Vortex flows normal to a flat surface

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ABSTRACT

Numerical results of axisymmetric vortex flows normal to a flat surface are presented for a variety of Reynolds numbers. The results are based on an extension of Prandtl's boundary layer theory, which permits a reduction of the Navier-Stokes equations to an ordinary boundary value problem that is transformed to a set of nonlinear Volterra integral equations. The solutions are compared with phenomena observed in hurricanes.

1. Introduction

The axisymmetric vortex motion of a viscous fluid which is produced by a very long rod normal to a flat surface (see Fig. 1), has been investigated in SCHWIDERSKI and LUGT (1962) on the basis of an extension of Prandtl's boundary layer theory. The Navier-Stokes equations have been reduced to a set of ordinary differential equations which are connected with appropriate boundary data. In the present paper the nonlinear boundary value problem is transformed into a set of nonlinear Volterra integral equations, which can be solved by an efficient iteration procedure.

Complete numerical results of vortex flows normal to a flat surface are displayed in the present paper for a selected variety of Reynolds numbers. Characteristic properties of these flows are pointed out and compared with phenomena observed in hurricanes. Although the vortex models considered are only very rough approximations to real hurricanes, the qualitative agreement is satisfactory.

2. The reduced Navier-Stokes equations

An axisymmetric vortex flow over a flat surface has been defined in SCHWIDERSKI and LUGT (1962) as a solution of the Navier-Stokes equations

$$uu_r + wu_z - \frac{v^2}{r} = -\frac{1}{\varrho}p_r + \nu\left[u_{rr} + \left(\frac{u}{r}\right)_r + u_{zz}\right], \quad (1)$$

$$uv_r + wv_z + \frac{uv}{r} = +\nu\left[v_{rr} + \left(\frac{v}{r}\right)_r + v_{zz}\right], \quad (2)$$

$$uw_r + ww_z = -\frac{1}{\varrho}p_z + \nu\left[w_{rr} + \frac{1}{r}w_r + w_{zz}\right], \quad (3)$$

$$(ru)_r + (rw)_z = 0 \quad (4)$$

by the boundary data

$$\left.\begin{matrix} r>0 \\ z=0 \end{matrix}\right\}: u=0,\ v=0,\ w=0, \quad (5)$$

$$\left.\begin{matrix} r=\infty \\ z<\infty \end{matrix}\right\}: u=0,\ v=0,\ w=0, \quad (6)$$

$$\left.\begin{matrix} r\to 0 \\ z>0 \end{matrix}\right\}: u\to 0,\ \frac{rv}{\Gamma}\to 1,\ \frac{w}{\Gamma A \log\frac{r_0}{r}}\to 1, \quad (7)$$

$$\left.\begin{matrix} r<\infty \\ z\to\infty \end{matrix}\right\}: u\to 0,\ \frac{rv}{\Gamma}\to 1,\ \frac{w}{\Gamma A \log\frac{r_0}{r}}\to 1. \quad (8)$$

In these equations u, v, and w denote the velocity components of the vortex flow which correspond to the coaxial cylindrical coordinate system (r, φ, z). The variable pressure and the constant density and kinematic viscosity of the fluid are designated by p, ϱ, and ν. While the vortex strength Γ and the radial extension r_0 of the axial logarithmic sink (see Fig. 1) are at one's disposal, the constant value of A must be determined simultaneously with the solution.

Guided by an extension of Prandtl's boundary layer theory a first order reduction of the Navier-Stokes equations to an ordinary boundary value problem has been found in SCHWI-

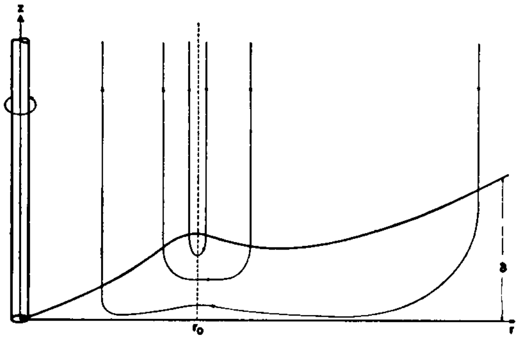

FIG. 1. Scheme of the secondary flow produced by a vortex motion normal to a flat surface.

DERSKI and LUGT (1962), which is valid in the vicinity of the line $r = r_0$. This approximation was achieved with the aid of the limiting line $z = \delta(r)$ of the boundary layer along the surface at $z = 0$. This limiting line was defined by the ε-condition

$$v(r, z) = \frac{\Gamma}{r}(1-\varepsilon) \tag{9}$$

and remained to be found together with the primary tangential velocity v.

After applying the following similarity transformation

$$r = r, \quad \zeta = \frac{z}{\delta(r)}, \quad R = \frac{\Gamma}{\nu} \ (= \text{Reynolds number}), \tag{10}$$

$$u = \frac{\Gamma}{r} U(\zeta), \quad v = \frac{\Gamma}{r} V(\zeta), \quad w = \Gamma A \log \frac{r_0}{r} W(\zeta),$$

$$\frac{p}{\varrho} = -\frac{\Gamma^2}{2r^2} P(\zeta), \tag{11}$$

$$U = -\dot{G}(\zeta), \quad W = \frac{1}{G_\infty}[G(\zeta) - \zeta \dot{G}(\zeta)] \tag{12}$$

one arrives at the following ordinary differential equations

$$\dddot{G} + \sigma^2 \zeta \ddot{G} = \frac{R\sigma^2}{4}(\dot{G}^2 + V^2 - P), \tag{13}$$

$$\ddot{V} + \sigma^2 \zeta \dot{V} = 0, \tag{14}$$

$$\dot{G}(G - \zeta \dot{G}) = \frac{1}{2\sigma^2} \dot{P}, \tag{15}$$

which must be integrated under the boundary conditions

$$\zeta = 0: \ G = 0, \ \dot{G} = 0, \ V = 0, \tag{16}$$

$$\zeta = \infty: \ G = G_\infty, \ V = 1, \ P = 1. \tag{17}$$

Simultaneously with this reduction one obtains the equation of the limiting line of the boundary layer in the vicinity of $r = r_0$ in the form (see Fig. 1)

$$z = \delta(r) = \sigma \frac{r_0}{2}\left(\frac{r}{r_0}\right)^2 \left(1 + 2 \log \frac{r_0}{r}\right). \tag{18}$$

The constant value A is

$$A = 2\frac{\sigma}{r_0} G_\infty. \tag{19}$$

where the characteristic parameter σ is determined by the ε-condition

$$\zeta = 1: \ V = 1 - \varepsilon. \tag{20}$$

For an integration of the remaining ordinary boundary value problem it is convenient to introduce the following new variables

$$\eta = \sigma \zeta \quad \text{and} \quad g = \sigma G. \tag{21}$$

This substitution leads to the differential equations

$$\dddot{g} + \eta \ddot{g} = \frac{R}{4}(\dot{g}^2 + V^2 - P), \tag{22}$$

$$\dot{g}(g - \eta \dot{g}) = \frac{1}{2} \dot{P} \tag{23}$$

with
$$V(\eta) = \operatorname{erf} \frac{\eta}{\sqrt{2}} = \frac{2}{\sqrt{\pi}} \int_0^{\eta/\sqrt{2}} e^{-t^2} dt, \tag{24}$$

which must be integrated under the boundary conditions

$$\eta = 0: \ g = 0, \ \dot{g} = 0, \tag{25}$$

$$\eta = \infty: \ g = g_\infty, \ P = 1. \tag{26}$$

3. A reduction to a Volterra integral equation

The ordinary boundary value problem defined by the equations (22) through (26) has been solved for the Reynolds number $R = 10$ by the Runge-Kutta method which started the integration with an assumed set of initial data that had to be improved successively until the boundary condition (26) was sufficiently met. The result of this integration has been displayed in SCHWIDERSKI and LUGT (1963). The inefficiency of this method is obvious because two initial conditions must be found which correspond to the two ignored boundary conditions (26).

For an efficient integration of the remaining ordinary boundary value problem it is convenient to transform the differential equations (22) and (23) with the boundary conditions (25) and (26) into an equivalent integral equation (TRICOMI, 1957). Assuming that the auxiliary function

$$D(\eta) = P(\eta) - \dot{g}^2(\eta) - V^2(\eta) \tag{27}$$

is known, then the differential equation (22) reduces to the linear equation

$$\dddot{g} + \eta \ddot{g} = -\frac{R}{4} D(\eta) \tag{28}$$

which is integrable by quadrature. Indeed, with the aid of the error function (24) one finds the general solution of (28) in the form

$$\dot{g}(\eta) = \lambda V(\eta) + \bar{\lambda} - \frac{R}{4}\int_0^\eta \frac{D(t)}{\dot{V}(t)}[V(\eta) - V(t)]\,dt. \tag{29}$$

The boundary conditions (25) and (26) lead then to the nonlinear Volterra integral equation

$$\dot{g}(\eta) = \lambda V(\eta) - \frac{R}{4}\int_0^\eta \frac{D(t)}{\dot{V}(t)}[V(\eta) - V(t)]\,dt \tag{30}$$

for $\dot{g}(\eta)$, where

$$\lambda = \frac{R}{4}\int_0^\infty \frac{D(t)}{\dot{V}(t)}[1 - V(t)]\,dt. \tag{31}$$

The functions $g(\eta)$ and $P(\eta)$ are obtained by direct integration of the equations (30) and (23) under the boundary conditions (25) and (26).

They are
$$g(\eta) = \int_0^\eta \dot{g}(t)\,dt \tag{32}$$

and
$$P(\eta) = \mu + g^2(\eta) - 2\int_0^\eta t\dot{g}^2(t)\,dt, \tag{33}$$

where
$$\mu = P(0) = 1 - g_\infty^2 + 2\int_0^\infty t\dot{g}^2(t)\,dt. \tag{34}$$

This concludes the transformation of the boundary value problem under consideration into a set of Volterra integral equations. It is, vice versa, not difficult to show that any solution of the integral equations is a solution of the boundary value problem. Thus the transformation applied is an identical transformation.

The integral equations (30), (32), and (33) have been numerically integrated for a variety of Reynolds numbers. Some selected solutions will be discussed in the following section. All solutions have been constructed by an iteration procedure (SCHWIDERSKI and LUGT, 1962), which successively improved an appropriate first approximation. It was found that the method is very efficient, provided the first approximation is sufficiently close to the correct solution.

For small Reynolds numbers the iterations can be started with

$$g(\eta) \equiv 0,\ P(\eta) \equiv 1 \tag{35}$$

as adequate first approximations. For larger Reynolds numbers the iterations may be started by solutions, which are obtained for smaller Reynolds numbers. In order to improve the rate of convergence of the iteration procedure, especially in cases where the iteration is started with a very crude approximation, it is helpful to average the outcoming solution with the entering approximation of the iteration by means of appropriate weighting factors.

4. Properties of vortex flows

Axisymmetric vortex flows normal to a flat surface have been computed and tabulated for various Reynolds numbers (SCHWIDERSKI and LUGT, 1963). Characteristic examples have been selected and plotted in Figs. 2 and 3.

The numerical results confirm the phenomena of vortex flows which were pointed out by SCHWIDERSKI and LUGT (1962). It may be emphasized that the numerical calculations of all vortex flows computed indicated no symptoms of instability of the flow near the line $r = r_0$ (see Fig. 1 and compare SCHWIDERSKI and LUGT (1963)). Accordingly, vortex flows seem to remain laminar and attached to the surface for all Reynolds numbers at least in the vicinity of $r = r_0$. Nevertheless, since the secondary flow is of "wake type" (SCHWIDERSKI and LUGT, 1963) within the cylinder $r = r_0$, separation of the

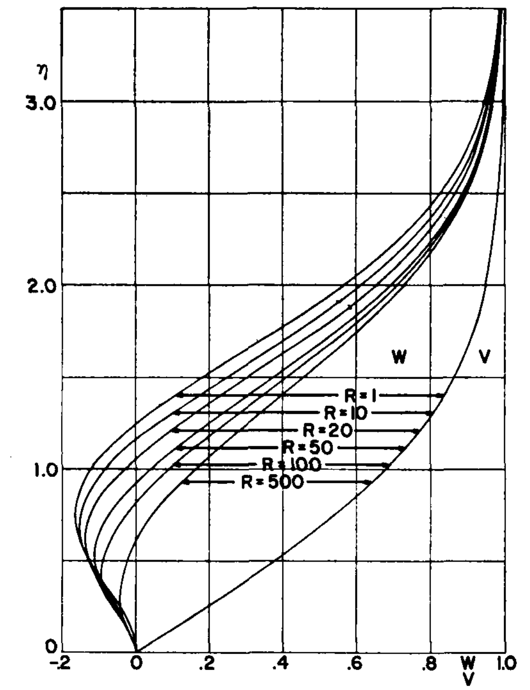

FIG. 2. The dimensionless velocity components V and W vs. the dimensionless variable η for different Reynolds numbers.

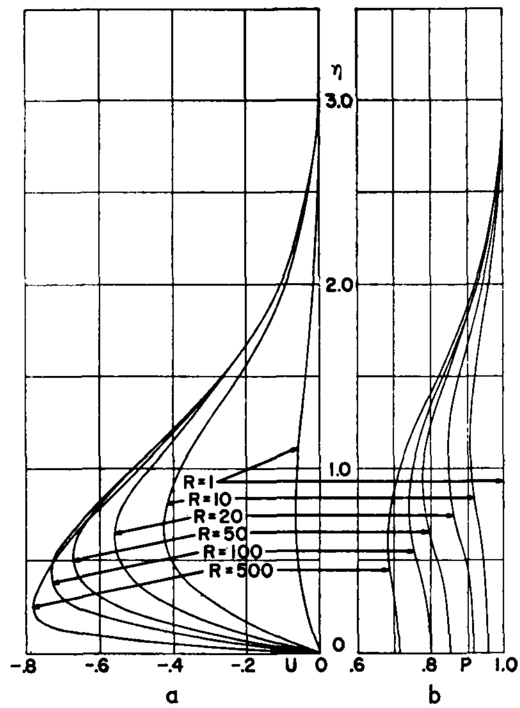

FIG. 3. The dimensionless velocity component U (3a) and the dimensionless pressure P (3b) vs. the dimensionless variable η for different Reynolds numbers.

flow should be expected to occur in the neighborhood of the vortex axis. However, outside the cylinder $r = r_0$ the secondary flow is of "stagnation type" and tends to prevent any flow separation. Thus, the instability in the motion, which is caused by a flow separation at the axis, seems to fade away when the flow changes its character. This explains the fact that the radial velocity U remains free of inflection points which characterize unstable flows.

As was explained by SCHWIDERSKI and LUGT (1962), vortex flows normal to a flat surface represent approximate models of hurricanes, provided exterior disturbances of the vortex flows other than those caused by the surface of the earth are excluded. Consequently, the numerical results may be compared with phenomena observed outside the cores of hurricanes. Despite the fact that real hurricanes are highly distorted by the tremendous rainfall inside the circle $r = r_0$, by the change of the density and the turbulence of the air, etc., the agreement with available observations appears to be satisfactory.

Indeed, the observations confirm the wake and stagnation character of the secondary flow, where the rainfall area coincides roughly with the circular region $r \leqslant r_0$. The rapidly increasing rainfall toward the core of a hurricane (ELIASSEN and KLEINSCHMIDT, 1957, p. 130) is indicated by the logarithmic increase of the axial velocity toward the vortex axis. The strong dependence of the secondary flow upon the Reynolds number R (see Figs. 2 and 3) indicates a radial shear stress at the surface, which is large compared with the tangential shear stress. This explains the very strong radial ocean waves produced by hurricanes which are known as ocean swells (RIEHL, 1954, p. 298). As can be deduced from Fig. 2, inside the cylinder $r = r_0$ the secondary flow near the surface converges to the surface before it finally leaves the vicinity of the surface in the normal direction (see Fig. 1). According to Riehl this remarkable phenomenon has also been observed in hurricanes (RIEHL, 1954, p. 320).

In order to get a rough idea of the boundary layer thickness near the core of a hurricane, one may consider a hurricane of average size, which has a rainfall area of about $2r_0 \approx 2000$ km in diameter (ELIASSEN & KLEINSCHMIDT, 1957, p. 130). If the limiting line of the boundary layer $z = \delta(r)$ (see eq. (18)) is determined to a relative accuracy of $\varepsilon_r = 1$ %, then the characteristic parameter σ was found to be $\sigma \approx 2.5$ (SCHWIDERSKI and LUGT, 1963). With these assumptions the boundary layer thickness at the edge of the hurricane core which is at $r \approx 30$ km from the vortex axis (RIEHL, 1954, p. 297), can be computed by equation (18), which yields $\delta \approx 9$ km. This result explains the observed thick boundary layers (PALMÉN and RIEHL, 1957), which are produced by the friction forces along the surface of the earth in hurricanes.

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