Baroclinic instability in two layer systems

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ABSTRACT

The stability properties of the two layer baroclinic model with arbitrary horizontal shear are discussed. Necessary conditions for instability are derived and show that the potential vorticity gradient must be both positive and negative. Bounds are placed on the propagation speed and growth rate of unstable waves by showing that the complex phase speed must lie in a certain semi-circle in the complex velocity plane. Sufficient conditions for instability are also discussed for a certain class of flows.

Introduction

In the dynamics of atmospheric and oceanic systems an important role is played by the release of energy by dynamically unstable mean flow systems. What is usually meant by such mean systems is a flow with negligible zonal variation. The perturbations which grow from such unstable systems have fundamentally two-energy sources, the horizontal (north-south) shear of the mean motion, and the available potential energy associated with a north-south density gradient and a vertical shear.

One of the simplest physically reproducible models that contains these elements is the two-layer system shown here in Fig. 1. The flow is taken to be gravitationally stable ($\varrho_1 < \varrho_2$) and the pressure is determined hydrostatically while ϱ is constant in each layer.

$$p_1 = \varrho_1 g(h - z) \quad (h > z > h_2), \tag{1a}$$

$$p_2 = \varrho_2(h_2 - z) g + \varrho_1 g h_1 \quad (h_2 > z > 0). \tag{1b}$$

Consider a situation where the flow in the system is only in the x direction and is a function of y only and is geostrophically balanced.

$$\varrho_1 f u_1 = -\frac{\partial p_1}{\partial y} = -\varrho_1 g \frac{\partial h}{\partial y}, \tag{2a}$$

$$u_1 - = \frac{g}{f}\frac{\partial h}{\partial y}. \tag{2b}$$

If $u_2 = 0$,

$$\frac{\partial p_2}{\partial y} = 0 = \varrho_1 g \frac{\partial h_1}{\partial y} + \varrho_2 g \frac{\partial h_2}{\partial y} = \varrho_1 g \frac{\partial h}{\partial y} + \frac{\varrho_2 - \varrho_1}{\varrho_2}\varrho_2 g \frac{\partial h_2}{\partial y}, \tag{3a}$$

$$\frac{\partial h_2}{\partial y} = -\frac{\varrho_1}{\varrho_2 - \varrho_1}\frac{\partial h}{\partial y} = +\frac{\varrho_1}{\Delta\varrho\, g} f\, u_1. \tag{3b}$$

In the usual case where $\Delta\varrho/\varrho << 1$ we see that a vertical shear produces a large tilt of the interface between the two layers and an energy source for the disturbance. The lines of constant pressure and density are not parallel and the flow is baroclinic.

1. Equations of motion

One can show that for a layer system the equations of motion in each layer are the conservation of potential vorticity and the conservation of volume.

$$\frac{D}{Dt}\frac{(\xi + f)}{h} = 0, \tag{1.1a}$$

$$\frac{D}{Dt}(h) + h \operatorname{div} \bar{u} = 0, \tag{1.1b}$$

where h is the layer thickness, $\xi = v_x - u_y$, and D/Dt is the two-dimensional substantial derivative.

The system becomes closed if we assume that

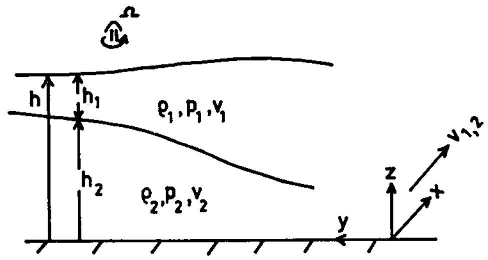

FIG. 1. The two-layer baroclinic flow system.

the frequency scale of the motion U/L (U is a characteristic velocity and L a characteristic length) is small compared to the earth rotation frequency $f/2$. In such systems the flow is geostrophically balanced during the motion and

$$\xi = v_x - u_y = \frac{p_{xx}}{\varrho f} + \frac{p_{yy}}{\varrho f}, \tag{1.2}$$

this assumption with the assumption of hydrostatic vertical balance closes the system.

For the mean state whose stability we wish to investigate we take: $\bar{u}_1 = U_1(y)$, $\bar{u}_2 = U_2(y)$, $\bar{v}_2 = \bar{v}_1 = 0$. We superimpose small perturbations on the system of the form (h is total height):

$$\left.\begin{aligned} h &= h(y)\, e^{ik(x-ct)}, \\ h_1 &= h_1(y)\, e^{ik(x-ct)} \quad \text{etc.} \end{aligned}\right\} \tag{1.3}$$

In these expressions c is in general a complex number and the linearized perturbation equations corresponding to (1.1) are

$$(U_1 - c)(D^2 - k^2)p_1 + \left(\beta - U_{1yy} - f\frac{H_{1y}}{H_1}\right)p_1 = \frac{f^2}{g'H_1}(U_1 - c)(p_1 - p_2), \tag{1.4 a}$$

$$(U_2 - c)(D^2 - k^2)p_2 + \left(\beta - U_{2yy} - f\frac{H_{2y}}{H_1}\right)p_2 = \frac{f^2}{g'H_2}(U_2 - c)(p_1 - p_2), \tag{1.4 b}$$

where $$h_1 = \frac{p_1 - p_2}{p_2 g'},$$

$$h = \frac{p_1}{\varrho g}, \quad D \equiv d/dy, \quad g' = \Delta\varrho/\varrho g$$

and we have assumed $\Delta\varrho/\varrho < 1$.

If we scale the variables in the following manner:

$$(x, y) = (x', y')L \quad (k = \alpha/L),$$

$$U'_{(1.2)} = U_0 U'_{(1.2)} \quad (c = U_0 c'),$$

$$H_y = \frac{U_0 f}{g'} H'_{2y}.$$

The non-dimensional equations are (after dropping the primes)

$$(D^2 - \alpha^2)p_1 + (\beta/R_0 - U_{1yy} - F_1 H_{1y})\frac{p_1}{(U_1 - c)} - F_1 p_1 = -F_1 p_2, \tag{1.5 a}$$

$$(D^2 - \alpha^2)p_2 + (\beta/R_0 - U_{2yy} - F_2 H_{2y})\frac{p_2}{(U_2 - c)} - F_2 p_2 = -F_2 p_1, \tag{1.5 b}$$

$$F_{1.2} = \frac{f^2L^2}{g'H_{1.2}} = \frac{f^2L^2}{U_0^2}\frac{U_0^2}{g'H_{1.2}} = \left(\frac{1}{R_0}\right)^2 F'_{r\,1.2},$$

$$R_0 = \frac{U_0}{fL},$$

$$F_{r\,1.2} = U_0^2/g'H_{1.2}.$$

For convenience define the y derivative of the mean potential vorticity in the i_{th} layer as:

$$\frac{\partial q_i}{\partial y} \equiv (\beta/R_0 - U'_{1yy} - F_i H_{iy}).$$

The phase velocity c is in general complex, $c = c_r + ic_i$ (c_r and c_i are real) and the flow is unstable if $c_i > 0$.

2. Necessary conditions for instability

The perturbation equations can be written as

$$(D^2 - \alpha^2)p^1 + \frac{\partial q_1}{\partial y}\frac{p_1}{U_1 - c} - F_1 p_1 = -F_1 p_2, \tag{2.1 a}$$

$$(D^2 - \alpha^2)p_2 + \frac{\partial q_2}{\partial y}\frac{p_2}{U_2 - c} - F_1 p_2 = -F_2 p_1. \tag{2.1 b}$$

Multiply (2.1 a) by p_1^* (complex conjugate of

[1] If $F_{1,2}$ is of order unity, we see that the mean flow is subcritical in the hydraulic sense.

p_1) and substract from that the conjugated equation to obtain

$$\frac{d}{dy}\left(p_1^* \frac{dp_1}{dy} - p_1 \frac{dp_1^*}{dy}\right) + 2c_i \frac{\partial q_1}{\partial y} \frac{|p_1|^2}{|U_1 - c|^2}$$

$$= -F_1(p_2 p_1^* - p_1 p_2^*). \qquad (2.2\,\text{a})$$

Similarly for layer two

$$\frac{d}{dy}\left(p_2^* \frac{dp_2}{dy} - p_2 \frac{dp_2^*}{dy}\right) + 2c_i \frac{\partial q_2}{\partial y} \frac{|p_2|^2}{|U_2 - c|^2}$$

$$= -F_2(p_1 p_2^* - p_2 p_1^*). \qquad (2.2\,\text{b})$$

Integrate (2.2a and b) between two points y_1 and y_2 where p_1 and p_2 go to zero. We eliminate the coupling term on the right-hand side and obtain (where $c = c_r + ic_i$)

$$2c_i\left[\int_{y_1}^{y_2} \frac{\partial q_1}{F_1 \partial y} \frac{|p_1|^2 dy}{|U_1 - c|^2} + \frac{\partial q_2}{\partial y} \frac{|p_2|^2 dy}{F_2 |U_2 - c|^2}\right] = 0$$

or

$$\boxed{2c_i \int_{y_1}^{y_2} \sum_{n=1}^{2} H_n \frac{\partial q_n}{\partial y} \frac{|p_n|^2 dy}{|U_n - c|^2} = 0.} \qquad (2.3)$$

Therefore if $c_i \neq 0$ the potential vorticity gradient must be positive in some regions and negative in others for instability to occur. It is not true that it must be zero some place in the fluid since the flow is discontinuous. It is clear also that equation (2.3) can be generalized to any multiple layer model. In the simplest baroclinic model where horizontal shears are neglected as well as β

$$\frac{\partial q_1}{\partial y} = -F_1 H_{1y} > 0 \quad \text{and constant,}$$

$$\frac{\partial q_2}{\partial y} = -F_2 H_{2y} < 0 \quad \text{and constant.}$$

In the case where $\Delta\varrho/\varrho \ll 1$ $H_{1y} \approx -H_{2y}$ equation (2.3) tells us then that

$$\frac{-|p_1|^2}{|U_1 - c|^2} + \frac{|p_2|^2}{|U_2 - c|^2} = 0 \qquad (2.4)$$

or

$$\frac{|\bar{p}_1|^2}{|\bar{p}_2|^2} = \frac{|U_1 - c|^2}{|U_2 - c|^2} = \frac{|U_1\,\text{rel}|^2}{|U_2\,\text{rel}|^2} = K_1,$$

so that

$$|p_1|^2 = K\,|U\,\text{rel}|^2 \qquad (2.5)$$

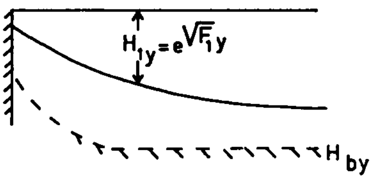

FIG. 2. Flow whose bottom slope removes the sufficient condition for stability.

and the amplitude of the pressure perturbation in each layer is proportional to the relative wind in that layer.

Consider the case where, neglecting β, $\partial q_1/\partial y = 0$ and $U_2 = 0$:

$$+\frac{d^2 U_1}{dy^2} - F_1 U_1 = 0,$$

since in non-dimensional units $H_{iy} = -U_1$.

So that in a region with a wall (at $y = 0$) marking the northernmost extent of the fluid:

$$U_1 \sim e^{+\sqrt{F_1}\,y} \quad (-\infty \leqslant y \leqslant 0).$$

Now $H_{2y} = -H_{1y} = U_1$ and does not change sign so that equation (2.3) yields $c_i = 0$, and the flow is stable. However, let us consider putting into our system a bottom slope of the form shown in Fig. 2.

In this case if the bottom slope is chosen correctly, $\partial q_2/\partial y$ is both positive and negative, and the flow will satisfy the necessary condition for instability. So the addition of a bottom slope which is in this case in the β "sense" appears to be a *destabilizing* influence. In any case, we can arbitrarily change the potential vorticity in the lowest layer, thus changing the stability properties of the flow, without altering the kinematics of the mean flow (e.g.) a symmetric flow over a symmetric (in y) mountain will have asymmetric stability properties, the flow being more unstable on the northern slope for a westerly shear.

Let us define the following functions for the unstable flow regime

$$\varphi_1 = \frac{p_1}{U_1 - c}; \quad \varphi_2 = \frac{p_2}{U_2 - c}. \qquad (2.6)$$

Then equations (2.1) may be re-written as (if we ignore β)

$$\frac{d}{dy}\frac{(U_1-c)^2}{F_1}\frac{d\varphi_1}{dy} - \alpha^2\frac{(U_1-c)^2}{F_1}\varphi_1 + (U_1-c)(U_1-U_2)\varphi_1$$
$$= (U_1-c)(p_1-p_2) = (U_2-c)(U_1-c)(\varphi_1-\varphi_2)$$
$$+ (U_1-U_2)\varphi_1(U_1-c), \qquad (2.7\text{ a})$$

$$\frac{d}{dy}\frac{(U_2-c)^2}{F_1}\frac{d\varphi_2}{dy} - \alpha^2\frac{(U_2-c)^2}{F_2}\varphi_2 + (U_2-c)(U_2-U_1)\varphi_2$$
$$= (U_2-c)(p_2-p_1) = (U_1-c)(U_2-c)(\varphi_2-\varphi_1)$$
$$+ U_2 - U_1)\varphi_2(U_2-c) \qquad (2.7\text{ b})$$

multiply (2.7 a) by φ_1^* and (2.7 b) by φ_2^* and integrate between y_1 and y_2, and add the resulting equations

$$- \int_{y_1}^{y_2} dy \left\{ \frac{(U_1-c)^2}{F_1}\left[\left|\frac{d\varphi_1}{dy}\right|^2 + \alpha^2|\varphi_1|^2\right] + \frac{(U_2-c)^2}{F_2}\left[\left|\frac{d\varphi_2}{dy}\right|^2 + \alpha^2|\varphi_2|^2\right]\right\}$$
$$= \int_{y_2}^{y_1}(U_1-c)(U_2-c)$$
$$\times \{|\varphi_1|^2 - \varphi_2\varphi_1^* - \varphi_1\varphi_2^* + |\varphi_2|^2\}\,dy$$
$$= \int_{y_1}^{y_2}(U_1-c)(U_2-c)|\varphi_1-\varphi_2|^2 dy$$
$$= -\int_{y_1}^{y_2}\{[(U_1-c)-(U_2-c)]^2 - (U_1-c)^2$$
$$-(U_2-c)^2\}\frac{|\varphi_1-\varphi_2|^2}{2}dy, \qquad (2.8)$$

$$+\int_{y_1}^{y_2} dy(U_1-c)^2[Q_1] + \int_{y_1}^{y_2}(U_2-c)^2[Q_2]dy$$
$$= +\int_{y_1}^{y_2}(U_1-U_2)^2 J dy, \qquad (2.9)$$

where Q_1, Q_2 and J are positive definite quantities.

The imaginary part of (2.9) is

$$c_i\int_{y_1}^{y_2}(U_1-c_r)Q_1dy + \int_{y_1}^{y_2}(U_2-c_r)Q_2dy = 0. \qquad (2.10)$$

The real part of (2.9) yields:

$$\int_{y_1}^{y_2} U_1^2Q_1dy + \int_{y_1}^{y_2} U_2^2Q_2dy - 2c_r\int_{y_1}^{y_2}(Q_1U_1$$
$$+ Q_2U_2)dy + c_r^2\int_{y_1}^{y_2}(Q_1+Q_2)dy$$
$$- c_i^2\int_{y_1}^{y_2}(Q_1+Q_2)dy\int_{y_1}^{y_2}(U_2-U_1)^2Jdy. \qquad (2.11)$$

Let a bar denote integration between y_1 and y_2.

Then using (2.10), (2.11) becomes

$$\overline{U_1^2Q_1} + \overline{U_2^2Q_2} = (c_r^2+c_i^2)\overline{(Q_1+Q_2)} + (U_2-U_1)^2J. \qquad (2.12)$$

Now let both U_1 and U_2 be bounded from above by a and from below by b. Then the following inequality holds

$$0 \geqslant (U_1-a)(U_1-b)Q_1 + (U_2-a)(U_2-b)Q_2$$
$$= \overline{U_1^2Q_1} + \overline{U_2^2Q_2} - (U_1Q_1 + U_2Q_2)(a+b)$$
$$+ ab(Q_1+Q_2). \qquad (2.13)$$

Substituting from (2.10) and (2.13):

$$0 \geqslant (c_r^2+c_i^2)\overline{(Q_1+Q_2)} - c_r(a+b)\overline{(Q_1+Q_1)}$$
$$+ ab(\overline{Q_1}+\overline{Q_2}) + \overline{(U_2-U_1)^2J} \qquad (2.14)$$

or equivalently, since $\overline{(U_2-U_1)^2J} \geqslant 0$

$$\left(\frac{a-b}{2}\right)^2 \geqslant \left(c_r - \left(\frac{a+b}{2}\right)\right)^2 + c_i^2. \qquad (2.15)$$

The complex phase speed lies within a semicircle of diameter $a-b$ (the difference between the velocity extremes) and the circle is centered on the real axis at $\frac{1}{2}(a+b)$ the median velocity (Fig. 3).

This type theorem was first proven by Howard for continuously stratified shear flows in two-dimensional flow in which the divergence is zero.

We can do more with this. Let

$$\left.\begin{aligned} b_1 \leqslant U_1 \leqslant a_1, \\ b_2 \leqslant U_2 \leqslant a_2. \end{aligned}\right\} \qquad (2.16)$$

Then instead of (2.13) we have

$$0 \geqslant \overline{U_1^2Q_1} + \overline{U_2^2Q_2} - \overline{U_1Q_1}(a_1+b_1)$$
$$- \overline{U_2Q_2}(a_2+b_2) + a_1b_1\overline{Q_2} + a_2b_2\overline{Q_2} \qquad (2.16\text{ b})$$

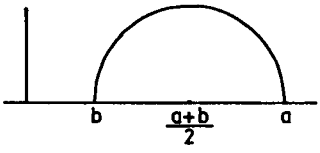

FIG. 3. Semi-circle in complex velocity plane in which the phase speed for unstable baroclinic waves must lie.

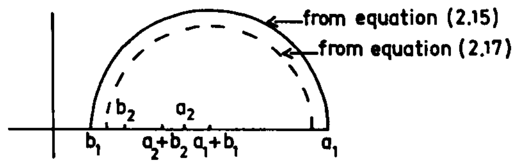

FIG. 4. Modified semi-circle theorem when minimum and maximum velocity are in the same layer.

or using the relations (2.10) and (2.11)

$$0 \geqslant (c_r^2 + c_i^2)\overline{(Q_1 + Q_2)} - (a_1 + b_1) c_r \overline{(Q_1 + Q_2)} + a_1 b_1 \overline{(Q_1 + Q_2)} + \overline{U_2 Q_2 (a_1 + b_1 - (a_2 + b_2))} + \overline{Q_2}(a_2 b_2 - a_1 b_1). \qquad (2.16\,c)$$

The inequality in (2.16c) is still valid if U_2 is replaced by its minimum value b_2, if $a_1 + b_1 \geqslant a_2 + b_2$ and if $a_1 + b_1 > 0$ (i.e. if the median value in layer 1 is greater than that in layer 2).

$$\left(c_r - \frac{a_1 + b_1}{2}\right)^2 + c_i^2 \leqslant \left(\frac{a_1 - b_1}{2}\right)^2 + \frac{\overline{Q}_2}{\overline{Q}_1 + \overline{Q}_2} \times (a_1 - b_2)(b_1 - b_2). \qquad (2.17)$$

If $b_1 = b_2$, (2.17) reduces to (2.15), since a_1 and b_1 are then the velocity bounds for both layers.

Consider the following case:

$$b_1 < b_2, \quad a_1 > > a_2.$$

Equation (2.15) would yield a semicircle bound as indicated by Fig. 4.

Equation (2.17) tells us that the limiting bound is actually less since $(\overline{Q}_2/[\overline{Q}_1 + \overline{Q}_2])/(a_1 - b_2)(b_1 - b_2) \leqslant 0$. If $b_2 < b_1$, then we have a limiting semicircle whose diameter is larger than in the case $b_2 > b_1$. This implies the flow is more unstable (or is at least capable of greater growth rates) if the minimum value of the velocity is in the lower layer and the maximum in the upper layer.

If β is reintroduced in the potential vorticity, equation (2.15) becomes ($c_i \neq 0$),

$$\left(\frac{a-b}{2}\right)^2 \geqslant \left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 + \frac{\beta/R_0}{\overline{Q}_1 + \overline{Q}_2}\left[\overline{\left(U_1 - \frac{a+b}{2}\right)|\varphi_1|^2} + \overline{\left(U_2 - \frac{a+b}{2}\right)|\varphi_2|^2}\right] \qquad (2.18)$$

The inequality still holds if U_1 and U_2 are replaced by b. Equation (2.18) can then be rewritten as

$$\left(\frac{a-b}{2}\right)^2 + \frac{\beta/R_0}{\overline{Q}_1 + \overline{Q}_2}\left[\left(\frac{a-b}{2}\right)\overline{(|\varphi_1|^2 + |\varphi_2|^2)}\right] \geqslant \left(c_r - \frac{a+b}{2}\right)^2 + c_r^2. \qquad (2.19)$$

It can be shown that

$$\overline{Q}_1 + \overline{Q}_2 \geqslant \left(\frac{\pi^2}{4a^2} + \alpha^2\right)\overline{(|\varphi_1|^2 + |\varphi_2|^2)}$$

where a is half the distance in non dimensional units between vertical walls bounding the horizontal extent of the flow. Equation (2.19) is then replaced by

$$\left(\frac{a-b}{2}\right)^2 + \frac{\beta(a-b)}{R_0 2(\pi^2/4a^2 + \alpha^2)} \geqslant \left(c_r - \frac{a+b}{2}\right)^2 + c_r^2 \qquad (2.20)$$

We see that the presence of β increases the semicircle size, especially for large scale systems (a large, α small) primarily of course to allow the retrogression of the waves due to the planetary vorticity gradient. A more detailed analysis shows that even with β present c_r must remain less than U max.

3. Sufficient conditions for instability

Consider the class of flows for which there exists extrema of the vorticity in layer 1 and the velocity is zero in layer 2. Further let the extrema in layer 1 occur for a point or points of constant velocity ($U_1 = c_s$).

If

$$\frac{\partial q_1}{\partial y}\Big/ U_1 - c_s \equiv K_1(y) < \infty;$$

$$\frac{\partial q_2}{\partial y}\Big/ -c_s \equiv K_2(y) < \infty, \qquad (3.1)$$

$$K_1(y)+K_2(y)>\frac{\pi^2}{2a^2}, \qquad (3.2)$$

where L is the distance between vertical walls bounding the horizontal extent of the flow, then we can show that a neutral solution of equations (2.1) exists with phase speed c_s.

Consider the functional A^2 defined by

$$A^2=\frac{\int_{y_1}^{y_2}dy\left[\frac{K_1(y)}{F_1}(p_1)^2+\frac{K_2(y)}{F_2}(p_2)^2-(p_1-p_2)^2-\left(\frac{\partial p_1}{\partial y}\right)^2\frac{1}{F_1}-\left(\frac{\partial p_2}{\partial y}\right)^2\frac{1}{F_2}\right]}{\int_{y_1}^{y_2}dy\left[\frac{(p_1)^2}{F_1}+\frac{(p_2)^2}{F_2}\right]}.$$

A^2 possesses a maximum $\leqslant K_{1\max}+K_{2\max}$. The consideration of the trial function $p_1=p_2=\cos(\pi y/2a)$ shows that this maximum is positive, (since $A^2>0$ for the trial function) and the maximum occurs where $\delta A^2/\delta p_1=0$ and $\delta A^2/\delta p_2=0$. These variational conditions are equivalent to the equations (2.1) (a) and (b) if $A^2_{\max}$ is associated with α^2 and $c=c_s$. A neutral solution therefore exists for $\hat{\alpha}=A_{\max}$ and $c=c_s$ and $p=\hat{p}$ satisfying the equations

$$(D^2-\hat{\alpha}^2)\hat{p}_1+\frac{\partial q_1}{\partial y}\frac{\hat{p}_1}{U_1-c_s}-F_1\hat{p}_1=-F_1\hat{p}_2, \qquad (3.4\text{ a})$$

$$(D^2-\hat{\alpha}^2)\hat{p}_2+\frac{\partial q_2}{\partial y}\frac{\hat{p}_2}{-c_s}-F_2\hat{p}_2=-F_2\hat{p}_1. \qquad (3.4\text{ b})$$

Consider some disturbance at a wave number α near $\hat{\alpha}$.

$$(D^2-\alpha^2)p_1+\frac{\partial q_1}{\partial y}\frac{p_1}{U_1-c}-F_1p_1=-F_1p_2, \qquad (3.5\text{ a})$$

$$(D^2-\alpha^2)p_2+\frac{\partial q_2}{\partial y}\frac{1}{-c}-F_2p_2=-F_2p_1. \qquad (3.5\text{ b})$$

Multlply (3.4 a) by $\hat{p}_1$, (3.4 b) by $\hat{p}_2$; (3.5 a) by p and (3.5 b) by p_2 an obtain after integration over the range (y_1, y_2):

$$0=\frac{1}{F_1}(\alpha^2-\hat{\alpha}^2)\overline{p_1\hat{p}_1}+\frac{1}{F_2}(\alpha^2-\hat{\alpha}^2)\overline{p_2\hat{p}_2}+\overline{\frac{\partial q_1}{\partial y}p_1\hat{p}_1\frac{c_s-c}{(U_1-c)(U_1-c_s)}}+\overline{\frac{\partial q_2}{\partial y}\frac{p_2\hat{p}_2(c-c_s)}{cc_s}} \qquad (3.6)$$

or

$$\frac{\alpha^2-\hat{\alpha}^2}{c-c_s}=\frac{\overline{\frac{K_1(y)p_1\hat{p}_1}{F_1(U_1-c)}}+\overline{\frac{K_2(y)p_2\hat{p}_2}{F_2(-c)}}}{\frac{\overline{p_1\hat{p}_1}}{F_1}+\frac{\overline{p_2\hat{p}_2}}{F_2}}. \qquad (3.7)$$

Now let $\alpha^2\to\hat{\alpha}^2$; $c\to c_s$ such that $c_i>0$. One obtains

$$\left(\frac{d\alpha^2}{dc}\right)_{c=c_s}=\lim_{c_i\to 0}\frac{\overline{\frac{K_1(y)(\hat{p}_1)^2}{F_1(U_1-c)}}+\overline{\frac{K_2(y)(\hat{p}_2)^2}{F_2(-c)}}}{\overline{(\hat{p}_1)^2}/F_1+\overline{(\hat{p}_2)^2}/F_2}. \qquad (3.8)$$

These integrals are discussed by LIN (1955) where it is shown then that

$$\left(\frac{d\alpha^2}{dc}\right)_{c=c_s}=A+iB, \qquad (3.9)$$

where all that is important here is that $B>0$, A and B bounded. We see that

$$\frac{dc}{d\alpha^2}=\frac{A-iB}{A^2+B^2}. \qquad (3.10)$$

So that if α^2 decreases slightly c_i becomes positive and the flow is unstable to wavelengths slightly longer than $\hat{\alpha}^2$.

As an example consider the flow (in non-dimensional units)

$$u_1=e^{-r^2y^2}. \qquad (3.11)$$

Now

$$\frac{\partial q_1}{\partial y}=(2r^2+F_1-r^4 4y^2)^{-y^2r^2}, \qquad (3.12\text{ a})$$

$$\frac{\partial q_2}{\partial y}=-F_2e^{-r^2y^2}. \qquad (3.12\text{ b})$$

The potential vorticity in the top layer vanishes at $y_c=\pm\sqrt{(F_1+2r^2)/2r^2}$. We see that this jet flow can satisfy the criteria listed at the beginning of this section so that a neutral solution exists with a phase speed

$$c_s=e^{-(F_1+2r^2)/4r^2}$$

and a neighboring unstable solution also exists.

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REFERENCES

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