The exact solution of the Rossby adjustment problem

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ABSTRACT

The exact solution of the ROSSBY (1938) adjustment problem is obtained. If a is the half-width of the initial current and λ is the radius of deformation, it is shown that Rossby's solution is asymptotically valid for $a/\lambda \ll 1$. We examine the asymptotic solution for $a/\lambda \gg 1$. The exact energetic relations are obtained. The numerical examples of Rossby are recalculated with the exact solution.

1. Introduction

ROSSBY (1938) examined the following problem. Suppose on a rotating earth a uniform current exists initially in a limited region of a homogeneous incompressible ocean of constant depth. Initially there are no horizontal pressure gradients. Owing to the Coriolis force the current undergoes an oscillation about an equilibrium position. During these oscillations a train of gravity–inertia waves are emitted transverse to the current. Ultimately the flow comes into geostrophic equilibrium. Rossby solved approximately the problem of determining the ultimate velocity field and shape of the free surface. He found that there is less total energy in the ultimate geostrophic flow than existed initially.

CAHN (1945) examined the transient problem by linearizing the governing equations. He found that equilibrium is approximately established after a few oscillations of the current.

BOLIN (1953) examined the transient problem in a stratified incompressible ocean. Vertical variations of density and current speed were considered. Apparently, certain distributions of density and vertical shear lead to a prolonged adjustment process.

MONIN and OBUKHOV (1959) have considered the adjustment process in a baroclinic atmosphere.

The significance of the adjustment problem has been very clearly stated by BOLIN (1953) and need not be restated here.

Returning to the Rossby problem, physical interest often centers on cases where the width of the current is large or the speed of gravity waves is small. However, Rossby's solution is not valid for such physical problems. If a is the half-width of the current and λ is the radius of deformation (to be defined in section 2), we shall show that Rossby's solution is valid for $a/\lambda \ll 1$.

In this paper we shall obtain the exact solution of the Rossby adjustment problem. Two asymptotic solutions will be obtained: (a) $a/\lambda \ll 1$ and (b) $a/\lambda \gg 1$. The exact energetic relations will be obtained. Finally, the numerical examples of Rossby will be recalculated using the exact solution.

We remark that our exact solution is considerably easier to obtain than Rossby's approximate solution.

2. Formulation of the Rossby adjustment problem

We consider a homogeneous incompressible ocean. In Fig. 1a, x increases into the paper and y increases to the left. The origin of y is not specified at this time. Initially, the ocean is of constant depth D_0 and there is a uniform current of speed u_0 into the paper. The current has a width $2a$. The current is flanked on two sides by a quiescent ocean. Owing to the Coriolis force the current will undergo an oscillation, but ultimately geostrophic equilibrium will be established. The problem is to determine the ultimate distribution of u and D, where u is the velocity into the paper and D is the total depth of the ocean.

We assume no variations of u or D in the x-direction. Then the x-equation of motion is

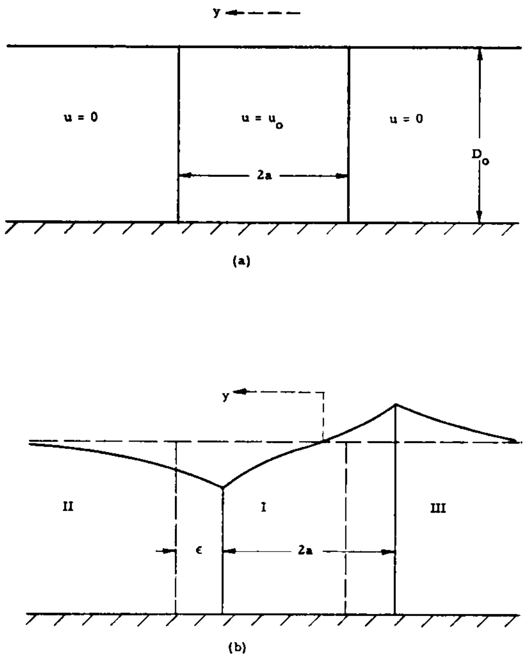

FIG. 1. The initial geometry is depicted schematically in (a). The ultimate balanced state is depicted in (b).

$$du/dt = fv$$

or

$$u - u_0 = f(y - y_0) \tag{2.1}$$

by simple integration. u_0 is the velocity of a particle when at its initial position y_0 and f is the Coriolis parameter assumed constant.

The flow is ultimately geostrophic so that

$$u = -\frac{g}{f}\frac{dD}{dy}, \tag{2.2}$$

where g is the acceleration of gravity.

Finally, the condition of mass conservation is

$$D = D_0 \frac{dy_0}{dy}. \tag{2.3}$$

Using (2.3), (2.2) becomes

$$u = -\frac{c^2}{f}\frac{d^2y_0}{dy^2}, \tag{2.4}$$

where $$c = \sqrt{gD_0}.$$

c is the speed of infinitesimal long waves in an ocean of depth D_0.

Entering (2.4) into (2.1) we obtain

$$\frac{d^2y_0}{dy^2} - \frac{y_0}{\lambda^2} = -\frac{y}{\lambda^2} - \frac{u_0(y)}{f\lambda^2}, \tag{2.5}$$

where $$\lambda = c/f.$$

λ is Rossby's "radius of deformation". Rossby did not use equation (2.5) as it stands.

3. Exact solution

In Fig. 1b the current has shifted a distance ε to the right. We distinguish three regions. In region I, $u_0(y)$ is a constant whilst in regions II and III, $u_0(y)$ is zero (see Fig. 1b). The solution of (2.5) in II and III is

$$\text{(II)} \quad y_0 - y = Ce^{-y/\lambda} + Fe^{y/\lambda},$$
$$\text{(III)} \quad y_0 - y = Ee^{y/\lambda} + Ge^{-y/\lambda},$$

where C, E, F, and G are undetermined constants.

We require that $y_0 - y$ tend to zero as y tends to $\pm\infty$. Thus in II, $F = 0$ and in III, $G = 0$. Adding the solution in I we obtain

$$\left.\begin{aligned} &\text{(I)} \quad y_0 - y = Ae^{y/\lambda} + Be^{-y/\lambda} + u_0/f, \\ &\text{(II)} \quad y_0 - y = Ce^{-y/\lambda}, \\ &\text{(III)} \quad y_0 - y = Ee^{y/\lambda}, \end{aligned}\right\} \tag{3.1}$$

where A and B are undetermined constants.

Using (2.3) we obtain the tentative solution for D:

$$\left.\begin{aligned} &\text{(I)} \quad D = D_0\left[1 + \frac{A}{\lambda}e^{y/\lambda} - \frac{B}{\lambda}e^{-y/\lambda}\right], \\ &\text{(II)} \quad D = D_0\left[1 - \frac{C}{\lambda}e^{-y/\lambda}\right], \\ &\text{(III)} \quad D = D_0\left[1 + \frac{E}{\lambda}e^{y/\lambda}\right]. \end{aligned}\right\} \tag{3.2}$$

We now establish the origin of y at the point in I where $D = D_0$. Thus $B = A$. From the symmetry of the problem it is clear that $E = C$. Also the boundaries of I are clearly at $y = \pm a$, as can be seen from Fig. 1b. We now determine A and C by requiring continuity of $y_0 - y$ and D at $y = a$. From (3.1) and (3.2) we obtain

$$\left.\begin{aligned} A = B &= -\frac{1}{2}\frac{u_0}{f}e^{-a/\lambda}, \\ C = E &= \frac{u_0}{f}\sinh a/\lambda. \end{aligned}\right\} \tag{3.3}$$

The final solution

$$\left.\begin{aligned} &(\mathrm{I}) \quad y_0 - y = \frac{u_0}{f}[1 - e^{-a/\lambda}\cosh y/\lambda], \\ &(\mathrm{II}) \quad y_0 - y = \frac{u_0}{f} e^{-y/\lambda}\sinh a/\lambda, \\ &(\mathrm{III}) \quad y_0 - y = \frac{u_0}{f} e^{y/\lambda}\sinh a/\lambda. \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} &(\mathrm{I}) \quad D = D_0 - \frac{u_0 D_0}{f\lambda} e^{-a/\lambda}\sinh y/\lambda, \\ &(\mathrm{II}) \quad D = D_0 - \frac{u_0 D_0}{f\lambda} e^{-y/\lambda}\sinh a/\lambda, \\ &(\mathrm{III}) \quad D = D_0 + \frac{u_0 D_0}{f\lambda} e^{y/\lambda}\sinh a/\lambda. \end{aligned}\right\} \tag{3.5}$$

We may use (2.2) to compute u. Thus

$$\left.\begin{aligned} &(\mathrm{I}) \quad u = u_0 e^{-a/\lambda}\cosh y/\lambda, \\ &(\mathrm{II}) \quad u = -u_0 e^{-y/\lambda}\sinh a/\lambda, \\ &(\mathrm{III}) \quad u = -u_0 e^{y/\lambda}\sinh a/\lambda. \end{aligned}\right\} \tag{3.6}$$

From (3.4) we may calculate ε since

$$\varepsilon = [y_0 - y]_{y=a},$$

whence

$$\varepsilon = \frac{1}{2}\frac{u_0}{f}[1 - e^{-2a/\lambda}]. \tag{3.7}$$

Now Rossby solved (2.5) in II and III but not in I. He assumed that u is constant in I. An arbitrary constant of integration he determined by requiring conservation of the absolute momentum in I. We shall investigate the consequences of his assumption in section 5.

4. The energetics

Let T be the total kinetic energy and V be the total gravitational potential energy. Then

$$\left.\begin{aligned} T &\equiv \int \tfrac{1}{2}\varrho u^2 dv, \\ V &\equiv \int \varrho g z dv, \end{aligned}\right\} \tag{4.1}$$

where ϱ is the density; dv, an infinitesimal fluid volume; and z, a vertical coordinate. Now the potential energy is defined to within an arbitrary constant. It is convenient to set $V = 0$ when $D = D_0$ everywhere. Performing the z integrations in (4.1) we obtain

$$\left.\begin{aligned} T &= \tfrac{1}{2}\varrho \int_{-\infty}^{\infty} u^2 D dy, \\ V &= \tfrac{1}{2}\varrho g \int_{-\infty}^{\infty} \eta^2 dy, \end{aligned}\right\} \tag{4.2}$$

where $\eta \equiv D - D_0$. From (4.2) it can be seen that the initial values of T and V are

$$T_0 = \varrho D_0 a u_0^2,$$

$$V_0 = 0.$$

Using (3.5) and (3.6) the integrals in (4.2) are easily evaluated. The results are

$$\left.\begin{aligned} T &= T_0\left[\tfrac{1}{2} e^{-2a/\lambda}\left(1 + \frac{\lambda}{2a}\sinh\frac{2a}{\lambda} + \frac{\lambda}{a}\sinh^2\frac{a}{\lambda}\right)\right] \\ V &= T_0\left[\tfrac{1}{2} e^{-2a/\lambda}\left(-1 + \frac{\lambda}{2a}\sinh\frac{2a}{\lambda} + \frac{\lambda}{a}\sinh^2\frac{a}{\lambda}\right)\right] \end{aligned}\right\} \tag{4.3}$$

The total energy $\mathcal{E} \equiv T + V$ is

$$\mathcal{E} = T_0\left[e^{-2a/\lambda}\left(\frac{\lambda}{2a}\sinh\frac{2a}{\lambda} + \frac{\lambda}{a}\sinh^2\frac{a}{\lambda}\right)\right] \tag{4.4}$$

5. Asymptotic solutions

We consider two asymptotic solutions:

(*a*) $a/\lambda < 1$. We expand our solution in a Mclaurin series in a/λ. In I we have

$$u_{\mathrm{I}} = u_0\left[1 - \frac{a}{\lambda} + \frac{1}{2}\frac{a^2}{\lambda^2}\left(1 + \frac{y^2}{a^2}\right)\right] + 0\left(\frac{a}{\lambda}\right)^3, \tag{5.1}$$

where $0(a/\lambda)^3$ means the remaining terms are of the order $(a/\lambda)^3$. The maximum counter currents are at $y = a^+$ and $y = -a^-$,

$$u(a^+) = u(-a^-) = -u_0\frac{a}{\lambda}\left[1 - \frac{a}{\lambda}\right] + 0\left(\frac{a}{\lambda}\right)^3. \tag{5.2}$$

The shift of the current is

$$\varepsilon = \frac{u_0}{f}\frac{a}{\lambda}\left[1 - \frac{a}{\lambda}\right] + 0\left(\frac{a}{\lambda}\right)^3. \tag{5.3}$$

The total energy is

$$\mathcal{E} = T_0\left[1 - \frac{a}{\lambda} + \frac{2}{3}\left(\frac{a}{\lambda}\right)^2\right] + 0\left(\frac{a}{\lambda}\right)^3. \tag{5.4}$$

Now Rossby assumed that u_{I} does not vary with y. But (3.6) shows that u_{I} varies as cosh y/λ. From (5.1) we see that the y variation is associated with terms of the second order in a/λ. Thus, we conclude that Rossby's results are correct only through terms of the first order in a/λ. His solution is thus applicable only for small values of a/λ.

(*b*) $a/\lambda \gg 1$. This approximation is based on

$$\left.\begin{aligned}\cosh\frac{a}{\lambda} &= \tfrac{1}{2}e^{a/\lambda}\\ \sinh\frac{a}{\lambda} &= \tfrac{1}{2}e^{a/\lambda}\end{aligned}\right\} \; a/\lambda \gg 1,$$

whence

$$\left.\begin{aligned}u(0) &= u_0 e^{-a/\lambda},\\ u(a^-) &= u(-a^+) = \tfrac{1}{2}u_0,\end{aligned}\right\} \tag{5.5}$$

$$u(a^+) = u(-a^-) = -\tfrac{1}{2}u_0, \tag{5.6}$$

$$\varepsilon = \frac{1}{2}\frac{u_0}{f}, \tag{5.7}$$

and

$$\mathcal{E} = \tfrac{1}{2}T_0(a/\lambda)^{-1}. \tag{5.8}$$

The asymptotic solution for $a/\lambda \gg 1$ cannot be correctly derived from Rossby's solution since his solution is valid only for small a/λ.

In order to clearly display the error in Rossby' solution, we have calculated the variation of ε versus a/λ. The results are displayed in Fig. 2. Rossby's solution is indicated by ε_R and is calculated from

$$\varepsilon_R = \frac{u_0}{f}\,\frac{1}{1 + \dfrac{\lambda}{a} + \dfrac{a}{3\lambda}}$$

(see Rossby, 1938) The behavior of Rossby's solution is incorrect when $a/\lambda > 1.75$. The Rossby error is 10 per cent at $a/\lambda = 2.50$. The correct solution very rapidly approaches its asymptotic value. For $a/\lambda > 3$, ε is essentially its asymptotic value, $\frac{1}{2}u_0/f$. Note that for large a/λ, ε_R is asumptotic to

$$\frac{3u_0}{f}(a/\lambda)^{-1},$$

which is erroneous.

6. Modification for a two-layer ocean

Fig. 3 indicates schematically the geometry of an initial simple current system in a two-layer ocean. The upper layer has a density ϱ and thickness D_0^* while the lower layer has a density ϱ' and thickness D_0'. A uniform current is confined to the upper layer and is of width $2a$.

Rossby assumed that $D_0' \gg D_0^*$ so that to

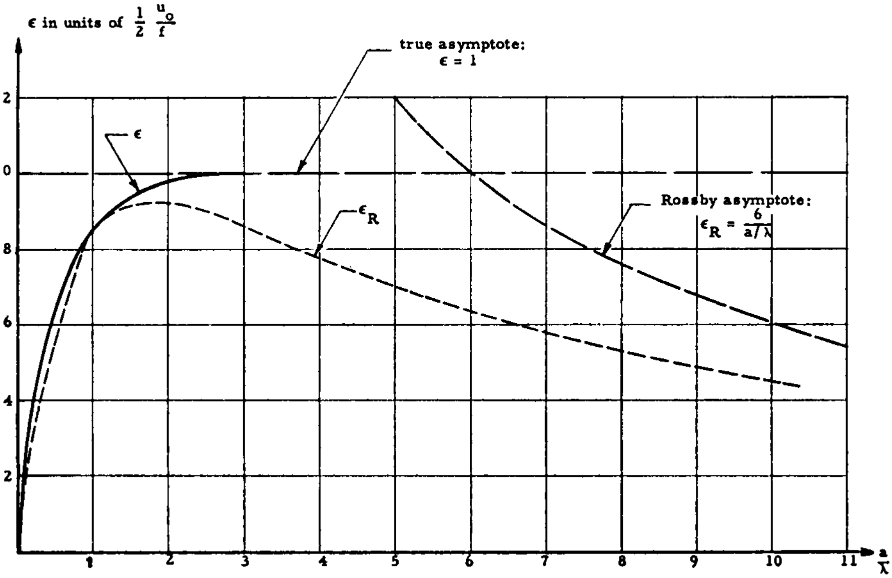

Fig. 2. Graphs of ε and ε_R vs a/λ. Note the large errors in ε_R when a/λ is large.

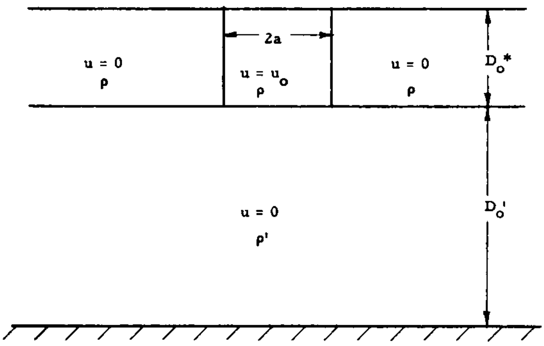

FIG. 3. Schematic picture of the initial geometry in a two-layer ocean.

first approximation the lower layer can be treated as inert. Define

$$g^* \equiv \frac{\varrho' - \varrho}{\varrho'} g,$$

$$\lambda^* \equiv \sqrt{g^* D_0^*}/f.$$

Then our previous results apply provided we replace g by g^*, λ with λ^*, D_0 with D_0^*, and D with D^*, where D^* is the ultimate thickness of the upper layer. The energetics are similarly treated if we replace η by η^*, where

$$\eta^* \equiv D^* - D_0^*.$$

7. Numerical examples

(*a*) *Single-layer ocean.* Rossby took

$D_0 = 2$ km,	Then
$g = 9.8$ msec^{-2},	$c = 140$ msec^{-1},
$f = 1.03 \times 10^{-4}$ sec^{-1},	$\lambda = 1360$ km,
$a = 100$ km,	$a/\lambda = 0.0735$.

Taking only the first order terms into account in section 5, sec. (*a*), we have

$$u_{\mathrm{I}} = 0.93\, u_0, \qquad \varepsilon = 0.07\, u_0/f,$$

$$u(a^+) = u(-a^-) = -0.07\, u_0, \qquad \mathcal{E} = 0.93\, T_0$$

From the second order terms we see that the error in u_{I}, $u(a^+)$, and ε is about 7 per cent while the error in $\mathcal{E}$ is about 4 per cent. These errors may be considered tolerable.

(*b*) *Two-layer ocean.* Here Rossby took

$$(\varrho' - \varrho)/\varrho' = 0.002, \quad D_0^* = 400 \text{ m}, \quad a = 100 \text{ km}.$$

Then $\quad \lambda^* = 27.2$ km, $\quad a/\lambda^* = 3.68$.

Here Rossby's results are erroneous. The correct numerical results are

$$\begin{aligned} u(0) &= 0.025\, u_0, & \varepsilon &= \tfrac{1}{2} u_0/f, \\ u(a^-) &= u(-a^+) = 0.505\, u_0, & \mathcal{E} &= 0.136\, T_0. \\ u(a^+) &= u(-a^-) = -0.495\, u_0, & & \end{aligned}$$

8. Conclusions

We have derived the exact solution of the Rossby adjustment problem and shown that Rossby's solution is valid only when $a/\lambda < 1$. We have obtained the exact energetic relations. Two asymptotic solutions have been obtained: (*a*) $a/\lambda < 1$ and (*b*) $a/\lambda \gg 1$. The numerical examples of Rossby have been recalculated and it has been shown that his errors for the single-layer ocean are probably tolerable, while his results for the two-layer ocean are definitely erroneous.

REFERENCES

BOLIN, B., 1953, The adjustment of a non-balanced velocity field towards geostrophic equilibrium in a stratified fluid. *Tellus*, **5**, pp. 373–385.

CAHN, A., 1945, An investigation of the free oscillations of a simple current system. *J. Meteor.*, **2**, pp. 113–119.

MONIN, A. S., and OBUKHOV, A. M., 1959, A note on general classification of motions in a baroclinic atmosphere. *Tellus*, **11**, pp. 159–162.

ROSSBY, C. G., 1938, On the mutual adjustment of pressure and velocity distribution in certain simple current systems, II. *J. Marine Res.*, **1**, pp. 239–263.