Determination of the stream function from vertical motion[1]

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ABSTRACT

By means of the solution of the linearized equation for the vertical components of the vorticity, the stream function is estimated using the data for the vertical motion. The exact method for solving the finite-difference analogue of the differential equation by spherical coordinates is discussed.

Функция тока определяется на основе решения линеаризированного уравнения для вертикальной составляющей вихря скорости с привлечением данных о вертикальных движениях в атмосфере. Дается метод точного решения конечно-разностного аналога дифференциального уравнения в сферических координатах.

1. The problem of determining vertical velocities according to the field of stream function (or according to the geopotentional field) has become classical. This method has been used because of the fact that the vertical component of wind speed in the atmosphere cannot be measured with instruments, and the meteorologists were obliged to express it through some characteristics of the atmosphere state, measured, for example, through the geopotentional. In theoretical and applied problems, however, we may meet with the inverse problem—the problem of determining the field of stream function according to the distribution of vertical velocities. As an example of such a type of the problem we shall take the problem of an air current flowing round a mountain barrier. In this problem we usually use the equation for the stream function integrated under appropriate boundary conditions.

For the large-scale motions, the boundary conditions may be used for determining the vertical currents over the mountains giving simple relations between a vertical component of current velocity and the form of the relief. Thus the problem is to determine the character of large-scale currents in the atmosphere in the surroundings of mountains under given vertical motions.

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To determine the stream function for this case the equation of vorticity linearized relative to zonal current was used (MUSAJELJAN, 1960)

$$\alpha\frac{\partial\Delta\psi}{\partial\lambda}+\eta(\alpha+\omega)\frac{\partial\psi}{\partial\lambda}-\frac{\nu}{a^2}\Delta\Delta\psi=-2\alpha_\xi(\alpha+\omega)a^2\cos\theta\frac{\partial\eta}{\partial\lambda}-\alpha^2a^2\sin\theta\frac{\partial^2\eta}{\partial\theta\partial\lambda},\qquad(1.1)$$

where λ is the geographical longitude; θ is complementary to the latitude $(\theta=\frac{1}{2}\pi-\varphi)$; α and α_ξ are the indices of the atmospheric circulation over the "mean level" and on the top of the mountain respectively; ω is the angular velocity of the earth's rotation; ψ is the stationary non-zonal part of the stream function on the mean level of the atmosphere; a, the mean radius of the earth, ν the coefficient of the horizontal turbulent mixing; $z=\xi(\theta,\lambda)$ is the equation for mountain surface; and $H=RT_0/g=8\times10^3\ m$ is "the height of the homogeneous atmosphere",

$$\Delta=\frac{1}{\sin\theta}\frac{\theta}{\partial\theta}\sin\theta\frac{\theta}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2};\ \eta=\frac{\xi}{H}$$

The use of this equation for the determination of the stream function in the climatological aspect (stationary problem) has given positive results (MUSAJELJAN, 1960). This result can be successfully applied to the solution of general non-stationary problems.

We assume that we can somehow determine the distribution of the verical component of wind velocity, i.e. the function $v_z = v_z(\theta, \lambda, z, t)$ is known, and the stream function $\psi = \psi(\theta, \lambda, z, t)$ is to be determined.

For this problem we shall take the non-linear vorticity equation, expressed as follows:

$$\frac{\partial \Delta\psi}{\partial t} + \frac{1}{a^2 \sin\theta}(\psi, \Delta\psi) + 2\omega\frac{\partial\psi}{\partial\lambda} = f(\theta, \lambda, z\ t), \qquad (1.2)$$

where $$(\psi, \Delta\psi) = \frac{\partial\psi}{\partial\lambda}\frac{\partial\Delta\psi}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\Delta\psi}{\partial\lambda},$$

$$f(\theta, \lambda, z, t) = \frac{2\omega a^2 \cos\theta}{\varrho} \cdot \frac{\partial \varrho v_z}{\partial z}.$$

We shall express this equation as a finite-difference equation in time, choosing the central differences;

$$\Delta\psi^{(1)} + \frac{\delta t}{2a^2\sin\theta}(\psi, \Delta\psi)^{(1)} + \omega\delta t\frac{\partial\psi^{(1)}}{\partial\lambda}$$
$$= \Delta\psi^{(0)} - \frac{\delta t}{2a^2 \sin\theta}(\psi, \Delta\psi)^{(0)} - \omega\delta t\frac{\partial\psi^{(0)}}{\partial\lambda}$$
$$+ \frac{\delta t}{2}(f^{(1)} + f^{(0)}). \qquad (1.3)$$

When $\psi^{(0)}$, $(\psi, \Delta\psi)^{(0)}$ and $f^{(0)}$ are known, this equation may be solved by an approximate method of iteration (Sadokov, 1960).

As the first step we shall take eq. (1.2), linearized relative to the zonal flow $\bar{U}_\lambda(\theta, z)$ under the conditions that

$$\psi = \bar{\psi}(\theta, z) + \psi'(\theta, z, \lambda, t).$$

In this case an exact solution of the finite-differential analogue of this equation might be constructed.

2. We shall express the linearized equation (1.2) as follows:

$$L(\psi') = \left(\frac{\partial}{\partial t} + \alpha(\theta, z, t)\frac{\partial}{\partial\lambda}\right)\Delta\psi'$$
$$+ 2(\bar{\alpha}(\theta, z, t) + \omega)\frac{\partial\psi'}{\partial\lambda} = f(\theta, \lambda, z, t). \qquad (2.1)$$

Now we shall discuss the method for solving eq. (1.2)[1] by means of the finite-difference method under the following initial and boundary conditions for an unknown function ψ':

$$\begin{aligned} &\text{at} \quad t = 0 \quad \psi'(\theta, \lambda, z, 0) = \psi^\circ(\theta, \lambda, z) \\ &\text{at} \quad \theta = 0 \quad \psi'(0, \lambda, z, t) \text{ limited.} \qquad (2.2) \\ &\text{at} \quad \theta = \pi/2 \quad \psi'(\pi/2, \lambda, z, t) \equiv 0 \end{aligned}$$

Furthermore, the function ψ' must satisfy the conditions of periodicity and continuity along λ, given by

$$P^{(u)}_{\lambda_r + 2\pi}(\psi') = P^{(u)}_{\lambda_r}(\psi'), \quad P^{(0)}(\psi') \equiv \psi' \quad (u = 0, 1 \ldots),$$

$$P^{(u)}_{\lambda_r} = \frac{\partial^u}{\partial\lambda^u}\bigg/_{\lambda = \lambda_r} \quad 0 \leqslant \lambda_r \leqslant 2\pi. \qquad (2.3)$$

For simplicity, the notation (′) on the function ψ' will be omitted in further discussion.

If we assume that

$$\psi(\theta, \lambda_j, z, t) = \psi_j(\theta, z, t); \quad f(\theta, \lambda_j, z, t) = f_j(\theta, z, t), \qquad (2.4)$$

then at every straight line $\lambda = \lambda_j = 2\pi j/n$ $(j = 1, 2, \ldots, n)$ we may write down the original eq. (2.1), and thus will obtain the system of differential equations as follows:

$$L_{\lambda = \lambda_j}(\psi) = f(\theta, \lambda_j, z, t) = f_j(\theta, z, t) \quad (j = 1, 2, \ldots, n). \qquad (2.5)$$

Now we introduce the complementary notation

$$P^{(u)}_{\lambda_j}(\psi) = \varphi^{(u)}_j(\theta, z, t), \quad \varphi^{(0)}_j \equiv \psi_j$$
$$(u = 1, 2, 3;\ j = 1, 2, \ldots, n) \qquad (2.6)$$

and from functions $\psi_j, \varphi_j^{(u)}$ $(u = 1, 2, 3)$, f_j we shall compose matrices

$$\Psi_{(\theta, z, t)} = \begin{Vmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{Vmatrix}, \quad \underset{(u = 1, 2, 3)}{\Phi^{(u)}_{(\theta, z, t)}} = \begin{Vmatrix} \varphi_1^{(u)} \\ \varphi_2^{(u)} \\ \vdots \\ \varphi_n^{(u)} \end{Vmatrix}, \quad F_{(\theta, z, t)} = \begin{Vmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Vmatrix}.$$

[1] Dependence of the function ψ' from z, and α and $\bar{\alpha}$ from z and t in eq. (2.1) is parametric. Eq. (2.1) at $f \equiv 0$ was described in (Checirda, 1961), where it is shown that dependence of the function α from θ is important.

Then, the equation system (2.5) may be expressed in the matrix form:

$$\frac{\partial}{\partial t}[L_1(\Psi)+\Phi^{(2)}]+\alpha[L_1(\Phi^{(1)})+\Phi^{(3)}]$$
$$+2(\tilde{\alpha}+\omega)\sin^2\theta\Phi^{(1)}=\sin^2\theta F \quad (2.7)$$
$$L_1=\sin\theta\frac{\partial}{\partial\theta}\sin\frac{\partial}{\partial\theta}.$$

It appears that eq. (2.7) will have sense if we can express matrix $\Phi^{(u)}$ through matrix Ψ. If we use the central finite differences to express $P_{\lambda j}^{(u)}(\Psi)$, then

$$\delta_\lambda\varphi_j^{(1)}=\sum_{p=1}^{q+1}B_p^{(q)}(\psi_{j+p}-\psi_{j-p})+O_1(\delta_\lambda^{2q+3}),$$
$$\delta_\lambda^2\varphi_j^{(2)}=-A_0^{(q)}\psi_j+\sum_{p=1}^{q+1}A_p^{(q)}(\psi_{j+p}+\psi_{j-p})+O_2(\delta_\lambda^{2q+2}),$$
$$\delta_\lambda^3\varphi_j^{(3)}=\sum_{p=1}^{q}C_p^{(q)}(\psi_{j+p}-\psi_{j-p})+O_3(\delta_\lambda^{2q+3}),$$
$$\delta_\lambda=\lambda_{j+1}-\lambda_j=\frac{2\pi}{n}. \quad (2.8)$$

A method for the determination of concrete expressions for the coefficient and residual terms in (2.8) may be found in (MIKELADZE, 1940). In our case q is an arbitrary constant, satisfying the following inequality:

$$1<1+q\leqslant\left[\frac{n}{2}\right], \quad (2.9)$$

where n is a number of points λ_j, dividing the interval $(0, 2\pi)$ into equal segments δ_λ. Further, as the residual terms in (2.8) are very small, they will be omitted.

If we introduce the square matrix of order n,[1]

$$D=\begin{Vmatrix} 0 & 1 & 0 \cdots\cdots\cdots & 0\\ 0 & 0 & 1 \quad 0 \cdots\cdots & 0\\ 0 & 0 & 0 \quad 1 \quad 0 \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 \cdots & \cdots & \cdots\; 0 & 1\\ 1 & 0 \cdots & \cdots\; 0 & 0\end{Vmatrix} \quad (2.10)$$

[1] When solving the boundary value problems according to the finite-difference method with the help of matrix D, it is possible to satisfy the periodic boundary conditions (2.3). The problem of such a type is discussed in NEMCHINOV (1962).

then according to equalities (2.8) and to conditions of periodicity and continuity, relations between matrices $\Phi^{(u)}$ and Ψ for functions ψ' (2.3) will be as follows:

$$\left.\begin{aligned}\delta_\lambda\Phi^{(1)}&=\sum_{p=1}^{q+1}B_p^{(q)}(D^p-D^{-p})\Psi,\\ \delta_\lambda^2\Phi^{(2)}&=-A_0^{(q)}\Psi+\sum_{p=1}^{q+1}A_p^{(q)}(D^p+D^{-p})\Psi,\\ \delta_\lambda^3\Phi^{(3)}&=\sum_{p=1}^{q}C_p^{(q)}(D^p-D^{-p})\Psi.\end{aligned}\right\} \quad (2.11)$$

It is possible to show that

$$D=WLW^*,\ W'=W,\ WW^*=W^*W=E, \quad (2.12)$$

where W is a fundamental matrix for matrix D, W^* is a Hermitian-conjugate matrix to matrix W, $L=\{\varkappa_1,\varkappa_2,\ldots,\varkappa_n\}$ is a diagonal matrix with elements—eigen numbers of matrix D, E is a unitary matrix. And

$$\left.\begin{aligned}W&=W_{(c)}+iW_{(s)},\\ W^*&=W_{(c)}-iW_{(s)},\\ L&=L_{(c)}^{(1)}+iL_{(s)}^{(1)},\end{aligned}\right\} \quad (2.13)$$

where

$$W_{(c)}=\|W_{jl}^{(c)}\|;\ W_{jl}^{(c)}=\frac{1}{\sqrt{n}}\cos\frac{2\pi jl}{n};$$

$$W_{(c)}=\|W_{jl}^{(s)}\|;\ W_{jl}^{(s)}=\frac{1}{\sqrt{n}}\sin\frac{2\pi jl}{n};\ (j,l=1,2,\ldots,n)$$

$$L_{(c)}^{(1)}=\{\varkappa_1^{(c)},\varkappa_2^{(c)},\ldots\varkappa_n^{(c)}\},\quad\left(\varkappa_j^{(c)}=\cos\frac{2\pi j}{n}\right),$$

$$L_{(s)}^{(1)}=\{\varkappa_1^{(s)},\varkappa_2^{(s)},\ldots\varkappa_n^{(s)}\},\quad\left(\varkappa_j^{(s)}=\sin\frac{2\pi j}{n}\right).$$

According to the properties of the trigonometric functions it follows that

$$W_{(c)}^2+W_{(s)}^2=E,\ W_{(c)}W_{(s)}=W_{(s)}W_{(c)}\equiv 0, \quad (2.14)$$
$$L^p=L_{(c)}^{(p)}+iL_{(s)}^{(p)},\quad L^{-p}=L_{(c)}^{(p)}-iL_{(s)}^{(p)}, \quad (2.15)$$

$$L_{(c)}^{(p)}=\{\varkappa_{jp}^{(c)}\},\quad\varkappa_{jp}^{(c)}=\cos\frac{2\pi jp}{n};$$

$$L_{(s)}^{(p)}=\{\varkappa_{jp}^{(s)}\},\quad\varkappa_{jp}^{(s)}=\sin\frac{2\pi jp}{n},$$

$$\varkappa_{jp}^{(c,s)}=\sqrt{n}\,W_{jp}^{(c,s)},$$

$$W_{jl+2}^{(c,s)} = 2\varkappa_j^{(c)} W_{jl+1}^{(c,s)} - W_{jl}^{(c,s)},$$
$$\varkappa_{j+2}^{(c,s)} = 2\varkappa_1^{(c)} \varkappa_{j+1}^{(c,s)} - \varkappa_j^{(c,s)} \quad (2.16)$$

at

$$W_{j0}^{(c)} = \frac{1}{\sqrt{n}}, \quad W_{j1}^{(c)} = \frac{1}{\sqrt{n}} \varkappa_j^{(c)} = \frac{1}{\sqrt{n}} \cos \frac{2\pi j}{n},$$
$$\varkappa_0^{(c)} = 1, \quad \varkappa_1^{(c)} = \cos \frac{2\pi}{n},$$

$$W_{j0}^{(s)} = 0, \quad W_{j1}^{(s)} = \frac{1}{\sqrt{n}} \varkappa_j^{(s)} = \frac{1}{\sqrt{n}} \sin \frac{2\pi j}{n},$$
$$\varkappa_0^{(s)} = 0; \quad \varkappa_1^{(s)} = \sin \frac{2\pi}{n}.$$

It is easy to see that if we know only the limited number of constants with high degree of exactitude—cos $(2\pi/n)$, sin $(2\pi/n)$—it is possible to determine all the elements of matrices $W_{(c)}$, $W_{(s)}$, and values $\varkappa_j^{(c)}$, $\varkappa_j^{(s)}$, $\varkappa_{jp}^{(c,s)}$, $W_{jp}^{(c,s)}$ using the recurrent formulae (2.16). As

$$\left.\begin{aligned} D^p + D^{-p} &= 2WL_{(c)}^{(p)} W^* \\ D^p - D^{-p} &= 2iWL_{(s)}^{(p)} W^* \end{aligned}\right\} \quad (2.17)$$

then equalities (2.11) may be expressed as follows:

$$\left.\begin{aligned} \delta_\lambda \Phi^{(1)} &= iW\Lambda^{(1)} W^* \Psi, \\ \delta_\lambda^2 \Phi^{(2)} &= W(\Lambda^{(2)} - A_0^{(q)} E) W^* \Psi, \\ \delta_\lambda^3 \Phi^{(3)} &= iW\Lambda^{(3)} W^* \Psi, \end{aligned}\right\} \quad (2.18)$$

where $\Lambda^{(1)} = \{\Lambda_j^{(1)}\}$, $\Lambda^{(2)} = \{\Lambda_j^{(2)}\}$, $\Lambda^{(3)} = \{\Lambda_j^{(3)}\}$ are the diagonal matrices with elements

$$\left.\begin{aligned} \Lambda_j^{(1)} &= 2\sum_{p=1}^{q+1} B_p^{(q)} \varkappa_{jp}^{(s)}, \\ \Lambda_j^{(2)} &= 2\sum_{p=1}^{q+1} A_p^{(q)} \varkappa_{jp}^{(c)}, \\ \Lambda_j^{(3)} &= 2\sum_{p=1}^{q} C_p^{(q)} \varkappa_{jp}^{(s)}. \end{aligned}\right\} \quad (2.19)$$

Excluding matrices $\Phi^{(u)}$ $(u = 1, 2, 3)$ from (2.7) with the help of (2.18), we now obtain the equation for matrix Ψ:

$$\frac{\partial}{\partial t}[\delta_\lambda^2 L_1(\Psi) + W(\Lambda^{(2)} - A_0^{(q)} E) W^* \Psi]$$
$$+ i\alpha[\delta_\lambda L_1(W\Lambda^{(1)} W^* \Psi) + \frac{1}{\delta_\lambda} W\Lambda^{(3)} W^* \Psi]$$
$$+ 2i\delta_\lambda(\tilde{\alpha} + W) \sin^2\theta W\Lambda^{(1)} W^* \Psi = \delta_\lambda^2 \sin^2\theta F. \quad (2.20)$$

As the elements of matrices W, W^* and Λ do not depend on the variable θ, they are commutative with an operator L_1. Then, introducing matrices

$$\begin{aligned} W^*\Psi &= \Psi^* = W_{(c)}\Psi - iW_{(s)}\Psi = \Psi^{(c)} - i\Psi^{(s)}, \\ W^* F &= F^* = W_{(c)} F - iW_{(s)} F = F^{(c)} - iF^{(s)} \end{aligned} \quad (2.21)$$

and muitiplying eq. (2.20) from the left by matrix W^*, and taking into consideration that $W^* W = E$, we obtain the canonic equation (2.20)

$$\delta_\lambda^2 L_1\left(\frac{\partial \Psi^*}{\partial t}\right) + (\Lambda^{(2)} - A_0^{(q)} E)\frac{\partial \Psi^*}{\partial t}$$
$$+ i\alpha\left[\delta_\lambda \Lambda^{(1)} L_1(\Psi^*) + \frac{1}{\delta_\lambda}\Lambda^{(3)}\Psi^*\right]$$
$$+ 2i\delta_\lambda(\tilde{\alpha} + \omega)\sin^2\theta \Lambda^{(1)}\Psi^* = \delta_\lambda^2 \sin^2\theta F^*, \quad (2.22)$$

which is the matrix system of n independent nonhomogeneous differential equations along variables θ and t, the right-hand components of matrix Ψ^* correspond to the components of matrix F^*.

If $\bar{\psi}^{(j)}$ and $\bar{F}^{(j)}$ are the components of matrices Ψ^* and F^*, then for $\bar{\psi}^{(j)}$ and $\bar{F}^{(j)}$ we shall have a system of independent equations

$$\delta_\lambda^2 L_1\left(\frac{\partial \bar{\psi}^{(j)}}{\partial t}\right) + (\Lambda_j^{(2)} - A_0^{(q)})\frac{\partial \bar{\psi}^{(j)}}{\partial t}$$
$$+ i\alpha\left[\delta_\lambda \Lambda_j^{(1)} L_1(\bar{\psi}^{(j)}) + \frac{1}{\delta_\lambda}\Lambda_j^{(3)}\bar{\psi}^{(j)}\right]$$
$$+ 2i\delta_\lambda(\tilde{\alpha} + \omega)\Lambda_j^{(1)}\sin^2\theta\bar{\psi}^{(j)} = \delta_\lambda^2\sin^2\theta\bar{F}^{(j)}, \quad (2.23)$$

instead of (2.22).

To simplify the notations we shall remove (j) from functions $\bar{\psi}^{(j)}$ and $\bar{F}^{(j)}$. Then we assume that

$$\left.\begin{aligned} \partial\bar{\psi} &= \frac{1}{\delta_t}(\bar{\psi}^1 - \bar{\psi}^0), \\ \bar{\psi} &= \gamma_1\bar{\psi}^1 + (1-\gamma_1)\bar{\psi}^0, \\ \bar{F} &= \gamma_2\bar{F}^1 + (1-\gamma_2)\bar{F}^0, \end{aligned}\right\} \quad (0 \leq \gamma_1,\ \gamma_2 \leq 1) \quad (2.24)$$

Then, instead of (2.23) we obtain the following differeniial equation:[1]

[1] Operator becomes singular at $\theta = 0$. According to KELDISH's (1951) investigation, however, eq. (2.25) will have the only and bounded solution at $\theta = 0$. Thus, the boundary conditions for the function ψ' give the unique solution of the boundary problem for eq. (2.25).

$$L_2(\bar{\psi}^1) = Z^{(1)}(\theta)\sin\theta\frac{d}{d\theta}\sin\theta\frac{d\bar{\psi}^1}{d\theta} + Z^{(2)}(\theta)\bar{\psi}^1 = Q(\theta), \tag{2.25}$$

where

$$Z^{(1)} = Z^{(11)} + i\gamma_1 Z^{(12)}(\theta) = \frac{\delta_\lambda^2}{\delta_t} + i\gamma_1\alpha(\theta, Z, 0)\,\delta_\lambda\Lambda_j^{(1)},$$

$$Z^{(2)} = Z^{(21)} + i\gamma_1 Z^{(22)}(\theta)$$

$$= \frac{1}{\delta_t}(\Lambda_j^{(2)} - A_0^{(q)}) + i\gamma_1(\Lambda_j^{(3)}\alpha(\theta, Z, 0)$$

$$+ 2\delta_\lambda(\tilde{\alpha} + \omega)\,\Lambda_j^{(1)}\sin^2\theta),$$

$$Q(\theta) = Z^{(3)}(\theta)\sin\theta\frac{d}{d\theta}\sin\theta\frac{d\bar{\psi}^0}{d\theta}$$

$$+ Z^{(4)}(\theta)\,\bar{\psi}^0 + \gamma_2\bar{F}^1 + (1-\gamma_2)\,\bar{F}^0$$

$$= Q^{(1)}(\theta) + iQ^{(2)}(\theta),$$

$$Z^{(3)}(\theta) = Z^{(11)} - i(1-\gamma_1)\,Z^{(12)}(\theta),$$

$$Z^{(4)}(\theta) = Z^{(21)} - i(1-\gamma_2)\,Z^{(22)}(\theta).$$

Now we shall solve eq. (2.25) also by means of a finite-difference method. For this purpose we shall divide the interval $(O, \pi/2)$ into segments $\delta_\theta = \theta_{m+1} - \theta_m = \pi/2\,(N+1)$, by means of points $\theta_m = \pi m/[2(N+1)]$ $(m = 1, 2, \ldots, N)$. Then it will be possible to show that the differential operator in (2.25) may be represented as

$$\delta_\theta^2 \sin\theta\frac{d}{d\theta}\sin\theta\frac{d\bar{\psi}^1}{d\theta}\bigg|_{\theta=\theta_m}$$

$$= G_m\bar{\psi}^1_{m-1} - K_m\bar{\psi}^1 + H_m\bar{\psi}^1_{m+1} + O(\delta_\theta^4), \tag{2.26}$$

where $\bar{\psi}^1(\theta_m) = \bar{\psi}^1_m,$

$$G_m = \sin^2\theta_m - \frac{\delta_\theta}{2}\sin 2\theta_m,$$

$$H_m = \sin^2\theta_m + \frac{\delta_\theta}{2}\sin 2\theta_m,$$

$$K_m = 2\sin^2\theta_m.$$

Note, that

$$G_1 = \sin^2\delta_\theta - \frac{\delta_\theta}{2}\sin 2\delta_\theta = \tfrac{1}{3}\delta_\theta^4 + \ldots = O(\delta_\theta^4). \tag{2.27}$$

So, as the values $O(\delta_\theta^4)$ and of the higher order are neglected in this problem, we assume that

$$G_1 \equiv 0. \tag{2.28}$$

Inserting for the operator in (2.25) the expression from (2.26) we obtain the following finite-differential equation relative to functions $\bar{\psi}^1_m = \psi_m^{1(c)} - i\psi_m^{1(s)}$

$$\bar{G}_m\bar{\psi}^1_{m-1} - \bar{K}_m\bar{\psi}^1_m + \bar{H}_m\bar{\psi}^1_{m+1} = Q_m^{(1)} + iQ_m^{(2)}, \tag{2.29}$$

where

$$\bar{G}_m = G_m^{(1)} + iG_m^{(2)}, \quad \bar{K}_m = K_m^{(1)} + iK_m^{(2)},$$

$$\bar{H} = H_m^{(1)} + iH_m^{(2)},$$

$$Q_m^{(1)} = \delta_\theta^2 Q^{(1)}(\theta_m), \quad G_m^{(1)} = G_m Z^{(11)},$$

$$Q_m^{(2)} = \delta_\theta^2 Q^{(2)}(\theta_m), \quad G_m^{(2)} = G_m Z^{12}(\theta_m),$$

$$K_m^{(1)} = K_m Z^{(11)} - \delta_\theta^2 Z^{(21)}(\theta_m), \quad H_m^{(1)} = H_m Z^{(11)},$$

$$K_m^{(2)} = \gamma_1(K_m Z^{(12)}(\theta_m) - \delta_\theta^2 Z^{(22)}(\theta_m)),$$

$$H_m^{(2)} = H_m Z^{(12)}(\theta_m).$$

Eq. (2.29) may be solved by means of the recurrence formulae (Godunov & Rjabenjkii, 1962) system as follows:

$$\bar{\beta}_{m+1} = \frac{\bar{G}_{m+1}\bar{H}_m}{\bar{K}_m - \bar{\beta}_m}, \quad \bar{\gamma}_{m+1} = \frac{\bar{\beta}_{m+1}}{\bar{H}_m}(\bar{\gamma} - Q_m^{(1)} - iQ_m^{(2)}),$$

$$\bar{\psi}^1_m = \frac{1}{\bar{G}_{m+1}}(\bar{\beta}_{m+1}\bar{\psi}^1_{m+1} + \bar{\gamma}_{m+1}) \quad (m = 1, 2, \ldots, N) \tag{2.30}$$

at the following initial values of functions $\bar{\beta}_1$, $\bar{\gamma}_1$ and $\bar{\psi}^1_{N+1}$:

$$\bar{\beta}_1 = 0; \quad \bar{\gamma}_1 = 0; \quad \bar{\psi}^1_{N+1} = 0. \tag{2.31}$$

The conditions of (2.31) are the result of (2.28) and of boundary conditions for the unknown function ψ'.

As $\bar{G}_m$, $\bar{K}_m$, $\bar{H}_m$, and $\bar{\psi}^1_m$ are complex, then according to (2.30) $\bar{\beta}_m$ and $\bar{\psi}_m$ will be complex too. Thus, in order that we can use the formulae (2.30) immediately, they should be expressed separately for real and imaginary parts of functions $\bar{\beta}_m$, $\bar{\gamma}_m$ and $\bar{\psi}^1_m$. Let

$$\left.\begin{aligned}\bar{\beta}_m &= \beta_m^{(1)} + i\beta_m^{(2)},\\ \bar{\gamma}_m &= \gamma_m^{(1)} + i\gamma_m^{(2)}\end{aligned}\right\} \tag{2.32}$$

and, as $\bar{\psi}^1_m = \psi_m^{1(c)} - i\psi_m^{1(s)}$, then probably instead of (2.30) we shall have the following system of recurrence formulae

$$\left.\begin{aligned}\beta^{(1)}_{m+1} &= \frac{1}{\Delta^{(1)}_m}[\xi_m \mu_{(1)m} + \zeta_m \mu_{(2)m}];\\ \beta^{(2)}_{m+1} &= \frac{1}{\Delta^{(1)}_m}[-\xi_m \mu_{(2)m} + \zeta_m \mu_{(1)m}]\end{aligned}\right. \quad (2.33)$$

at

$$\beta^{(1)}_1 = \beta^{(2)}_1 \equiv 0,$$

where

$$\xi_m = G^{(1)}_m H^{(1)}_m - G^{(2)}_m H^{(2)}_m,$$

$$\zeta_m = G^{(1)}_m H^{(2)}_m + G^{(2)}_m H^{(1)}_m,$$

$$\mu_{(1,2)m} = K^{(1,2)}_m - \beta^{(1,2)}_m;$$

$$\Delta^{(1)}_m = \mu^2_{(1)m} + \mu^2_{(1)m},$$

$$\gamma^{(1)}_{m+1} = \frac{1}{\Delta^{(2)}_m}[\nu_m(\gamma^{(1)}_m - Q^{(1)}_m) - \tau_m(\gamma^{(2)}_m - Q^{(2)}_m)];$$

$$\gamma^{(2)}_{m+1} = \frac{1}{\Delta^{(2)}_m}[\nu_m(\gamma^{(2)}_m - Q^{(2)}_m) + \tau_m(\gamma^{(1)}_m - Q^{(1)}_m)], \quad (2.34)$$

at

$$\gamma^{(1)}_1 = \gamma^{(2)}_1 = 0,$$

where

$$\nu_m = \beta^{(1)}_{m+1} H^{(1)}_m + \beta^{(2)}_{m+1} H^{(2)}_m;$$

$$\tau_m = -\beta^{(1)}_{m+1} H^{(2)}_m + \beta^{(2)}_{m+1} H^{(1)}_m;$$

$$\Delta^{(2)}_m = [H^{(1)}_m]^2 + [H^{(2)}_m]^2;$$

$$\psi^{1(c)}_m = \frac{1}{\Delta^{(3)}_m}[\sigma_m G^{(1)}_{m+1} + \varrho_m G^{(2)}_{m+1}];$$

$$\psi^{1(s)}_m = \frac{1}{\Delta^{(3)}_m}[-\sigma_m G^{(2)}_{m+1} + \varrho_m G^{(1)}_{m+1}] \quad (2.35)$$

at

$$\psi^{1(c)}_{N+1} = \psi^{1(s)}_{N+1} \equiv 0,$$

where

$$\sigma_m = \beta^{(1)}_{m+1}\psi^{1(c)}_{m+1} + \beta^{(2)}_{m+1}\psi^{1(s)}_{m+1} + \gamma^{(1)}_{m+1};$$

$$\varrho_m = -\beta^{(1)}_{m+1}\psi^{1(s)}_{m+1} + \beta^{(2)}_{m+1}\psi^{1(c)}_{m+1} + \gamma^{(2)}_{m+1},$$

$$\Delta^{(3)}_m = [G^{(1)}_{m+1}]^2 + [G^{(2)}_{m+1}]^2.$$

System (2.33)–(2.35) must be solved n times depending upon the number of different values $\Lambda^{(1)}_j$, $\Delta^{(2)}_j$ and $\Lambda^{(3)}_j$ $(j = 1, 2, \ldots, n)$. As a result of this solution we shall have two sets of values for every index m:

$$\left.\begin{aligned}&\psi^{1(c)}_{m,j} \quad (j = 1, 2, \ldots, n;\ m = 1, 2, \ldots, N),\\ &\psi^{1(s)}_{m,j} \quad (j = 1, 2, \ldots, n;\ m = 1, 2, \ldots, N).\end{aligned}\right\} \quad (2.36)$$

From this we may construct two vectors, fixing index m every time:

$$\Psi^{1(c)}_m = \Psi^{1(c)}(\theta_m, z, \delta_t) = \left\| \begin{matrix} \psi^{1(c)}_{m,1} \\ \psi^{1(c)}_{m,2} \\ \vdots \\ \psi^{1(c)}_{m,n} \end{matrix} \right\|,$$

$$\Psi^{1(s)}_m = \Psi^{1(s)}(\theta_m, z, \delta_t) = \left\| \begin{matrix} \psi^{1(s)}_{m,1} \\ \psi^{1(s)}_{m,2} \\ \vdots \\ \psi^{1(s)}_{m,n} \end{matrix} \right\|$$

$$(m = 1, 2, \ldots, N).$$

But according to (2.21)

$$\Psi^{1(c)}(\theta_m, z, \delta_t) = W_{(c)}\Psi(\theta_m, z, \delta_t);$$

$$\Psi^{1(s)}(\theta_m, z, \delta_t) = W_{(s)}\Psi(\theta_m, z, \delta_t).$$

Then, multiplying matrix $\Psi^{1(c)}_m$ by $W^{(c)}$ from the left, and matrix $\Psi^{1(s)}_m$ by $W_{(s)}$, we obtain

$$W_{(c)}\Psi^{1(c)}_m + W_{(s)}\Psi^{1(s)}_m = W^2_{(c)}\Psi(\theta_m, z, \delta_t) + W^2_{(s)}\Psi(\theta_m, z, \delta_t). \quad (2.37)$$

But according to (2.14)

$$W^2_{(c)} + W^2_{(s)} = E$$

then

$$\Psi(\theta_m, z, \delta_t) = W_{(c)}\Psi^{1(c)}_m + W_{(s)}\Psi^{1(s)}_m. \quad (2.38)$$

After the multiplication of matrices is accomplished at the right-hand of equality (2.38), we obtain the components of matrix $\Psi(\theta_m, z, \delta_t)$:

$$\psi(\theta_m, \lambda_j, z, \delta_t) \quad (m = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n), \quad (2.39)$$

which are the unknown solutions of the problem concerning the determination of the stream function ψ' at the given descrete grid of points (θ_m, λ_j) $(m = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n)$. The procedure of solving the problem being discussed here, is a direct method of realization of the finite-differential analogue of the boundary problem according to the determination of ψ', it includes a relatively small number of necessary arithmetic operations and it is stable to the round-off errors.

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