Trapping of Low Frequency Oscillations in an Equatorial "Boundary Layer"

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ABSTRACT

A thin layer of water (thickness h) and its spherical boundaries (mean radius R) rotate with uniform angular velocity Ω. Using an asymptotic expansion in the parameter $h/R \ll 1$ we investigate the inertial oscillations of "large scale" disturbances at the equator. It is found that the vertical constraints on the motion inhibit the lateral (poleward) propagation of low frequency energy. Solutions of the non-separable eigenvalue problem give standing oscillations whose longest period is $(R/h)^{\frac{1}{2}}(2\pi/\Omega)$ and which are self-confined within a distance of order $(Rh)^{\frac{1}{2}}$ on either side of the equator. This axial symmetric calculation is probably valid for disturbances whose east-west wave-length is large compared with the width of the "boundary layer" $(Rh)^{\frac{1}{2}}$. It is suggested that a trapped wave can transfer energy to a mean flow at the equator, as the result of the correlation between the velocity components of the eigenfunction.

In the bottom water of the equatorial ocean, where the stratification is small, there might be oscillations whose period is of the order of a month and which extended some 300 km north and south of the equator. When the static stability is appreciable the effect can only be expected if, on this time scale, the fluid dissipates temperature perturbations much more rapidly than momentum.

1. Introduction

According to the classical theory of inertial oscillations it is possible for a fluid, which is enclosed by latitude walls in an equatorial channel, to have oscillatory modes whose period is much larger than one day (BJERKNES *et al.*, 1933; PROUDMAN, 1942). We propose to show that a natural trapping effect (no latitude walls) is possible at the equator, provided there is a mechanism for reflecting energy which is propagated vertically. Consider two spherical shells whose radial separation (h) is small compared to the mean radius R. These horizontal boundaries, and the incompressible fluid contained between them, are rotating about the polar axis at the uniform rate Ω (rad/sec). The general equation for the normal modes of oscillation may be set down in spherical coordinates (SOLBERG, 1936). Aside from the fact that the general solution is not known, we shall use a boundary layer analysis to isolate a particular part of the complete spectrum. The simplified form of the spherical equations for motions which are confined to an equatorial region are known (VERONIS, 1963) and need only be sketched below. One starts with the Eulerian perturbation equations

$$w'(i\omega') - 2\Omega \cos\phi u' = -\frac{\partial P'}{\partial r}, \tag{1}$$

$$u'(i\omega') + 2\Omega \cos\phi w' - 2\Omega \sin\phi v' = -\frac{1}{r\cos\phi}\frac{\partial P'}{\partial \theta}, \tag{2}$$

$$v'(i\omega') + 2\Omega \sin\phi u' = -\frac{1}{r}\frac{\partial P'}{\partial \phi}, \tag{3}$$

$$\frac{\partial}{\partial r} w' r^2 \cos\phi + \frac{\partial}{\partial \phi} r v' \cos\phi + \frac{\partial}{\partial \theta} r u' = 0, \tag{4}$$

where $w'(r, \theta, \phi)\, e^{i\omega' t}$ is the radial (r) component of an oscillation whose time (t) frequency is ω', u' is the component of motion in the eastward direction (θ), v' is the northward ($+\phi$) component and P' is the pressure perturbation divided by the constant density.

The scaling transformations and subsequent asymptotic expansion given below are based on the premise that there are disturbances at the equator for which all the coriolis forces are of the same order of magnitude in h/R. Let

$$\left.\begin{array}{c} r = R + z', \quad z' = hz, \quad \phi = y\sqrt{h/R}, \quad \theta = x\sqrt{h/R} \\ u' = u, \quad v' = v, \quad w' = \sqrt{h/R}w, \quad P' = 2h\Omega p \\ \omega' = 2\Omega\sqrt{h/R}\omega, \end{array}\right\} \quad (5)$$

If $h/R < 1$ then $\sin \phi = (h/R)^{\frac{1}{2}}(y + O(h/R))$ and the asymptotic form of (1–4) becomes

$$-u + \frac{\partial p}{\partial z} = 0, \quad (6)$$

$$i\omega u + w - yv + \frac{\partial p}{\partial x} = 0, \quad (7)$$

$$i\omega v + yu + \frac{\partial p}{\partial y} = 0, \quad (8)$$

$$\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0 \quad (9)$$

correct to order h/R. These equations include all the coriolis terms expanded about their equatorial values, but the vertical acceleration $(\partial w'/\partial t)$ is missing in (6) because w' is small compared to u' and $\partial/\partial t$ is small compared to Ω. The central question is: Do there exist solutions of (6–9) which satisfy the boundary condition

$$\{u, v, w\} \to 0 \text{ for } |y| \gg 1, \quad (10)$$

$$w(x, y, 0) = w(x, y, -1) = 0. \quad (11)$$

If so then the foregoing assertions are valid and there will be trapped modes of frequency $\omega' \sim 2\Omega\sqrt{h/R}$. To demonstrate this we will consider the axial symmetric case $(\partial/\partial x = 0)$. This solution will probably be valid for long waves too (i.e. $\partial/\partial x < 1$) but there may be additional modes for this case which are not revealed by the axial-symmetric calculation.

2. Axial-symmetric oscillations

When $\dfrac{\partial}{\partial x} = 0$ eq. (9) allows us to introduce a stream function

$$w = \psi_y \quad v = -\psi_z$$

and from (7) it follows that

$$i\omega u = -(\psi_y + y\psi_z).$$

By eliminating p from (8), (6) and substituting the above relations in the result we get the following self-adjoint equation

$$\left.\begin{array}{c} (y^2 - \omega^2)\dfrac{\partial^2\psi}{\partial z^2} + \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial}{\partial y} y \dfrac{\partial\psi}{\partial z} + \dfrac{\partial}{\partial z} y \dfrac{\partial\psi}{\partial y} = 0, \\ \psi(y, 0) = \psi(y, -1) = 0, \\ \psi(\pm\infty, z) = 0. \end{array}\right\} \quad (12)$$

The characteristics of this partial differential equation are the parabolas: $z = \frac{1}{2}(y \pm \omega)^2 +$ constant, and it may be verified by direct substitution that the general solution of (12) is

$$\psi = F[\tfrac{1}{2}(y \pm \omega)^2 - z],$$

where F is any function that is twice differentiable. An elementary solution of this type is

$$A(\alpha, \omega)\, e^{i\omega t} e^{i\alpha[\frac{1}{2}(y-\omega)^2 - z]}$$

and although this satisfies "reasonable" z-boundary conditions it is not an admissible eigenfunction because of its behavior at $y = \infty$. We will superimpose an infinite number of these, corresponding to the free parameter α, in order to satisfy the boundary conditions in (12).

3. The Eigenvalues

A solution of (12) which has the same value on $z = 0$ and $z = -1$, and which is anti-symmetric to the equator is

$$\psi_A = \sum_{n=1}^{M} b_n(M)\, e^{-2n\pi i z}\{e^{n\pi i(y+\omega)^2} - e^{n\pi i(y-\omega)^2}\}. \quad (13)$$

This will represent an eigenfunction of our problem if it is possible to choose the arbitrary co-efficients $b_n(M)$ such that $\psi_A(y, 0)$ vanishes as $M \to \infty$, and such that $\lim \psi_A(y, z)$ vanishes for large y. The formal iteration scheme for determining the M co-efficients in (13) is as follows. Consider an arbitrary interval from $y = 0$ to $y = Y$ and distribute the $M - 1$ points

$$y_1 = \omega, \; y_2, \dots y_m \dots y_{M-1} \quad (14)$$

on the top boundary and inside this interval. This subdivision will be chosen such that: a) the critical point $y_1 = \omega$ is included for all M, b) none of the remaining y_m is commensurate with ω, c) the maximum length of a sub-interval is less than twice (say) Y/M.

Now require that $\psi_A(y, 0)$ vanish at each of these points. From (13) we then obtain the following $M-1$ linear equations

$$0 = \psi_A(\omega, 0) = \sum b_n(M)[e^{n\pi i 4\omega^2} - 1] \qquad (15)$$

$$0 = \psi_A(y_m, 0) = \sum b_n[e^{n\pi i(y_m+\omega)^2} - e^{n\pi i(y_m-\omega)^2}] \qquad (16)$$

$$m = 2, \ldots M-1$$

For the M^{th} condition we specify that the value of $\partial^2\psi/\partial y^2$ at the critical point $(\omega, 0)$ shall be zero:

$$\left.\begin{aligned} 0 = & -16\omega^2\pi^2 \sum n^2 b_n e^{n\pi i 4\omega^2} \\ & + 2\pi i \sum n b_n [e^{n\pi i 4\omega^2} - 1] \end{aligned}\right\} \qquad (17)$$

Equations (15), (16), (17) have a solution only if its $M \times M$ determinant vanishes, and this leads to an equation for ω at each stage of the iteration. Now notice from (15) that one row of this determinant will be identically zero if

$$4\omega^2 = 2, 4, 6, 8 \ldots \qquad (18)$$

Using any one of these even integers will assure a solution for $b_n(M)$ at each iteration.

With regard to the convergence of (13) we *will now show that for any Y, no matter* how large, it is possible to choose M so that $|\psi_A(y, 0)|$ is uniformly small between the $(M-1)$ zeroes. Note that the co-efficient of b_n in (16) is never zero zince y_m is incommensurate with $y_1 = \omega$. Likewise the co-efficient of b_n in (17) is $-16\omega^2\pi^2 n^2 \neq 0$. Therefore the rank of the matrix of (15)–(17) is $(M-1)$, or can be made so by a slight modification in the distribution of the zeroes. A specific value for the b_n may be found at each stage once we introduce a suitable normalization condition. For this we have chosen

$$\sum_1^M n^2 |b_n(M)| = 1 \qquad (19)$$

because this assures that all second derivatives of ψ_A are bounnded by a value which is independent of M. This allows us to compute the maximum value of $\partial\psi_A/\partial y$ between any two zeroes of $\psi_A(y_m, 0)$. When the finite sum in (13) is differentiated we get

$$\left|\frac{\partial\psi_A}{\partial y}\right| < 2\pi \sum n |b_n| \{|y+\omega| + |y-\omega|\}$$
$$< 4\pi(Y+\omega)\sum n|b_n| < 4\pi(Y+\omega).$$

Let $2Y/M$ be the maximum spacing between any two successive zeroes. Then the *maximum* value of ψ_A between any two zeroes is:

$$\max|\psi_A| < \int dy \left|\frac{\partial\psi_A}{\partial y}\right| < 4\pi(Y+\omega)\left(\frac{2Y}{M}\right).$$

Therefore, given a Y and an ε we may insure that all values of $\psi_A(y,0)$ are less than ε by choosing a number of terms greater than $M = 8\pi Y(Y+\omega)/\varepsilon$. This completes the proof that the z-boundary conditions may be satisfied to any required accuracy.

It remains to prove that the boundary condition at infinity is also satisfied. The underlying reason for this is that when $z \neq 0$ the two terms which constitute the right hand side of (13) produce "beats" in the y direction and, as we shall now show, the beat frequency approaches zero as $y \to \infty$. Introduce the coordinate η instead of y into (13) by means of the transformation

$$(\eta-\omega)^2 \equiv (y-\omega)^2 - 2z \quad (z \text{ negative})$$

and therefore

$$(y+\omega)^2 - 2z = (\eta+\omega)^2 + \gamma$$

where $\gamma = 4\omega[\sqrt{(\eta-\omega)^2 + 2z} - (\eta-\omega)]$.

Eq. (13) then becomes

$$\psi_A[y(\eta, z), z] = \sum_1^M b_n(M)\{e^{n\pi i[(\eta+\omega)^2+\gamma]} - e^{n\pi i(\eta-\omega)^2}\}.$$

When the value of $\psi_A(\eta, 0)$ is subtracted from both sides we get

$$\psi_A(y, z) - \psi_A(\eta, 0) = \sum_1^M b_n(M) e^{n\pi i(\eta+\omega)^2}\{e^{n\pi i\gamma} - 1\}$$
$$= 2i\sum_1^M b_n(M) e^{n\pi i(\eta+\omega)^2} e^{n\pi i\gamma/2} \sin\left(\frac{n\pi\gamma}{2}\right).$$

But γ is a real function because $(\eta-\omega)^2+2z>0$, and since $|\sin(n\pi\gamma/2)/n\pi\gamma/2|\leq 1$ we obtain the following inequality

$$|\psi_A(y,z)-\psi_A(\eta,0)|<\pi|\gamma|\sum_1^M n|b_n(M)|.$$

According to (19) the summation on the right is less than unity and therefore

$$\begin{aligned}|\psi_A(y,z)-\psi_A(\omega+\sqrt{(y-\omega)^2-2z},0)|\\ &<4\pi\omega[\sqrt{(y-\omega)^2-2z}-(y-\omega)]\\ &<\frac{4\eta\omega(-z)}{(y-\omega)}+O\left(\frac{1}{(y-\omega)^2}\right).\end{aligned}$$

For large values of $y<Y$ the term on the right is of order $1/y$. The term $\psi_A(\eta,0)$ on the left may be made arbitrarily small independent of y by choosing M sufficiently large. It therefore follows that

$$|\psi_A(y,z)|<\frac{4\pi\omega(-z)}{y-\omega}+O\left(\frac{1}{(y-\omega)^2}\right)$$

and we conclude that when the z boundary conditions are satisfied the magnitude of the eigenfunction decreases more rapidly than y^{-1}.

The eigenfunctions which are symmetric to the equator are treated in a similar way. Thus

$$\psi_S(y,z)=\sum_{n=1}^M B_{2n-1}(M)\,e^{-2\pi i z(2n-1)}\left\{e^{\pi i(2n-1)(y+\omega)^2}+e^{\pi i(2n-1)(y-\omega)^2}\right\} \quad (20)$$

is solution of (12) and the condition $\psi_S(\omega,z)=0$ is satisfied for all M if

$$4\omega^2=1,3,5,7,\ldots \quad (21)$$

The remainder of the argument is identical with that used for the antisymmetric modes.

4. Conclusion

A quasi-horizontal equatorial disturbance in a homogeneous fluid will have restoring forces which are small compared with its inertia because of the preponderant motion parallel to the axis of rotation. However if no latitude walls are present an arbitrary equatorial disturbance will tend to radiate energy to the poles in a manner similar to, but more complicated than, other wave-guide phenomenon. We have shown that certain disturbances will not radiate at all. These normal modes are self confined to the equator and may be excited by a resonant source in that region. The lowest frequency that has been found in this study is $\omega'=\Omega\sqrt{h/R}$. This mode is symmetric to the equator and has vertical nodes at latitudes $\phi=\frac{1}{2}\sqrt{h/R}$, $\frac{3}{2}\sqrt{h/R}$, $\frac{5}{2}\sqrt{h/R}$,... The amplitude of the stream function decreases more rapidly than the inverse distance from the equator and the motion in each cell is undoubtedly asymmetric due to the non-separability. This last point is of particular interest when the east-west wavelength is large compared with the thickness of the equatorial boundary layer and not infinite, as is the case for the purely internal mode considered above. The non-linear Reynolds stress associated with correlated velocity components of the wave will tend to transfer energy to a mean flow at the equator. We are then led to speculate on geophysical energy sources for the trapped mode. Could the fortnightly tidal oscillation drive a subharmonic inertial oscillation in the neutrally stratified bottom water of the equatorial ocean? Might there be disturbances drifting slowly across the equator from higher latitudes and supplying energy to this different type of equatorial motion?

While it should be possible to realize this oscillation in the laboratory by reducing density variations and damping forces, it might be argued that the effect cannot be expected in the stratified ocean or atmosphere. A relatively small amount of static stability is required, in an adiabatic motion, to produce buoyant restoring forces which are much larger than the vertical component of the coriolis force. However when one is theorizing on long period geophysical oscillations it is not clear that the adiabatic approximation is more satisfactory than the barotropic approximation that has been made here. It is believed that there might be other processes in the geophysical complex which "open up" this degree of freedom and allow a small but systematic action of the vertical component of the coriolis force which modulates the higher frequency effects over long periods of time.

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