A note on the propagation of tidal waves in a viscous fluid in a rotating system

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ABSTRACT

The results presented in this note are an extension of classical results on tidal wave propagation. It is shown that the classical results obtained for an inviscid fluid are slightly modified when small viscosity is present. When large viscosity is present, however, it is shown that viscosity and rotation can interact to either prohibit, or encourage, wave propagation.

1. Introduction

It has long been known that the effect of viscosity on any fluid motion is characterized by energy dissipation. LAMB (1932), for example, in discussing the propagation of tidal waves in a non-rotating system, has shown that viscosity damps waves which are capable of propagating, and prevents the propagation of these waves if the viscosity is sufficiently large. This latter situation occurs in highly viscous fluids such as treacle or pitch. These results display one role played by viscosity, namely that of inhibiting or preventing wave motion.

More recently, in the field of hydrodynamic stability, it has been shown that viscosity can play a much different role. In plane Poiseuille flow it is known that viscosity has the dual role of damping wavelike disturbances, while at the same time extracting energy from the basic flow and thus producing instability (LIN, 1955). For this particular example viscosity is, in fact, the only mechanism of instability, and the corresponding inviscid flow is stable. In the problem now to be considered it will be shown that viscosity can again have a dual role.

2. The governing equations

Let us first denote the fluid velocity in the orthogonal directions (x,y,z) by the standard symbols (u,v,w), where the z-axis is vertically upwards. In order to obtain the tidal wave equations, we shall, as is customary, neglect the substantial derivative Dw/Dt in the vertical momentum equation, and neglect the convective terms in the horizontal momentum equations. The effect of viscosity on the vertical component of the motion is also neglected, which is in agreement with the neglected term $\partial w/\partial t$. The viscosity is, however, retained in the horizontal momentum equations, which leads to the viscous system (continuity and momentum)

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0, \tag{1}$$

$$p = -\varrho g(z-\zeta), \tag{2}$$

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \zeta}{\partial x} = \nu\nabla^2 u, \tag{3}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \zeta}{\partial y} = \nu\nabla^2 v. \tag{4}$$

The density and kinematic viscosity are denoted by ϱ and ν, respectively, while the Coriolis parameter, assumed constant, is denoted by f. The fluid surface is located at $z=\zeta$, and at any point the depth of the fluid is given as $z=-H$.

We shall, in order to seek tidal wave solutions of equations (1) through (4), write the time dependence of u, v, and ζ, in the form

$$\left.\begin{aligned} u &= e^{i\sigma t}\,\bar{u}(x,y),\\ v &= e^{i\sigma t}\,\bar{v}(x,y),\\ \zeta &= e^{i\sigma t}\,\bar{\zeta}(x,y).\end{aligned}\right\} \tag{5}$$

In terms of these new variables, which have spatial dependence only, we find from (3) and (4),

$$\nu^2\nabla^4\bar{u} - 2i\sigma\nu\nabla^2\bar{u} + (f^2 - \sigma^2)\bar{u}$$

$$= g[\nu\nabla^2(\partial\bar{\zeta}/\partial x) - i\sigma(\partial\bar{\zeta}/\partial x) - f(\partial\bar{\zeta}/\partial y)], \quad (6)$$

where $\bar{v} = f^{-1}[i\sigma\bar{u} + g(\partial\bar{\zeta}/\partial x) - \nu\nabla^2\bar{u}]$. (7)

We shall further restrict our attention to the propagation of waves travelling in the x-direction in a viscous fluid of constant depth $H(x,y) = H_0$. Thus we consider the particular forms

$$\left.\begin{aligned} \bar{\zeta} &= Ae^{-ikx}, \\ \text{and} \qquad \bar{u} &= Be^{-ikx}, \end{aligned}\right\} \quad (8)$$

where A and B are constants. The horizontal component of the velocity $\bar{v}(x)$ is then determined by (7). From the relations (6) and (8) it follows that the equation

$$\sigma^3 - 2i\nu k^2\sigma^2 - (gH_0k^2 + \nu^2k^4 + f^2)\sigma + igH_0\nu k^4 = 0 \quad (9)$$

governs the propagation of waves of the form (8), where the continuity equation (1) requires

$$A = H_0kB/\sigma. \quad (10)$$

Since all of the results which follow will be derived from properties of the basic equation (9), it is desirable to find a suitable non-dimensional form. Because particular interest will center about large and small viscosity, we first define the non-dimensional viscosity

$$N = g^{-\frac{1}{2}}H_0^{-\frac{3}{2}}\nu \quad (11)$$

so that the other parameters become

$$\left.\begin{aligned} \sigma^* &= g^{-\frac{1}{2}}H_0^{\frac{1}{2}}\sigma, \\ f^* &= g^{-\frac{1}{2}}H_0^{\frac{1}{2}}f, \\ k^* &= H_0k, \end{aligned}\right\} \quad (12)$$

where the starred quantities are non-dimensional. For convenience we shall hereafter delete the asterisks, and thus in all equations which follow it will be understood that the quantities are non-dimensional. Equation (9) can then be written

$$\sigma^3 - 2iNk^2\sigma^2 - (k^2 + N^2k^4 + f^2)\sigma + iNk^4 = 0. \quad (13)$$

3. The effect of viscosity

The nature of the roots to (13) displays most clearly the effect of viscosity. These roots can be examined more easily by first making the change of variable

$$\sigma = -i\gamma \quad (14)$$

which leads to an equation having all real, positive coefficients,

$$\gamma^3 + 2Nk^2\gamma^2 + (k^2 + N^2k^4 + f^2)\gamma + Nk^4 = 0. \quad (15)$$

The discriminant Δ of this equation, which is a cubic in the variable γ, can be determined by classical methods (LOVITT, 1939). After some algebraic manipulations, we find for the discriminant

$$\Delta = N^2(1 - 4N^2f^2)k^8 + 4(5N^2f^2 - 1)k^6$$

$$- 4f^2(2N^2f^2 + 3)k^4 - 12f^4k^2 - 4f^6. \quad (16)$$

There are three cases of interest:

(i) $\Delta < 0$. One of the roots is real, while the other two roots are complex conjugates of each other.

(ii) $\Delta = 0$. All of the roots are real, and two of the roots are equal.

(iii) $\Delta > 0$. All of the roots are real and distinct.

Let us now consider a system of tidal waves of the form (8) which have been set up in a viscous fluid in a rotating system. In all three of the aforementioned cases it can be mathematically shown, as would be anticipated on physical grounds, that viscosity dissipates energy and hence damps the motion. (The proof follows immediately from the fact that all of the coefficients of equation (15) are real and positive.) If the discriminant Δ is negative, as in the first case, the effect of viscosity is simply to damp the waves as they move in the x-direction. In particular, if we consider the case of small viscosity and recall that the tidal wave approximation requires

$$k \ll 1, \quad (17)$$

we find $\Delta \cong -4f^6 - 12f^4k^2$. (18)

Thus in this situation the tidal waves possess a similar behavior to that found for a slightly viscous fluid in the absence of rotation.

TABLE 1. *Summary of wave character (Section 3).*

Viscosity	For $f=0$	For $f\neq 0$
$N=0$	Propagating	Propagating
$0<N\ll 1$	Propagating, damped	Propagating, damped
$N\gg 1$	Non-propagating	(i) $N^2f^2\gg 1$ Propagating, damped (ii) $N^2f^2\ll 1$ Non-propagating

We shall now consider the case of a highly viscous fluid such as treacle or pitch. In the absence of rotation it has been shown by Lamb that there exists "a slow creeping of the fluid towards a state of equilibrium with a horizontal surface", and thus tidal waves are not capable of propagating. When rotation is present, we observe from (16) that, if the fluid is sufficiently viscous, the discriminant can be approximated by the expression

$$\Delta \cong -4N^4f^2k^8. \tag{19}$$

Again the discriminant is negative, and thus we find the somewhat surprising result that viscosity can interact with rotation, and thus lead to tidal wave propagation. It must, of course, be admitted that the viscosity must be exceedingly large in order that the expression (16) can be approximated by (19) in spite of the smallness of k.

If the discriminant is positive, or if it vanishes, tidal waves are not capable of propagating. We may now consider the case of large viscosity and small rotation described by the relations

$$\left.\begin{aligned} & N^2f^2 \ll 1, \\ \text{while} \qquad & N \gg 1. \end{aligned}\right\} \tag{20}$$

From (16) we find

$$\Delta \cong N^2k^8, \tag{21}$$

which shows that the classical results for the case of large viscosity and no rotation remain valid in the presence of the small rotation described by (20).

A summary of the results derived in this section is given in Table 1.

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