Steady plane fronts in a rotating fluid

By PIERRE WELANDER, *Swedish Natural Science Research Council and International Meteorological Institute in Stockholm*

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ABSTRACT

A derivation is given of the equations describing a steady plane front in a rotating differentially heated fluid bounded by a horizontal non-conducting wall. The kinematic viscosity ν and the thermal conductivity $\varkappa$ are assumed small and of the same order.

The thickness of the front is of order $(\nu/\Omega)^{\frac{1}{2}}$, where Ω is the angular velocity, and the characteristic length along the front is of order $(g\alpha/\Omega^2)\,\mathrm{tg}\,\varepsilon\Delta T$, where g is the acceleration of gravity, α the coefficient of thermal expansion, ε the slope of the front, and ΔT the temperature variation across the front. There is a circulation in a vertical plane perpendicular to the front surface driven by the friction boundary layer. In this plane Coriolis acceleration, non-linear acceleration and friction forces are all of the same order. Considering the mass and momentum transport equations and requiring the solution to be gravitationally stable, one can show that there must be a frictional inflow towards the front surface from both the cold and the warm side. A similarity form satisfies the front equations and leads to a system of ordinary differential equations. Some qualitative features of the solution can be deduced, but the complete solution, requiring an analysis of the transition region where the front joins the friction boundary layer, is still lacking.

The result obtained is in qualitative agreement with observations of fronts in some laboratory experiments, and with observations of natural fronts in the atmosphere and the oceans. In the latter cases the parameters ν and $\varkappa$ in the theory should stand for the turbulent coefficients of viscosity and conductivity.

1. Background

In the atmosphere and the oceans one observes different types of fronts, or discontinuity surfaces. Across these surfaces very rapid variations of temperature and tangential velocity occur. Best known are the atmospheric fronts, that become particularly well developed in the outbreaks of cold polar air over the continental areas. Temperature variations of 10°C per kilometer and more have been recorded in such cases. In the ocean a similar phenomenon occurs in regions where cold and warm water masses meet. Oceanic fronts with a temperature variation of the order of 1°C over a few meters can be found at the edge of the Gulf Stream. The existence of such a front is shown very clearly in some photos of the Gulf Stream (STOMMEL, H., 1958).

The atmospheric fronts have been studied a great deal in the past. The intensification of the temperature gradient occurring at the border between air masses of different temperatures seems at least in part to be an effect of the horizontal deformation associated with the large-scale eddy motion, that creates a stretching and thereby also a crowding of the isotherms (BERGERON, T., 1928; PETTERSSEN, S., 1936). More recently theoretical investigations of the circulation induced by release of latent heat and by ground friction have advanced the idea of a front formation through a similar deformation mechanism in a vertical plane (SAWYER, J. S., 1956; ELIASSEN, A., 1959).

The oceanic fronts have been studied much less, but a similarity in certain features of the oceanic and the atmospheric fronts has been established. The thermodynamic processes are, however, quite different in the atmosphere and the oceans. One has thus an indication that these processes are not the most important factor in explaining the formation of the front.

Fronts similar to those observed in the atmosphere and the oceans occur in a certain type of laboratory experiments, where a rotating fluid is differentially heated (FALLER, A., 1956). A

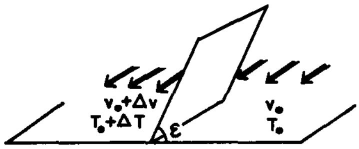

FIG. 1. Margules solution.

detailed study of such laboratory fronts would certainly be of great interest, and offers a possibility of making a direct comparison with different theoretical models.

Obviously there is a need for more basic studies, both theoretical and experimental, of the front problem. In the present paper the special problem of a steady front is treated by the methods of theoretical hydrodynamics. The problem is formulated in the simplest possible way. The starting point is the well-known class of ideal fluid solutions valid for a rotating and stratified fluid, where a horizontal density variation is balanced by the Coriolis forces of a transverse motion. As a particular case we have here the "Margules solution", where two homogeneous fluid layers sustain a plane sloping discontinuity surface through a jump in transverse velocity (Fig. 1). While the ideal fluid regime allows an infinity of solutions, it is expected that the corresponding problem for a real fluid, characterized by a small viscosity and conductivity, has a unique solution.

It is assumed that for small viscosity and conductivity one can find a solution close to a Margules case, and that noticeable deviations from this case occur only close to the discontinuity surface and the walls. Since the present model is based on the Navier-Stokes equation and the molecular heat transport equation, it should be possible to compare the theoretical prediction with laboratory-produced laminar fronts. In the rea atmosphere and oceans there is, however, always turbulence present, and to make any comparison with the present model one must replace the molecular values of viscosity and conductivity entering in the theory by the effective turbulent (or eddy) values. Certainly one can get some feeling for the value of the theory from such a comparison, but the approach is not entirely satisfactory.

Physically, there is some evidence that turbulence created in a fluid becomes noticeably modified in the neighbourhood of a front surface, and that this has feedback effects on the front itself. As an example, in the formation of sharp horizontal discontinuity surfaces in the ocean, such as the "shallow thermocline", the turbulence intensity seems to depend critically on the stratification. To maintain a certain vertical heat flux the vertical temperature gradient must vary inversely with the effective conductivity. If an increase of the temperature gradient occurs locally, this may cut down the turbulent intensity through the increased stratification and thus decrease the effective conductivity, and this will in turn require an even larger temperature gradient. In this way an amplification may proceed until ultimately a limit is set by the molecular conduction. Laboratory experiments that show this effect have been made, by heating a fluid at the surface while turbulence is generated by a vibrating grid. The aforementioned horizontal fronts are of a different type than the sloping fronts treated in this study, in which the Coriolis effects come in critically, but the above effects of turbulence may well be present also in our case and require important modifications. An ultimate theory of the natural fronts must then include both the equations for the gross-motion and the equations for the turbulent fluctuations that determine the effective values of viscosity and conductivity in different parts of the fluid. It is also likely that a realistic model must include three-dimensional and non-steady effects, all of which are left out in this study. Thus the present theory can hope to explain only some of the most basic features of the front phenomenon.

2. Formulation of the problem

Consider a fluid rotating with uniform angular velocity Ω around a vertical axis. Let (x,y,z) be a rectangular coordinate system that follows the rotation and that has the z-axis directed upwards. The fluid is kept at fixed temperatures T_0 and $T_0 + \Delta T$ at $x = +\infty$ and $x = -\infty$, respectively. We also fix the transverse velocities v_0 and $v_0 + \Delta v$ at these points. The velocity and the normal heat flux vanish along a horizontal boundary $z = 0$ (Fig. 2). The problem is further restricted by the following assumptions:

(i) The rotation is assumed so small and the gravity so strong that centrifugal forces can

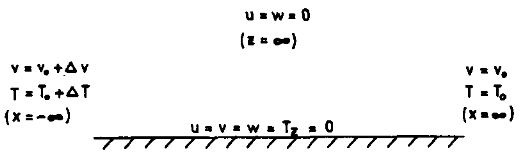

FIG. 2. Boundary conditions.

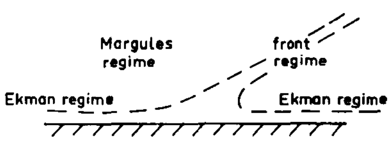

FIG. 3. The different regimes.

be neglected, and the equipotentials can be approximated by plane surfaces (the "f-plane" used in meteorologic and oceanographic modelling).

(ii) The temperature variations are so small that the Boussinesq approximation can be justified (density linearly dependent on temperature, the density is considered as a constant except when combined with gravity).

(iii) The coefficients of viscosity and conductivity are small (in a sense to be defined later), and the Prandtl number is of order one.

(iv) The motion is steady and independent of the y-coordinate.

(v) The solution can be given in terms of three regimes: a "front regime", an "Ekman regime", and an interior "Margules regime" (Fig. 3).

3. Basic equations

The problem stated in the previous section is described by the following set of equations.

$$(uu)_x + (wu)_z - fv = -\frac{1}{\varrho_0} p_x + \nu \nabla^2 u, \qquad (1)$$

$$(uv)_x + (wv)_z + fu = \nu \nabla^2 v, \qquad (2)$$

$$(uw)_x + (ww)_z = -\frac{1}{\varrho_0} p_z + g\alpha T + \nu \nabla^2 w, \qquad (3)$$

$$u_x + w_z = 0, \qquad (4)$$

$$uT_x + wT_z = \varkappa \nabla^2 T. \qquad (5)$$

The conditions at infinity are

$$T = \Delta T, \quad v = v_0 + \Delta v \quad \text{when} \quad x \to -\infty, \qquad (6)$$

$$T = 0, \quad v = v_0 \quad \text{when} \quad x \to +\infty.$$

The conditions at the ground are

$$\left.\begin{aligned} u = v = w = 0 \\ T_z = 0 \end{aligned}\right\} \quad \text{at} \quad z = 0. \qquad (7)$$

(u,v,w) is the velocity, p the pressure minus the hydrostatic pressure $-g\varrho_0 z$, T the temperature after subtracting T_0, α the coefficient of thermal expansion, g the acceleration of gravity, ν the kinematic viscosity, $\varkappa$ the thermal conductivity, and f twice the angular velocity Ω. Finally, ∇^2 stands for $\partial^2/\partial x^2 + \partial^2/\partial z^2$.

4. Equations for the front

In the front regime it is convenient to use coordinates (x', y', z') obtained by rotating the (x,y,z)-system an angle ε around the y-axis, so that the front surface falls along the x'–y'-plane (Fig. 4). We choose ε in the interval 0 to $\frac{1}{2}\pi$. Furthermore we translate the coordinate system with the constant velocity v_0 in the y-direction. This is done to simplify the later scaling of the equations.

The equations (1–5) become then

$$(u'u')_{x'} + (w'u')_{z'} - f_1 v' = -\frac{1}{\varrho_0} p'_{x'} + g_2 \alpha T' + \nu \nabla'^2 u', \qquad (8)$$

$$(u'v')_{x'} + (w'v')_{z'} + f_1 u' - f_2 w' = \nu \nabla'^2 v', \qquad (9)$$

$$(u'w')_{x'} + (w'w')_{z'} + f_2 v' = -\frac{1}{\varrho_0} p'_{z'} + g_1 \alpha T' + \nu \nabla'^2 w', \qquad (10)$$

$$u'_{x'} + w'_{z'} = 0, \qquad (11)$$

$$u' T'_{x'} + w' T'_{z'} = \varkappa \nabla'^2 T', \qquad (12)$$

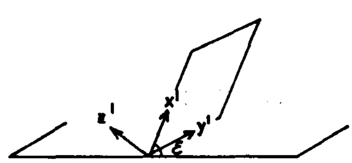

FIG. 4. The x-y-z system.

where
$$\left.\begin{aligned} g_1 &= g\cos\varepsilon, & f_1 &= f\cos\varepsilon, \\ g_2 &= g\sin\varepsilon, & f_2 &= f\sin\varepsilon. \end{aligned}\right\} \tag{13}$$

The boundary conditions are

$$T' = 0, \quad u' = v' = w' = 0 \quad \text{when} \quad z' \to -\infty,$$

$$T' = \Delta T, \quad u' = w' = 0, \quad v' = \Delta v \quad \text{when}\; z' \to +\infty. \tag{14}$$

Infinite values of z' are here to be interpreted as "very large compared to the front thickness". Since the equations will be applied only to the front region, the ground conditions do not enter. Of course, the complete solution can be obtained only through a proper joining of the front region and the Ekman region next to the ground; this problem is discussed in Sect. VI.

In the equations (9–12) we can eliminate the pressure, and introduce a stream function ψ' by the relations

$$u' = -\psi'_{z'}, \quad w' = \psi'_{x'}. \tag{15}$$

This gives us the following set of equations, which forms the starting point for the further investigations of the front:

$$J'(\psi', \nabla'^2\psi') - \nu\nabla'^4\psi' = -g_2\alpha T'_{z'} + g_1\alpha T'_{x'} - f_1 v'_{z'} - f_2 v'_{x'}, \tag{16}$$

$$J'(\psi', v') - \nu\nabla'^2 v' = f_1\psi'_{z'} + f_2\psi'_{x'}, \tag{17}$$

$$J'(\psi', T') - \varkappa\nabla'^2 T' = 0. \tag{18}$$

Here
$$\nabla'^2 = \frac{\partial^2}{\partial^2_{x'}} + \frac{\partial^2}{\partial^2_{z'}},$$

and J' stands for the Jacobian with respect to x' and z'.

5. The ideal fluid case

For an ideal fluid we have $\nu = \varkappa = 0$. A solution to the system of equations (1–5) is then given by

$$\left.\begin{aligned} u &= w = 0, \\ v &= v_0(x) + \frac{g\alpha}{f}\int_{z_0}^{z} T_x(x, z)\,dz, \end{aligned}\right\} \tag{19}$$

where the temperature distribution $T(x,z)$ and $v_0(x)$ are arbitrary. Since $w = 0$, a horizontal boundary can be included. If the isotherms make the angle ε_T with the horizontal (in the general case ε_T is a function of x and z), we have $\operatorname{tg}\varepsilon_T = -T_x/T_z$. Introducing this expression in (19) and integrating over a range (z_0, z_1) we have by the mean value theorem

$$v_1 = v_0 - \frac{g\alpha}{f}\operatorname{tg}\varepsilon_T^*(T_1 - T_0), \tag{20}$$

where ε_T^* is the angle of some intermediate isotherm. In the limiting case where the temperature variation is concentrated in a discontinuity surface the formula gives a relation between the jump in the transverse velocity $\Delta v = v_1 - v_0$ and the jump in the temperature $\Delta T = T_1 - T_0$ occurring at the interface between the two fluid masses, known as "Margules rule".

$$\Delta v = -\frac{g\alpha}{f}\operatorname{tg}\varepsilon\Delta T. \tag{21}$$

ε represents the slope of the interface. The parameters ε, ΔT and Δv may vary along the front. If we have the convention $z_1 > z_0$, $0 < \varepsilon < \frac{1}{2}\pi$, we must obviously have $\Delta T > 0$, since otherwise the fluid is gravitationally unstable. Margules rule will then require that $\Delta v < 0$. One further restriction on the transverse velocity will be mentioned later.

The simplest case, that we will call the "Margules solution", occurs when the interface is plane and the temperature and velocity are uniform within each layer.

In the (x', y', z')-system this solution is

$$\psi' = 0,\; T' = 0,\; v' = 0 \quad \text{for} \quad z' < 0,$$

$$\psi' = 0,\; T' = \Delta T,\; v' = -\frac{g\alpha}{f}\operatorname{tg}\varepsilon\Delta T \quad \text{for} \quad z' > 0. \tag{22}$$

Margules solution is assumed to describe the ideal fluid regime of our real fluid problem, at sufficient distance from the front surface and from the ground.

6. The Ekman layer

At the horizontal wall friction effects will occur within a distance of the order of $(\nu/f)^{\frac{1}{2}}$ (the Ekman depth). The hydrodynamic equations for this boundary region are, outside the front itself,

$$\left.\begin{aligned} -fv &= -\frac{1}{\varrho_1}p_x + \nu u_{zz}, \\ fu &= \nu v_{zz}, \\ w &= 0, p_z = 0. \end{aligned}\right\} \quad (23)$$

Introducing a complex velocity $W = u + iv$ one can write the two first equations of (23) in the form

$$\nu W_{zz} - ifW = \frac{1}{\varrho_0}p_x \quad (24)$$

and the solution satisfying the condition $W = 0$ at $z = 0$ is

$$W = \frac{i}{f\varrho_0}p_x\left(1 - e^{\left(\frac{if}{\nu}\right)^{\frac{1}{2}}z}\right) \quad (25)$$

This is the well-known Ekman spiral (Ekman, V. W, 1905). There is a transport of fluid in the x-direction

$$\psi_E = -\frac{p_x}{f\varrho_0}\left(\frac{\nu}{2f}\right)^{\frac{1}{2}}.$$

On the right and the left side of the front p_x takes on different values and one finds a net transport divergence in the Ekman layer

$$\psi_{E_1} - \psi_{E_0} = -\frac{(p_{x_1} - p_{x_0})}{f\varrho_0}\left(\frac{\nu}{2f}\right)^{\frac{1}{2}} = -(v_1 - v_0)\left(\frac{\nu}{2f}\right)^{\frac{1}{2}}. \quad (26)$$

Since $v_1 - v_0$ is negative in our case, $\psi_{E_1} - \psi_{E_0}$ is positive and fluid leaves the Ekman layer to go into the front. Depending on the values of v_0 and v_1 different cases may occur, as shown in Figs. 5*a–c*. In all cases there is a net upward motion in the front. It can now be shown that only the case 5*c* is physically possible. The case 5*a* is ruled out because the fluid is unstable. The Ekman flow leaving the front region passes under the coldest fluid, and it must carry with it at least some temperature excess which generates a gravitational instability. The case 5*b* is ruled out by momentum transport arguments that are developed in the Appendix. It is shown that the magnitude of the v-velocity

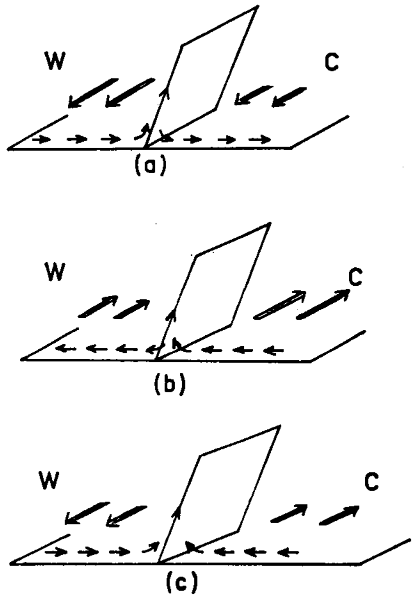

FIG. 5. Different cases for the frictional transports. (*a*) Inflow from the warm sector, outflow to the cold sector. (*b*) Inflow from the cold sector, outflow to the warm sector. (*c*) Inflow from both the cold and the warm sectors.

must be larger in the warm sector than in the cold sector, and this condition is fulfilled only in the case 5*c*. The conclusion is that we must have a frictional flow towards the front both from the warm and the cold sector. In a laboratory experiment described in the literature, where observations of the boundary layer flow close to fronts have been made, such a double-sided flow towards the front has also been observed (FALLER, A., 1956).

7. Scaling of the front equations

In the equations (16–18) and the boundary condition (14) valid for the front layer, we introduce new dependent and independent variables, denoted by a star, according to the linear transformation

$$\left.\begin{aligned} x' &= Lx^* \\ z' &= \delta z^* \\ T' &= TT^* \\ \psi' &= \Psi\psi^* \\ v' &= Vv^* \end{aligned}\right\} \quad (27)$$

L, δ, T, Ψ, and V are constants that represent the characteristic amplitudes of the variables. We search for transformations that make the equations and boundary conditions expressed in the star-variables of order unity, meaning that the coefficients occurring are pure numbers of order unity. If all coefficients can be made of order unity, the original equations are consistent, and no simplifications are allowed in this system. If some coefficients become much smaller than unity, one may neglect these terms, and arrive at a correspondingly simplified model.[1]

Of the possible transformations some may lead to trivial systems, or systems that do not possess any solution at all, and these have, of course, to be disregarded. If the original problem has a unique solution, one should be led to a single consistent scaling.

In the present problem we assume from the beginning that a front, i.e. a free boundary layer, actually exists. Thus we require:

$$\delta << L. \tag{28}$$

On the basis of (28) we can directly make some simplifications in the front equations: ∇'^2 and ∇'^4 can be replaced by $\partial^2/\partial z'^2$ and $\partial^4/\partial z'^4$, respectively, and in the right-hand sides of (16) and (17) the x'-derivatives of T', ψ' and v' can be neglected.[2]

The equations (16–18) and the boundary condition (14) transform now to

$$\frac{\Psi^2}{L\delta^3} J^*(\psi^*, \psi^*_{z^*z^*}) - \frac{\nu\Psi}{\delta^4} \psi^*_{z^*z^*z^*z^*} = -\frac{g_2\alpha T}{\delta} T^*_{z^*} - \frac{f_1 V}{\delta} v^*_{z^*}, \tag{29}$$

$$\frac{\Psi V}{L\delta} J^*(\psi^*, v^*) - \frac{\nu V}{\delta^2} v^*_{z^*z^*} = \frac{f_1\Psi}{\delta} \psi^*_{z^*}, \tag{30}$$

$$\frac{\Psi T}{L\delta} J^*(\psi^*, T^*) - \frac{\varkappa T}{\delta^2} T^*_{z^*z^*} = 0, \tag{31}$$

$$\left.\begin{aligned} &T^* = 0,\ \psi^* = 0,\ v^* = 0 \quad \text{when} \quad z^* \to -\infty, \\ &T^* = \frac{\Delta T}{T},\ \psi^* = 0,\ v^* = \frac{\Delta v}{V} \quad \text{when} \quad z^* \to +\infty. \end{aligned}\right\} \tag{32}$$

Two of the scaling constants can be directly found. In the first place to make the boundary conditions of order unity we require

$$\left.\begin{aligned} &T \sim \Delta T, \\ &V \sim \frac{g\alpha}{f} \operatorname{tg} \varepsilon \Delta T. \end{aligned}\right\} \tag{33}$$

The last relation follows from the requirement that we should be in the neighbourhood of the Margules solution.

Scaling V according to the Margules rule implies that the two terms in the right-hand side of (29) automatically become of the same order. The order of the other terms relative to these depends on the choice of the remaining scaling constants Ψ, L and δ. An inspection of equation (31) shows that we must choose $\Psi T/L\delta \sim \varkappa T/\delta^2$, i.e.

$$\Psi \sim \frac{\varkappa L}{\delta}. \tag{34}$$

To prove this suppose that $\Psi T/L\delta >> \varkappa T/\delta^2$. Then (31) would reduce to the equation $J^*(\psi^*, T^*) = 0$. But since the Prandtl number is of order one, it follows that the friction terms in (29) and (30) also drop out, hence all effects of conduction and friction are absent in the front, contrary to our basic assumption. Next suppose that $\Psi T/L\delta << \varkappa T/\delta^2$. Then (31) is reduced to the equation $T^*_{z^*z^*} = 0$, that is inconsistent with the condition that T^* tends to different constant values when $z^* \to -\infty$ and $z^* \to \infty$. Only the possibility expressed by (34) remains. The Coriolis acceleration, the non-

[1] The argument used here is that a system of equations and boundary conditions in which all coefficients are pure numbers of order unity possesses a solution that is also of order unity, meaning that the dependent variables (and the derivatives of these occurring) have amplitudes of order unity within unit ranges of the independent variables. This may, of course, not always be fulfilled, and it is thus necessary to check the approximations made at the end. However, it should be noted that some assumption of this type is always implied when we neglect terms by scaling arguments.

[2] One may question the neglect of the x'-derivatives compared to the z'-derivatives on the basis that the latter terms are multiplied by sin ε, which is a small number in all atmospheric and oceanic cases. However, within the framework of the theory the neglect is consistent. We choose a fixed value of ε, consider the corresponding Margules ideal fluid solution, and look for a neighbouring solution when a sufficiently small viscosity and conductivity is introduced.

linear acceleration and the friction force must all be of the same order.

In (30) we cannot have $\Psi V/L\delta << f_1 \Psi/\delta$. For since both terms in the left-hand side are of the same order, the equation would then reduce to $\psi^*_{z^*} = 0$, i.e. ψ^* would be constant in the z^*-direction. Such a solution could certainly not be joined to the outside Margules and Ekman regions. It follows that $\Psi V/L\delta \gtrsim f_1 \Psi/\delta$, i.e. $V \gtrsim f_1 L$.

Two more conditions are obtained by considering the Ekman boundary layer. The first condition simply says that the net transport in the Ekman layer given by (26), and which must go into the front, sets the scale Ψ:

$$\Psi \sim V\left(\frac{\nu}{f}\right)^{\frac{1}{2}}. \tag{35}$$

At this stage we note that the previous condition $V \gtrsim f_1 L$ together with (34) and (35) leads to the result that $\delta \lesssim (\nu/f)^{\frac{1}{2}}$, i.e. the front can at most have a thickness of the order of the Ekman layer.

One more condition is obtained from a theorem on the momentum transport balance, derived in the Appendix. It states that the momentum transport in the x-direction $M = \int u^2 dz$, computed across the front, must equal the difference in the same momentum transports computed for the Ekman layers on both sides of the front. u is of the order Ψ/δ in the front and of orders v_0 and v_1 in the right and the left Ekman layer, respectively. The thickness of the front is δ and the thickness of the Ekman layer is $(\nu/f)^{\frac{1}{2}}$. It follows that

$$\left(\frac{\Psi}{\delta}\right)^2 \delta \sim (v_1^2 - v_0^2)\left(\frac{\nu}{f}\right)^{\frac{1}{2}}. \tag{36}$$

If we assume that v and Δv are of the same order V (both are externally prescribed) the order of magnitude of Ψ is

$$\Psi \sim V\delta^{\frac{1}{2}}\left(\frac{\nu}{f}\right)^{\frac{1}{4}}. \tag{37}$$

From (34), (35) and (37) we can determine Ψ, L, and δ:

$$\left.\begin{aligned} \Psi &\sim \frac{g\alpha}{f}\operatorname{tg}\varepsilon\Delta T\left(\frac{\nu}{f}\right)^{\frac{1}{2}}, \\ L &\sim \frac{g\alpha}{f^2}\operatorname{tg}\varepsilon\Delta T, \\ \delta &\sim \left(\frac{\nu}{f}\right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{38}$$

(in the expression for L use has been made of the assumption that the Prandtl number is of order one). By (33) and (38) all the scales are determined.

It may be of some interest to compute the scales L and δ numerically in some hypothetical cases. In Table I are given some values that may be thought to apply in a laboratory experiment, in an atmospheric front, and in an oceanic front. The effective (turbulent) values of ν given in the last two cases are, of course, uncertain. However, it should be noted that the friction does not enter at all in the determination of the scale L along the front.

If we determine the scaling constants so that the order of magnitude relations (33, 38) become exact, the transformed equations and boundary conditions take on the following special form

$$J^*(\psi^*, \psi^*_{z^*z^*}) - \psi^*_{z^*z^*z^*z} = -\cos\varepsilon(T^*_{z^*} + v^*_{z^*}), \tag{39}$$

$$J^*(\psi^*, v^*) - v^*_{z^*z^*} = \cos\varepsilon\,\psi^*_{z^*}, \tag{40}$$

$$J^*(\psi^*, T^*) - \frac{\varkappa}{\nu}T^*_{z^*z^*} = 0, \tag{41}$$

$$\left.\begin{aligned} &T^* = 0, v^* = 0, \psi^* = 0 \quad \text{when} \quad z^* \to -\infty, \\ &T^* = 1, v^* = \frac{\Delta v}{\frac{g\alpha}{f}\operatorname{tg}\varepsilon\Delta T}, \psi^* = 0 \quad \text{when} \quad z^* \to \infty. \end{aligned}\right\} \tag{42}$$

In this formulation the only free parameters are the front angle ε and the Prandtl number $\nu/\varkappa$. We note that in the equations of motion the Coriolis acceleration, the non-linear acceleration and the friction term are all of the same order. The parameter

$$\frac{\Delta v}{\frac{g\alpha}{f}\operatorname{tg}\varepsilon\Delta T}$$

plays the role of an eigenvalue. In the case of an ideal fluid its value is exactly 1 (Margules rule). In a real fluid a value close to 1 is expected, but the exact value must come from the theory itself. In our original formulation ΔT and Δv were given, thus it is in fact ε that plays the role of eigenparameter.

8. Non-frontal solutions

It has earlier been assumed that the solution can be described in terms of a "front regime", an "Ekman regime" and an interior "Margules regime" (Fig. 3). In a complete description we should also include a "transition regime", describing the interaction of the front regime and the Ekman regime. For this regime the analysis becomes quite complicated. We are satisfied in the present case to apply certain integral relations in the transition regime, such as to prescribe the total mass and momentum transports, and no attempts are made to obtain the detailed solution there.

One general question of interest is whether or not the solution to the problem posed in 2 must be of front-type, as is assumed in (v), or if other types of solutions can exist. The following general arguments that only front-type solutions exist are presented. At $x = -\infty$ and $x = \infty$ we should have homogeneous conditions, and ordinary Ekman layers do exist. They have a difference in transports of order $V(\nu/f)^{\frac{1}{2}}$ and a transport of at least that magnitude must separate from the boundary and go into the interior of the fluid. If an interior region in which friction and diffusion play a role is not of boundary layer type, we can, by definition, scale the x and z coordinates by the same length. In the full heat conduction equation $J(\psi, T) - \varkappa\nabla^2 T = 0$ the length scale will now drop out. We find that $\Psi \sim \varkappa$, or $\Psi \sim \nu$ (since the Prandtl number is of order one), if the nonlinear term and the conduction term are of the same order. This must indeed be the case. For if the nonlinear term dominates, all friction and conduction effects in the problem can be neglected, which case is trivial. On the other hand, if the conduction term dominates, no solution at all can be found that satisfies the given boundary conditions. Now, from the Ekman layer a differential transport part of the order $V(\nu/f)^{\frac{1}{2}}$ must separate, and thus internal transports of at least that order must occur. When the viscosity is small enough, these results are necessarily inconsistent.

Although this forms no rigorous proof that the solution must be of front type, it demonstrates certainly that the front solution is the natural choice in the interior when matching to a boundary friction flow.

9. A similarity transformation

The system of equations (39–41) allows solution forms that separate in x^* and z^*. It is easily seen that the dependence on x^* must be linear. Thus we put

$$\psi^* = (x^* - x_0^*)M(z^*), \tag{43}$$

$$v^* = (x^* - x_0^*)N(z^*), \tag{44}$$

$$T^* = (x^* - x_0^*)Q(z^*). \tag{45}$$

Inserting in (39–41) yields

$$MM''' - M'M'' - M^{\mathrm{IV}} = -\cos\varepsilon(Q' + N'), \tag{46}$$

$$MN' - M'N - N'' = \cos\varepsilon\, M', \tag{47}$$

$$MQ' - M'Q - \frac{\varkappa}{\nu}Q'' = 0. \tag{48}$$

For small z^* one can use Taylor expansions for M, N and Q. These will together contain 8 arbitary constants. One finds

$$M = a_0 + a_1 z^* + a_2 z^{*2} + a_3 z^{*3}$$
$$+ \tfrac{1}{24}[6a_0 a_3 - 2a_1 a_2 + \cos\varepsilon(b_1 + c_1)]z^{*4} + \dots, \tag{49}$$

$$N = b_0 + b_1 z^* + \tfrac{1}{2}(a_0 b_1 - a_1 b_0 - \cos\varepsilon\, a_1)z^{*2} + \dots, \tag{50}$$

$$Q = c_0 + c_1 z^* + \frac{\nu}{2\varkappa}(a_0 c_1 - a_1 c_0)z^{*2} + \dots, \tag{51}$$

where a_i, b_i, c_i are the free constants.

For large positive and negative z^* one can generate asymptotic expansions for M, N and Q. An expansion where the stream function M tends to zero, while N and Q go to constant nonzero values N_0 and Q_0, is obtained by writing

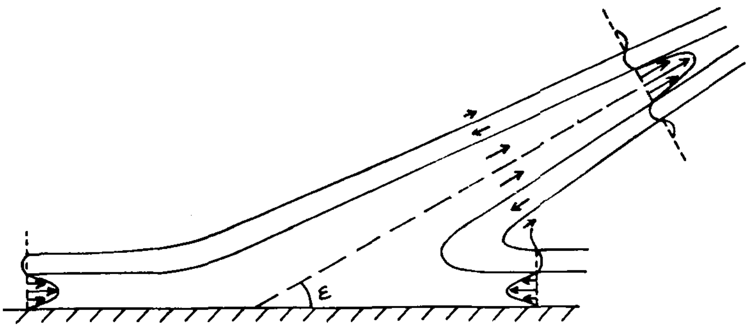

FIG. 6. Sketched streamline pattern corresponding to the similarity solution.

$$\left.\begin{aligned} M &= M_1 + M_2 + \dots, \\ N &= N_0 + N_1 + N_2 + \dots, \\ Q &= Q_0 + Q_1 + Q_2 + \dots, \end{aligned}\right\} \qquad (52)$$

where each term in the series should be small compared to the preceding for sufficiently large z^*. The second order terms M_1, N_1, Q_1 are found from the system

$$\left.\begin{aligned} M_1^{IV} &= \cos\varepsilon(Q_1' + N_1') \\ -(N_0 M_1' - N_1'') &= \cos\varepsilon M_1' \\ Q_0 M_1' - \frac{\varkappa}{\nu} Q_1'' &= 0 \end{aligned}\right\} \qquad (53)$$

Requiring that M_1, N_1 and Q_1 go to zero at $z^* = \infty$ one derives

$$M_1^{IV} + \alpha^4 M_1 = 0, \qquad (54)$$

where $$\alpha^4 = \frac{\nu Q_0}{\varkappa}\cos\varepsilon + (N_0 + \cos\varepsilon)\cos\varepsilon. \qquad (55)$$

If $\alpha^4 > 0$ one has a solution that decays for large positive z^*,

$$M_1 = C \exp\left(-\frac{\alpha}{\sqrt{2}} z^*\right) \cos\left(\frac{\alpha}{\sqrt{2}} z^* + \varphi\right), \qquad (56)$$

where C and φ are arbitrary. If $\alpha^4 < 0$, one has an exponentially decaying solution with one arbitrary constant. The solutions for N and Q are of the same type and they do not generate further free constants. Going to higher order terms in the expansion we can express all new constants in terms of C and φ. A similar solution is valid for large negative z^*.

To make a complete joining of the Taylor expansion and the asymptotic expansions for both positive and negative z^* altogether 16 constants are needed (we need continuity in M, M', M'', M''', N, N', Q, Q'). In general, only the case $\alpha^4 > 0$ will be possible, and this put certain restrictions on the parameters. The complete joining is, of course, complicated to make, and so far has not been attempted. A rough estimate considering only the leading terms yields a streamline pattern of the form seen in Fig. 6. The conditions in the transition region are not known at all, and the streamlines are simply interpolated over that region. It is of interest to note that the equation (56) is in form identical with the equation for the stream function in the Ekman layer, and gives the possibility of joining the streamline pattern of the front and the Ekman layer in a smooth way. The similarity solution for the front considered here is the only existing one of the form $A(x^*)$ $B[z^* k(x^*)]$ (meaning that the amplitude A and the front thickness k^{-1} are variable along the front). The solution is not quite satisfactory, since the boundary conditions at infinity cannot be exactly satisfied. In the Margules region the temperature and the transverse velocity will not tend to constant values but to values that vary with x^*. This variation is small over distances of the order of the boundary layer thickness but will, of course, affect the overall solution. For this reason an attempt is being made to tackle the original equations (39–41) by a numerical method. This will also have the advantage that the transition region can be included. Some preliminary results indicate that the general picture presented here is correct, but detailed solution is still lacking.

APPENDIX

Momentum transport equation

We start from the momentum equation (1) in Sect. 3 and subtract from this the Margules solution, valid at great x. This latter solution is defined by the geostrophic relation $fv_0 = (1/\varrho_0)p_{x_0}$ for negative z', and by $fv_1 = (1/\varrho_0)p_{x_1}$ for positive z'. If the deviations in v and p are denoted by v'' and p'', we have

$$(uu)_x + (wu)_z - fv'' = -\frac{1}{\varrho_0}p_x'' + \nu\nabla^2 u, \qquad (1)$$

or

$$\frac{\partial}{\partial x}\left(u^2 + \frac{p''}{\varrho_0} + \nu u_x\right) = -(wu)_z + fv'' + \nu u_{zz}. \qquad (1\,a)$$

Integrating vertically from $z=0$ (at the wall) to $z=\infty$ one has

$$\frac{\partial}{\partial x}\int_0^\infty \left(u^2 + \frac{p''}{\varrho_0}\right)dz = f\int_0^\infty v''dz - \nu(u_z)_{z=0}, \qquad (2)$$

since $\int_0^\infty udz$ is a constant (w should go to zero at infinity) and $(u_z)_{z=\infty} = 0$. Introducing the momentum transport $M = \int_0^\infty (u^2 + p''/\varrho_0)dz$ and the ground stress τ_0 and integrating from $x = x_a$ to $x = x_b$, where x_a, x_b lie on each side of the transition region (Fig. 7), one finds

$$M_b - M_a = f\int_{x_a}^{x_b} dx \int_0^\infty v''dz - \frac{1}{\varrho_0}\int_{x_a}^{x_b}\tau_0 dz. \qquad (3)$$

M_a has a contribution from the Ekman layer, M_b has contributions from both the Ekman layer and the front layer. There is no contribution from the Margules region, since u, v'' and p'' vanish there. We want to show that the terms in the right-hand side of (3) are negligible. We first consider the order of the Ekman contribution to M. In the Ekman layer, of thickness $(\nu/f)^{\frac{1}{2}}$, the u-velocity is of order V, and further $p''=0$. Thus the Ekman contribution is of order $V^2(\nu/f)^{\frac{1}{2}}$ (a more precise estimate is $(v_1^2 - v_0^2)(\nu/f)^{\frac{1}{2}}$). The velocity v'' is of order V both in the Ekman layer and the front layer. The horizontal range of integration $x_b - x_a$ as well as the vertical range of integration from which an essential contribution is obtained should be a few times the boundary layer thickness. For this we use the Ekman depth $(\nu/f)^{\frac{1}{2}}$ (the front layer thickness is known to be at most of that order, see Section 7). The Coriolis term in the right-hand side of (3) is then at most of order $fV\ (\nu/f)$, and its magnitude relative to the Ekman contribution of M is at most of order $(f\nu)^{\frac{1}{2}}/V$, that is of order δ/L as found by (34), (35) in Sect. 7, and the condition the Prandtl number is of order one. Thus the Coriolis term in (3) can be neglected. The ground stress τ_0 is by the Ekman theory of order $fV\varrho_0(\nu/f)^{\frac{1}{2}}$ and the second term in the right-hand side of (3) is at most of order $fV(\nu/f)$ and again negligible against the Ekman contribution to M. We will finally note that p''/ϱ_0 is negligible also in the front region. For the pressure gradient p_x'' can at most be of the order of the total Coriolis acceleration, i.e. of order $f\varrho_0 V$, and integrating the gradient from the Margules region, where p'' vanishes, one finds that in the front p''/ϱ_0 can at most be of order $fV(\nu/f)^{\frac{1}{2}}$, that is again a negligible quantity.

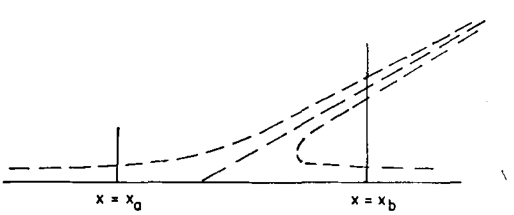

FIG. 7. Integration ranges in the momentum transport computation.

We can thus conclude that the difference in $\int u^2dz$ computed over the Ekman layers on both sides of the front must be balanced by the corresponding integral computed across the front, to the first order for small ν. It follows from this that $v_1^2 > v_0^2$, i.e. the geostrophic velocity must have larger absolute value on the warm side than on the cold side of the front.

The above derivation assumes that the dimensions of the transition region are at most of the order of the individual boundary layers, and that the ground stress in the transition region is at most of the order of the Ekman ground stress. These assumptions seem quite plausible but their ultimate justification can, of course, come only from a knowledge of the complete solution to the problem.

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Table 1.

(C.G.S.-units except L and S)

	g	f	α	ΔT	ε	ν	L	δ
Laboratory case	10^3	1	2×10^{-4}	50°	1	10^{-2}	10 cm	1 mm
Atmospheric case	10^3	10^{-4}	3×10^{-3}	10°	10^{-2}	10^5	30 km	300 m
Oceanic case	10^3	10^{-4}	2×10^{-4}	5°	10^{-2}	10^2	10 km	10 m

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