness Δp, bounded by the $n+1$ isobaric surfaces $p_0, p_2, \ldots, p_{2n}$, numbered from the ground upward. Thus p_0 represents surface pressure and $p_{2n}=0$. The odd subscripts from 1 to $2n$-1 denote the n layers (See Fig. 1).

We define a vertical average

$$\overline{(\quad)} = \frac{1}{n}\sum_{i=1}^{n}(\overline{\quad})_{2i-1} \tag{1}$$

so that the vertical average of the horizontal wind vector is:

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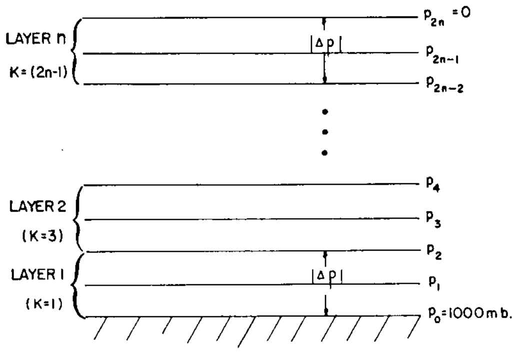

FIG. 1. The vertical division of the atmosphere; here $p_j = \frac{1}{2}(p_{j+1} + p_{j-1})$ $(j+1, 2, \ldots, n-1)$.

$$\overline{\mathbf{V}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{V}_{2i=1}. \tag{2}$$

We define a wind "shear" by

$$\mathbf{V}'_{2k-2} = a(\mathbf{V}_{2k-3} - \mathbf{V}_{2k-1}), \quad k = 2, 3, \ldots, n \tag{3}$$

and where a is a constant.

The system (2), (3) may be written in expanded form and solved for the wind at any level in the model in terms of the mean wind and the wind shears,

$$\mathbf{V}_{2i-1} = \overline{\mathbf{V}} + \sum_{k=2}^{n} C_{2k-2,\,2i-1}\mathbf{V}'_{2k-2}, \tag{4}$$

where the $C_{2k-2,\,2i-1}$ are constants. For example, when $n=2$,

$$\left.\begin{aligned} \mathbf{V}_1 &= \overline{\mathbf{V}} + C_{2,1}\mathbf{V}'_2, \\ \mathbf{V}_3 &= \overline{\mathbf{V}} + C_{2,3}\mathbf{V}'_2. \end{aligned}\right\}$$

Solving, we find

$$C_{2,1} = \tfrac{1}{2}a = -C_{2,3}.$$

When $\quad a = \frac{1}{2},\ C_{2,1} = 1 = -C_{2,3},$

which is the usual two-level case.

We also have

$$\zeta_{2i-1} = \bar{\zeta} + \sum_{k=2}^{n} C_{2k-2,\,2i-1}\zeta'_{2k-2}, \tag{5}$$

where ζ is the relative vorticity.

We shall use the vorticity equation in the form

$$\frac{\partial\zeta}{\partial t} + \mathbf{V}\cdot\nabla\zeta + \mathbf{V}\cdot\nabla f - f_0\frac{\partial\omega}{\partial p} = 0, \tag{6}$$

where f_0 = constant. This form of the equation is the simplest baroclinic form which possesses an energy invariant. Here $\omega = \dfrac{\partial p}{\partial t}$, f is the Coriolis parameter.

The equations for an n-level model may be written

$$\left.\begin{aligned}
&\frac{\partial\zeta_1}{\partial t} + \mathbf{V}_1\cdot\nabla\zeta_1 + \mathbf{V}_1\cdot\nabla f - f_0\frac{\omega_2-\omega_0}{\Delta p} = 0, \\
&\frac{\partial\zeta_3}{\partial t} + \mathbf{V}_3\cdot\nabla\zeta_3 + \mathbf{V}_3\cdot\nabla f - f_0\frac{\omega_3-\omega_2}{\Delta p} = 0, \\
&\qquad\vdots \\
&\frac{\partial\zeta_{2n-3}}{\partial t} + \mathbf{V}_{2n-3}, \nabla\zeta_{2n-3} + \mathbf{V}_{2n-3}\cdot\nabla f \\
&\qquad - f_0\frac{\omega_{2n-2}-\omega_{2n-4}}{\nabla p} = 0, \\
&\frac{\partial\zeta_{2n-1}}{\partial t} + \mathbf{V}_{2n-1}, \nabla\zeta_{2n-1} + \mathbf{V}_{2n-1}\cdot\nabla f \\
&\qquad - f_0\frac{\omega_{2n}-\omega_{2n-2}}{\Delta p} = 0.
\end{aligned}\right\} \tag{7}$$

We assume as boundary conditions on ω:

$$\left.\begin{aligned} \omega &= 0 \quad \text{at} \quad p = p_0\ (\omega_0 = 0), \\ \omega &= 0 \quad \text{at} \quad p = p_{2n}\ (\omega_{2n} = 0). \end{aligned}\right\} \tag{8}$$

If we add equations (7), we get

$$\frac{\partial\bar{\zeta}}{\partial t} + \frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_{2i=1}\cdot\nabla\zeta_{2i=1} + \overline{\mathbf{V}}\cdot\nabla f = 0. \tag{9}$$

If we note that,

$$\overline{\mathbf{V}\cdot\nabla\zeta} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_{2i-1}\cdot\nabla\zeta_{2i-1} \tag{10}$$

then equation (9) may be written

$$\frac{\partial\bar{\zeta}}{\partial t} + \overline{\mathbf{V}\cdot\nabla(\zeta+f)} = 0 \tag{11}$$

which is exactly the form taken by the vorticity equation when the bar operator is defined by

$$\overline{(\quad)} = \frac{1}{p_0}\int_0^{p_0} (\quad)\, dp \tag{12}$$

and the latter is applied to (6)

The latter is the form usually used in vertically integrated models.

We wish to express the middle term of (9) in terms of barred and primed quantities in order to compare this equation with the n-parameter vertically integrated vorticity equation

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{V}_{2i-1}\cdot\cdot\nabla\zeta_{2i-1} = \overline{\mathbf{V}}\cdot\nabla\bar{\zeta}$$

$$+\frac{1}{n}\left\{\begin{aligned} &\overline{\mathbf{V}}\cdot\nabla\sum_{k=2}^{n}\left(\sum_{i=1}^{n} C_{2k-2,2i-1}\right)\zeta'_{2k-2} \\ &+\left[\sum_{k=2}^{n}\left(\sum_{i=1}^{n} C_{2k-2,2i-1}\mathbf{V}'_{2k-2}\right)\right]\cdot\nabla\bar{\zeta} \\ &+\left(\sum_{k=2}^{n} C_{2k-2,1}\mathbf{V}'_{2k-2}\right)\cdot\nabla \\ &\times\left(\sum_{k=2}^{n} C_{2k-2,1}\zeta'_{2k-2}\right) \\ &+\left(\sum_{k=2}^{n} C_{2k-2,3}\mathbf{V}'_{2k-2}\right)\cdot\nabla \\ &\times\left(\sum_{k=2}^{n} C_{2k-2,3}\zeta'_{2k-2}\right) \\ &+\dots+\left(\sum_{k=2}^{n} C_{2k-2,2n-1}V'_{2k-2}\right)\cdot\nabla \\ &\times\left(\sum_{k=2}^{n} C_{2k-2,2n-1}\zeta'_{2k-2}\right)\end{aligned}\right\}$$

$$\begin{aligned} &= \overline{\mathbf{V}}\cdot\nabla\bar{\zeta} + \overline{\mathbf{V}}\cdot\nabla\sum_{k=2}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} C_{2k-2,2i-1}\right)\zeta'_{2k-2} \\ &+\sum_{k=2}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} C_{2k-2,2i-1}\right)\mathbf{V}'_{2k-2}\cdot\nabla\bar{\zeta} \\ &+\sum_{k=2}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} C^2_{2k=2,2i-1}\right)\mathbf{V}'_{2k-2}\cdot\nabla\zeta'_{2k-2} \\ &+\sum_{\substack{j,k=2\\ j\neq k}}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} C_{2k-2,2i-1}C_{2j-1,2i-1}\right) \\ &\qquad \mathbf{V}'_{2k-2}\cdot\nabla\zeta'_{2j-2}. \end{aligned} \tag{13}$$

If we apply the bar operator (1) to the expression (4) for $\mathbf{V}_{2i-1}$, we have

$$\overline{\mathbf{V}} = \overline{\mathbf{V}} + \sum_{k=2}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} C_{2k-2,2i-1}\right)\mathbf{V}'_{2k-2} \tag{14}$$

whence
$$\frac{1}{n}\sum_{i=1}^{n} C_{2k-2,2i-1} = \overline{C_{2k-2}} = 0. \tag{15}$$

Therefore the first two terms in the bracket of (13) are zero, and the vorticity equation (9) may now be written

$$\begin{aligned} &\frac{\partial\bar{\zeta}}{\partial t} + \overline{\mathbf{V}}\cdot\nabla(\bar{\zeta}+f) \\ &+\sum_{k=2}^{n}\overline{C^2_{2k-2}}\mathbf{V}'_{2k-2}\cdot\nabla\zeta'_{2k-2} \\ &+\sum_{\substack{j,k=2\\ j\neq k}}^{n}\overline{C_{2k-2}C_{2j-2}}\mathbf{V}'_{2k-2}\cdot\nabla\zeta'_{2j-2} = 0. \end{aligned} \tag{16}$$

The differential equation for the wind shear is given by

$$\frac{\partial\mathbf{V}}{\partial p} = -\frac{R}{f_0 p}\mathbf{k}\times\nabla T, \tag{17}$$

where T = temperature, R = gas constant.

In finite-difference form

$$\frac{\mathbf{V}_{2k-1}-\mathbf{V}_{2k-3}}{p_{2k-1}-p_{2k-3}} \approx -\frac{R}{f_0\, p_{2k-2}}\mathbf{k}\times\nabla T_{2k-2}. \tag{18}$$

With $\Delta p = (p_{2k-3}-p_{2k-1})$, and definition (3), we write

$$\left.\begin{aligned} \mathbf{V}'_{2k-2} &= -\frac{aR\Delta p}{f_0\, p_{2k-2}}\mathbf{k}\times\nabla T_{2k-2}, \\ z'_{2k-2} &= -\frac{aR\Delta p}{f_0\, p_{2k-2}}\nabla^2 T_{2k-2}. \end{aligned}\right\} \tag{19}$$

Substitution of (19) into (16) yields

$$\begin{aligned} &\frac{\partial\bar{\zeta}}{\partial t} + \overline{\mathbf{V}}\cdot\nabla(\bar{\zeta}+f) + \left(\frac{aR\Delta p}{f_0}\right)^2 \\ &\times\left[\begin{aligned} &\sum_{k=2}^{k}\frac{\overline{C^2_{2k-2}}}{p^2_{2k-2}}J(T_{2k-2},\nabla^2 T_{2k-2}) \\ &+\sum_{\substack{j,k=2\\ j\neq k}}^{n}\frac{\overline{C_{2k-2}C_{2j-2}}}{p_{2k-2}\,p_{2j-2}}J(T_{2k-2},\nabla^2 T_{2j-2})\end{aligned}\right] = 0. \end{aligned} \tag{20}$$

It will be shown later that the vorticity equation (20) for the vertically averaged flow in a multi-level model has the same mathematical form as the vorticity equation for the vertically integrated flow in a parameterized model.

The first law of thermodynamics for adiabatic flow is

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T + \sigma\omega = 0, \tag{21}$$

where
$$\sigma = \frac{\partial T}{\partial p} - \frac{\varkappa T}{p}. \tag{22}$$

For any level

$$\frac{\partial T_{2k-2}}{\partial t} + \mathbf{V}_{2k-2} \cdot \nabla T_{2k-2} + \sigma_{2k-2}\,\omega_{2k-2} = 0. \tag{23}$$

We may combine (4) and (19) to write

$$\mathbf{V}_{2i-1} = \overline{\mathbf{V}} - \frac{aR\Delta p}{f_0} \sum_{k=2}^{n} \frac{C_{2k-2,2i-1}}{p_{2k-2}} \mathbf{k} \times \nabla T_{2k-2}. \tag{24}$$

Then

$$\mathbf{V}_{2j-2} = \overline{\mathbf{V}} = \frac{aR\Delta p}{f_0} \sum_{k=2}^{n} \frac{C_{2k-2,2j-2}}{p_{2k-2}} \mathbf{k} \times \nabla T_{2k-2}, \tag{25}$$

where
$$C_{2k-2,2j-2} = \frac{C_{2k-2,2j-1} + C_{2k-2,2j-3}}{2} \tag{26}$$

$$\left(\text{i.e., } \mathbf{V}_{2j-2} = \frac{\mathbf{V}_{2j-1} + \mathbf{V}_{2j-3}}{2}\right).$$

Then the first law (23) becomes

$$\frac{\partial T_{2k-2}}{\partial t} + \overline{\mathbf{V}} \cdot \nabla T_{2k-2} - \frac{aR\Delta p}{2f_0} J\left[\sum_{j,k=2}^{n} \frac{C_{2k-2,2j-2}}{p_{2k-2}} T_{2j-2}, T_{2k-2}\right] + \sigma_{2k-2}\,\omega_{2k-2} = 0 \tag{27}$$

σ_{2k-2} may be computed from the finite difference form of (22), i.e.,

$$\sigma_{2k-2} = \frac{T_{2k} - T_{2k-4}}{-2\Delta p} - \frac{\varkappa T_{2k-2}}{p_{2k-2}} \tag{28}$$

but it will have to be assumed that T at both $p = p_0$ and $p = 0$ are known.

Equation (27) will have to be solved for each of the T_{2k-2} required by (20). In order to do so we shall need ω_{2k-2}.

The ω equation is obtained by differentiating the vorticity equation (6) with respect to p, taking the Laplacian of (21), and adding. If use is made of (17), one obtains

$$\nabla^2(\sigma\omega) - \frac{f_0^2 p}{R} \frac{\partial^2 \omega}{\partial p_2} = -\frac{f_0 p}{R} \frac{\partial}{\partial p} [\mathbf{V} \cdot (z + f)] - \nabla^2(\mathbf{V} \cdot \nabla T). \tag{29}$$

We wish to apply this equation at any level $(2k-2)$, and then sum all such equations over k.

At $(2k-2)$,

$$\nabla^2(\sigma\omega)_{2k-2} - \frac{f_0^2 p_{2k-2}}{R} \frac{\partial^2 \omega_{2k-2}}{\partial p^2} = -\frac{f_0 p_{2k-2}}{R}\left[\frac{\partial \mathbf{V}_{2k-2}}{\partial p} \cdot \nabla(\zeta + f)_{2k-2} + \mathbf{V}_{2k-2} \cdot \nabla \frac{\partial \zeta_{2k-2}}{\partial_p}\right] - \nabla^2(\mathbf{V} \cdot \nabla T)_{2k-2}. \tag{30}$$

If we make use of

$$\frac{\partial^2 \omega_{2k-2}}{\partial p^2} = \frac{\omega_{2k} - 2\omega_{2k-2} + \omega_{2k-4}}{(\Delta p)^2} \tag{31}$$

and (25), we may write (30) as

$$\nabla^2(\sigma\omega)_{2k-2} - \frac{f_0^2 p_{2k-2}}{R(\Delta p)^2}(\omega_{2k} - 2\omega_{2k-2} + \omega_{2k-4}) = \mathbf{k} \times \nabla T_{2k-2} \cdot \nabla\left[\bar{\zeta} + f - \left(\frac{aR\Delta p}{f_0 p_{2k-2}}\right) \sum_{k=2}^{n} C_{2k-2,2j-2} \nabla^2 T_{2k-2}\right] + \left[\overline{\mathbf{V}} - \left(\frac{aR\nabla p}{f_0 p_{2k-2}}\right) \sum_{k=2}^{k} C_{2k-2,2j-2} \mathbf{k} \times \nabla T_{2k-2}\right] \cdot \nabla(\nabla^2 T_{2k-2}) - \nabla^2(\mathbf{V} \cdot \nabla T)_{2k-2}. \tag{32}$$

We note that

$$\left.\begin{aligned} \overline{\sigma\omega} &= \frac{1}{n-1} \sum_{k=2}^{n} (\sigma\omega)_{2k-2}, \\ \overline{T} &= \frac{1}{n} \sum_{k=2}^{n} T_{2k-2}. \end{aligned}\right\} \tag{33}$$

Sum (32) over all k from $k = 2$ to $k = n$. We make use of the boundary conditions (8), and the relations

$$\left.\begin{aligned} p_{2k} &= (n-k)\,\Delta p, \\ p_{2k} + p_{2k-4} - 2p_{2k-2} &= 0. \end{aligned}\right\} \tag{34}$$

We finally obtain

$$\nabla^2\overline{(\sigma\omega)} + \frac{f_0^2}{R\Delta p}\frac{n}{n-1}\omega_2 = J(\overline{T},\bar{\zeta}+f)$$
$$-\frac{2aR\Delta p}{f_0}\sum_{k=2}^{n} J\left(\overline{\frac{C_{2k-2,2k-2}}{p_{2k-2}}}T_{2k-2},\nabla^2 T_{2k-2}\right)$$
$$-\frac{aR\Delta p}{f_0}\sum_{\substack{j,k=2\\ j\neq k}}^{n} J\left(\overline{\frac{C_{2k-2,2j-2}}{p_{2j-2}}}T_{2k-2},\nabla^2 T_{2j-2}\right)$$
$$-\nabla^2(\overline{\mathbf{V}}\cdot\nabla\overline{T})+\overline{\mathbf{V}}\cdot\nabla(\nabla^2\overline{T}). \qquad (35)$$

Equations (20), (27), (35) represent the general multi-level system, which has been reformulated in terms of averaged variables. For purposes of comparison, we now derive the vertically integrated system.

3. The vertically integrated equations

In order to derive the vertically integrated equations, it is necessary to assume the variations of certain quantities with height. The variation of wind, static stability, geopotential may all be derived if the variation of temperature with height is known. We shall define the temperature at any level as a linear combination of the average temperatures of the layers into which the fluid is divided. Thus,

$$T(x,y,p,t) = T^*(p) + \sum_{k=2}^{n}\alpha_{2k-2}(p)T_{2k-2}(x,y,t). \qquad (36)$$

where n is the number of layers into which the atmosphere is divided. The sum is defined such that

$$\Sigma = 0 \quad \text{when} \quad n=1. \qquad (37)$$

Thus for $n=1$,

$$T(x,y,p,t) = T^*(p) \qquad (38)$$

which is the barotropic case.

$T^*(p)$ is a standard temperature distribution which could be taken to be a "normal" or adiabatic distribution, or any other, in fact. The $\alpha_{2k-2}(p)$ represent dimensionless weighting functions, where the index $2k-2$ refers to the interface between two layers. It is possible to assume analytical expressions for the $\alpha_{2k-2}(p)$, but, for ultimate use in actual prediction, they should be determined empirically.

If we integrate (17) with respect to p between limits p and p_0, apply the operator (12) to the result, and substract this result from the unbarred form, we find

$$\mathbf{V} = \overline{\mathbf{V}} + \frac{R}{f_0}\mathbf{k}\times\nabla\left[\left(\overline{\int_{p_0}^{p}\frac{T}{p}dp}\right) - \left(\int_{p_0}^{p}\frac{T}{p}dp\right)\right]. \qquad (39)$$

Substituting (36),

$$\mathbf{V} = \overline{\mathbf{V}} + \frac{R}{f_0}\mathbf{k}\times\nabla\left[A_1(p) + \sum_{k=2}^{n}A_{2k-2}(p)T_{2k-2}\right], \qquad (40)$$

where

$$\left.\begin{aligned} A_1(p) &= \left(\overline{\int_{p_0}^{p}\frac{T^*(p)}{p}dp}\right) - \int_{p_0}^{p}\frac{T^*(p)}{p}dp,\\ A_{2k-2}(p) &= \left(\overline{\int_{p_0}^{p}\frac{\alpha_{2k-2}(p)}{p}}\right) - \int_{p_0}^{p}\frac{\alpha_{2k-2}(p)}{p}dp. \end{aligned}\right\} \qquad (41)$$

If we had applied the bar operator (12) to the vorticity equation (6), and had made use of the boundary conditions (8), we would have obtained equation (11), in which the bar is defined by (12). Upon substitution of (40) into (11), we obtain

$$\frac{\partial\bar{\zeta}}{\partial t} + \overline{\mathbf{V}}\cdot\Delta(\bar{\zeta}+f) + \left(\frac{R}{f_0}\right)^2$$
$$\left[\begin{aligned} &\sum_{k=2}^{n}\overline{A_{2k-2}^2}J(T_{2k-2},\Delta^2 T_{2j-2})\\ &+\sum_{\substack{j,k=2\\ j\neq k}}^{n}\overline{A_{2k-2}A_{2j-2}}J(T_{2k-2},\nabla^2 T_{2j-2}) \end{aligned}\right] = 0. \qquad (42)$$

We note that (42) and (20) are exactly the same mathematical form. Indeed, it is possible to adjust the coefficients so that they are identical. This procedure would perhaps provide a means for assessing the correctness of the assumptions about variations with height in parameterized models.

We observe that, in order to render the two vorticity equations mathematically equivalent, it was necessary to assume that the temperature varies with height according to (36). Modelling assumptions for vertically integrated models are not always stated in this fashion. It would seem that the statement (36) (or its equivalent) is a more appropriate form on which to base vertically integrated models, for then the vorticity equation becomes mathematically equivalent to that for the combined multi-level equations.

The first law is derived exactly as before, except that now (40) is used in place of (25) in equation (23). Substituting (40) into (23), we find

$$\frac{\partial T_{2k-2}}{\partial t} + \overline{\mathbf{V}} \cdot \nabla T_{2k-2} + \frac{R}{f_0} J\left[\sum_{2=j}^{n} A_{2j-2}(p)\, T_{2j-2}, T_{2k-2}\right] + \sigma_{2k-2}\,\omega_{2k-2} = 0. \quad (43)$$

Equation (43) is mathematically identical with equation (27). Here σ_{2k-2} is given by

$$\sigma_{2k-2} = \sigma^*(p) + \sum_{k=2}^{n}\left[\frac{d}{dp}\alpha_{2k-2}(p) - \frac{\varkappa\alpha_{2k-2}(p)}{p_{2k-2}}\right]T_{2k-2}. \quad (44)$$

It remains to derive the vertically integrated ω equation. We use (29) in the form

$$\nabla^2(\sigma\omega) - \frac{f_0^2 p}{R}\frac{\partial^2\omega}{\partial p^2} = -\frac{f_0 p}{R}\frac{\partial \mathbf{V}}{\partial p}\cdot\nabla(\zeta+f) - \frac{f_0 p}{R}\mathbf{V}\cdot\nabla\frac{\partial\zeta}{\partial p} - \nabla^2(\mathbf{V}\cdot\nabla T). \quad (45)$$

We apply the bar operator (12) to (45), and make use of (40) and (36):

$$\Delta^2\overline{(\sigma\omega)} - \frac{f_0^2}{R}\left(\frac{\partial\omega}{\partial p}\right)_{p_0} = J(\overline{T}, \overline{\zeta}+f) - \frac{2R}{f_0}\sum_{k=2}^{n} J\left[\overline{\left(p_{2k-2}A_{2k-2}\frac{dA_{2k-2}}{dp}\right)}T_{2k-2}, \nabla^2 T_{2k-2}\right] - \frac{2R}{f_0}\sum_{\substack{j,k=2\\ j\neq k}}^{n} J\left[\overline{\left(p_{2k-2}A_{2k-2}\frac{dA_{2k-2}}{dp}\right)}T_{2k-2}, \nabla^2 T_{2j-2}\right] + \overline{\mathbf{V}}\cdot\nabla(\nabla^2\overline{T}) - \nabla^2(\overline{\mathbf{V}}\cdot\nabla\overline{T}), \quad (46)$$

where we have made use of

$$p\overline{\frac{dA}{dp}} = -1.$$

A comparison between (35) and (46) shows that the right sides differ only in the numerical values of the coefficients, but the mathematical forms are the same. The left sides, however, differ somewhat in their mathematical forms. If we require that the two differential operators be similar, then we wish proportionality between the second terms on the left, i.e.,

$$\frac{n}{(n-1)\Delta p}\,\omega_2 = -C\left(\frac{\partial\omega}{\partial p}\right)_{p_0}. \quad (47)$$

In vertically integrated models, equation (46) is carried somewhat further by specifying the variation of ω with p, i.e.,

$$\omega(x,y,p,t) = F(p)\,\overline{\omega}(x,y,t). \quad (48)$$

Then, if we wish mathematical equivalence between the vertically integrated ω equation and the vertically averaged multi-level ω equation, we require, in view of (47),

$$F(p_2) = -\frac{C(n-1)}{n}\Delta p F'(p_0) \quad (49)$$

Since $F(p)$ must also satisfy the conditions

$$\left.\begin{aligned} F(0) &= 0,\\ F(p_0) &= 0,\\ \overline{F} &= 1 \end{aligned}\right\} \quad (50)$$

we have four conditions on $F(p)$, so that $F(p)$ is expressible as a cubic. For example, when $n=2$,

$$F(p) = \frac{6}{C}\left[2(1+C)\left(\frac{p}{p_0}\right)^3 - (3+4C)\left(\frac{p}{p_0}\right)^2 + (1+2C)\left(\frac{p}{p_0}\right)\right]. \quad (51)$$

If $C=-1$, then

$$F(p) = -6\left(\frac{p}{p_0}\right)^2 + 6\left(\frac{p}{p_0}\right) \quad (52)$$

which is the well-known parabolic case.

If $C=6$,

$$F(p) = 14\left(\frac{p}{p_0}\right)^3 - 27\left(\frac{p}{p_0}\right)^2 + 13\left(\frac{p}{p_0}\right), \quad (53)$$

4. Some simple computations

In order to assess the validity of the cubic profile, the following procedure was adopted. In the ω equation (45) we have considered ω

as a function of x and p only, and have computed the $\omega(p)$ profile under certain assumptions, for a two-parameter model. First, we have assumed that the variation of horizontal temperature gradient with height is linear, i.e., the thermotropic assumption

$$\nabla T(x, y, p, t) = \frac{2p}{p_0} \nabla \bar{T}(x, y, t). \tag{54}$$

This assumption leads to $\overline{A^2} = \frac{1}{3}$ in the vertically integrated vorticity equation, which is fairly close to the coefficient $\frac{1}{4}$ in the vertically averaged multi-level vorticity equations.[1]

If we now assume

$$\left.\begin{aligned} &\bar{\mathbf{V}} = \mathbf{k} \times \nabla \bar{\Psi} = \mathbf{k} \times \nabla(-Uy + A \sin mx), \\ &\bar{T} = \frac{dT}{dy} y + B \sin mx, \\ &\omega = F(p) \cos mx, \\ &\sigma^* = \frac{\sigma_0}{p}, \sigma_0 = \text{constant}, \end{aligned}\right\} \tag{55}$$

we find

$$\left.\begin{aligned} &\frac{d^2F}{dp^2} + \frac{aF}{p} = b + C\left(\frac{2p}{p_0} - 1\right) \\ &F(p_0) = F(0) = 0 \end{aligned}\right\} \tag{56}$$

whose solution, when normalized so that $F/\bar{F} = 1$, with appropriate values of the constants, is

$$\frac{F}{\bar{F}} = 2.45\left(\frac{p}{p_0}\right)^{1.348} + 5.12\left(\frac{p}{p_0}\right)^2 - 7.57\left(\frac{p}{p_0}\right)^3. \tag{57}$$

Thus the solution is cubic, so that the mathematical form is similar to (53), which is required for exact equivalence. However, a plot of (57), when compared with that of (53) (see Fig. 2), shows very large differences between the two curves. This must mean that the assumption (54), while it yields reasonably close mathematical equivalence in the vorticity equation, leads to a large mathematical discrepancy in the ω equation. For purposes of comparison, the parabolic profile (52) is shown on the same diagram.

For further comparison, we have repeated the above calculation with a different profile for ∇T i.e.,

$$\nabla T(x, y, p, t) = (1 + \varkappa)\left(\frac{p}{p_0}\right)^{\varkappa} \nabla \bar{T}(x, y, t) \tag{58}$$

This form is consistent with the requirement that $\sigma = \sigma(p)$ only, which is required for ener-

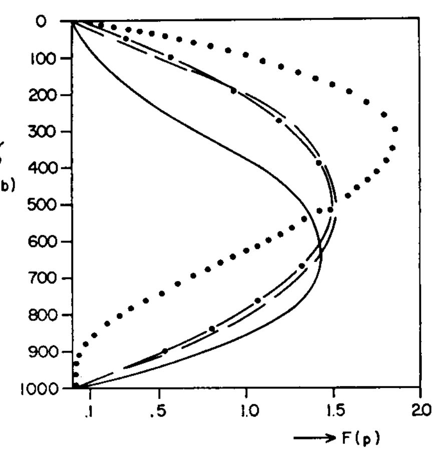

FIG. 2. The variation of ω with p for various models. ———— $\overline{A_2^2} = 0.333$; —— —— $\overline{A_2^2} = 0.634$; —●— parabola; ● ● ● ● ● cubic.

[1] If we let $\alpha(p) = 2p/p_0$ in (36), then (39) becomes

$$\mathbf{V} = \bar{\mathbf{V}} + \left[1 - \left(\frac{2p}{p_0}\right)\right] \frac{R}{f_0} \mathbf{k} \times \nabla T_2.$$

In the 2-level case,

$$\mathbf{V}_1 = \bar{\mathbf{V}} + \mathbf{V}_2',$$
$$\mathbf{V}_3 = \bar{\mathbf{V}} - \mathbf{V}_2'$$

and $$\mathbf{V}_2' = -\frac{(p_1 - p_3)}{2p_2} \frac{R}{f_0} \mathbf{k} \times \nabla T_2$$

in view of (19), with $a = \frac{1}{2}$,

then $$\mathbf{V} = \bar{\mathbf{V}} + A(p) \frac{R}{f_0} \mathbf{k} \times \nabla T_2,$$

where $$A(p) = \begin{cases} -\dfrac{(p_1 - p_3)}{2p_2} & \text{when } p = p_1 \\ \dfrac{(p_1 - p_3)}{2p_2} & \text{when } p = p_3 \end{cases}$$

If we further require that A vary linearly with p, then $A(p) = [1 - 2(p/p_0)]$, which is the same as in the vertically integrated model. The quantity $\overline{A^2} = \frac{1}{3}$ when computed from (41), while $\overline{A^2} = \frac{1}{4}$ when computed from $\frac{1}{2}[(A^2)_{p_1} + (A^2)_{p_3}]$. Thus the two formulations represent the same physical models, but the coefficients have different numerical values due to the method of averaging.

getic consistency, while the form (54) is inconsistent in this sense, BERKOFSKY (1956). The form (58) leads to $\overline{A^2} = 0.634$, which departs radically from the coefficient 1/4 *in the* vorticity equation, which may mean that the two vorticity equations in this case correspond to physically different systems (i.e., their thermodynamics are different).

When we make assumptions (55) and (58) in (45), we find

$$\left.\begin{aligned} &\frac{d^2F}{dp^2} + a\frac{F}{p} = b_1, p^{\varkappa-1} + c_1, p^{2\varkappa-1} \\ &F(p_0) = F(0) = 0 \end{aligned}\right\} \quad (59)$$

whose solution is

$$\frac{F}{\overline{F}} = -11.18\left(\frac{p}{p_0}\right)^{1.348} + 29.39\left(\frac{p}{p_0}\right)^{1.788} - 18.21\left(\frac{p}{p_0}\right)^{1.576} \quad (60)$$

The graph of (60) is also shown in Fig. 2. We see that this curve almost exactly coincides with the parabola, and is somewhat closer to the cubic required for mathematical exactness than is the previous solution.

Inasmuch as calculations by KNIGHTING (1960) have shown that the parabolic variation seems to be reasonably well satisfied in the large-scale motions (in regions where no rapid development is taking place), the profile (60) would seem to correspond more to physical reality than the profile (57). Thus we may conclude the following, as far as two-parameter models are concerned:

a. The *thermotropic* model, which is thermodynamically inconsistent, possesses a vorticity equation which is almost exactly mathematically equivalent to the vertically averaged two-level vorticity equation.

b. The ω-equation for this inconsistent thermotropic model leads to a variation of ω with height which is both mathematically inequivalent with the two-level model, and physically inequivalent to the real atmosphere.

c. The thermodynamically consistent thermotropic model (see equation 58) possesses a vorticity equation which departs mathematically from the vertically averaged two-level vorticity equation.

d. The ω-equation for the thermodynamically consistent thermotropic model leads to a variation of ω with height which agrees better with the two-level model than the inconsistent ω profile and also agrees better with the real atmosphere.

Thus, if a choice is to be made, it would appear to this writer that the consistent thermotropic model, even though only a two-parameter model, and even though departing from exact mathematical equivalence with the two-level model in the vorticity equation, is capable of capturing somewhat more of the features of the real atmosphere than the two-level or inconsistent thermotropic models. Whether this conjecture is true for models with more than two parameters remains to be investigated.

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