# SHORTER CONTRIBUTION

## A note on gravity reduction to a spherical surface

By ARNE BJERHAMMAR, *Institute of Geodesy, The Royal Institute of Technology, Stockholm*

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If gravity anomalies are known at the surface of a sphere then the disturbance potentials can be computed on the surface of the sphere and in space by the aid of a series expansion

$$T_j = \frac{1}{4\pi}\frac{1}{r_j}\iint_S \Delta g_0 \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\,dS, \tag{1}$$

where $r_j$ = the distance between the origin and the point in space,
$r_0$ = the radius of the sphere,
$S$ = the surface of the sphere,
$\Delta g_0$ = the gravity anomaly at the surface of the sphere.

In the normal geodetic problem the gravity anomalies are not known at any sphere but instead at the physical surface of the earth. Therefore, we have to find the spatial gravity anomalies $\Delta g_j$ which belong to a given sphere. This we can do by the aid of the boundary condition

$$-\frac{\partial T_j}{\partial m} + \frac{T_j}{\gamma}\frac{\partial \gamma}{\partial m} = \Delta g_j, \tag{2}$$

where $T_j$ = the disturbance potential outside the sphere.

For $m = r_j$, we obtain

$$\frac{\partial T}{\partial m} = \frac{\partial T}{\partial r_j} = \frac{-1}{4\pi\cdot r_j 2}\iint_S \Delta g_0 \sum \frac{(2n+1)(n+1)}{(n-1)} \times \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\,dS. \tag{3}$$

Thus we have

$$\Delta g_j = \frac{1}{4\pi r_j^2}\iint_S \Delta g_0 \sum \left(\frac{(2n+1)(n+1)}{n-1} + \frac{r_j}{\gamma}\frac{\partial\gamma}{\partial m}\frac{2n+1}{n-1}\right)\left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\,dS. \tag{4}$$

In the most important application of the formula, $\Delta g_j$ represents the gravity anomalies at the surface of the non-spherical earth. Then $\Delta g_0$ represents the gravity anomalies at a reference sphere. The relation is interesting because it makes it possible for us to reduce the gravity anomalies from a physical surface to a mathematical reference sphere without making use of any hypothesis or any guessing of the density. Furthermore the equation can be solved without any integration over the rugged physical surface and without any computations of unattainable surface normals.

For the further analysis we accept the usual approximation

$$\frac{1}{\gamma}\cdot\frac{\partial\gamma}{\partial m} = -\frac{2}{r_j}. \tag{5}$$

Thus we obtain the simple expression

$$\Delta g_j = \frac{1}{4\pi r_j^2}\iint_S \Delta g_0 \sum (2n+1)\left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\,dS. \tag{6}$$

Only when $r_0$ is smaller than $r_j$ we have a convergent series expansion. The formula is still valid in the divergent case but this application requires special treatment for infinite $n$-values. Another method is to select a reference sphere with radius smaller than the shortest distance from the origin to the physical surface. Then we have to note that the gravity anomalies computed inside the surface have no physical meaning. They are only mathematical parameters.

A numerical evaluation of our series expansion gives (cf. Bjerhammar, 1960)

$$\sum_{n=2}^{\infty}(2n+1)x^n P_n(\cos\omega)=\frac{1}{r}-3x\cos\omega-\frac{2(x^2-x\cos\omega)}{r^3}-1 \tag{7}$$

where $$x=\frac{r_0}{r_j}\quad r=\frac{r_{ij}}{r_j}.$$

$r_{ij}$ = the distance between the actual point and the running point.

Thus we get the final solution

$$\Delta g_j=\frac{1}{4\pi r_j^2}\iint_S \Delta g_0 \times\left(\frac{1}{r}-3x\cos\omega-\frac{2(x^2-x\cos\omega)}{r^3}-1\right)dS. \tag{8}$$

This equation is the integral equation necessary for a computation of the reduced gravity values from given gravity values on the physical surface. It is supposed to be related with the reduction method of Rudzki and the isostatic reduction method. The principal difference is that our integral equation does not make use of any hypothesis or any guessed values of the density.

In a practical application of the method one can use equation (8) as a method for the determination of *iterative* values of $\Delta g_0$. The $x$-values will be smaller than unity and $\Delta g_j$ has to be computed from approximate $\Delta g_0$. The values of $\Delta g_0$ are just inserted as approximations in (8) for the first check. If the $\Delta g_j$ now computed show any difference with respect to the known $\Delta g_j$, the procedure is repeated with the discrepancy used as a correction. The whole procedure can be repeated till the wanted accuracy is obtained. Normally only two operations should be needed and it is supposed to be sufficient with integration over a surface less than 1°.

## REFERENCES

Bjerhammar, A., 1960, *Solution of the Resolvent Equation by the Aid of Spherical Harmonics for Non-spherical Surface*. Communications from the Royal Institute of Technology, Stockholm.

Bæschlin, C. F., 1948, *Lehrbuch der Geodäsie*, Zürich, pp. 713–718.