An entraining jet model for cumulo-nimbus updraughts

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ABSTRACT

A model of a cumulo-nimbus updraught is presented, based on a steady-state, turbulent, condensing plume, entraining environmental air according to the simple law that the inflow velocity at any height is proportional to the upward velocity of the plume.

With this model, the shape of the cloud as well as its other properties follow from the dynamics, and cloud development is size-dependent. The virtual temperature excess at cloud base is taken as zero; this seems the only appropriate assumption for a steady-state plume model. The thermodynamical treatment is made with few approximations, and the effect of freezing on the dynamics is included in a simple fashion. Some preliminary calculations result in realistic cloud shapes and reasonable values for the internal properties.

1. Introduction

In their description of the succession of convection cells which make up a thunderstorm, BYERS & BRAHAM (1949) have pointed out the important effect which the formation and fall of precipitation have on the circulation of air in the cell. It goes without saying that the growth of the precipitation particles themselves depends on the air motions. In thunderstorms, therefore, the problems of convection dynamics and the growth of precipitation can hardly be separated. In the case of large hail, which has a terminal velocity comparable with the updraught speeds found in cumulo-nimbi, theories aimed at describing particle growth depend critically on the assumptions made about updraught speeds and their variation in space and time. Thus convection dynamics is important not only because of its inherent interest and because of its significance for the vertical transport of heat, moisture and momentum, emphasized by RIEHL (1950) and MALKUS (1952), but also in connection with the growth of thunderstorm precipitation, and especially of large hail.

Although in the later mature stage of a cell's history the general dynamics of the system is influenced by precipitation so that the problem becomes extremely complex, there is a period of initial growth, the "cumulus stage" of BYERS & BRAHAM (1949), in which these effects may be negligible. It is obvious that a study of the whole problem must be preceded by consideration of this earlier, simpler phase. Even this, however, presents problems about which there is at present no general agreement. The purpose of this paper is to present a treatment (following MORTON, 1957) which differs from some earlier formulations in being based on a clearer physical model and involving no disposable parameters. An important feature of the model is that it predicts the shape of the cloud as well as its other properties. Cloud development is found to depend primarily on the properties of the atmosphere and the mass flux of the updraught as it passes through the condensation level.

2. A steady-state entraining jet

DISCUSSION OF VARIOUS MODELS

According to the measurements of the Thunderstorm Project (BYERS & BRAHAM 1949), a thunderstorm updraught is most commonly found, at any rate in its lower layers, to have a width of about 5000 ft. During the cumulus stage, it extends continuously from cloud base to cloud top, often at a height exceeding 25,000 ft. This rather tall and slender current immediately suggests an analogy with a buoyant, turbulent plume or jet. Indeed, the majority of treatments of thunderstorm convection have

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been in terms of the steady-state jet model introduced by SCHMIDT (1947) and STOMMEL (1947). A notable exception is the discussion of MALKUS (1960) of the growth of cumulonimbus towers in a hurricane, using the spherical vortex model of LEVINE (1959). This approach is a form of the bubble or thermal model of SCORER & LUDLAM (1953) which in fact constitutes the only alternative so far proposed to the steady-state jet model. LUDLAM (1958) also has envisaged the updraughts in a thunderstorm as a succession of thermals or bubbles which form in the vicinity of cloud base. It is difficult, however, to form any clear physical picture of this process as a representation of a continuing current, except by regarding the individual bubbles as an alternative way of describing the general turbulence present in a plume.

Each of the two main types of mixing model, the jet or plume and the bubble, contains features which one feels should be incorporated in a full theory of cloud growth, but each has certain shortcomings. The plume theories disregard any development in time of the depth or width, and they neglect any possible mixing at the top of the cloud. The bubble theories, on the other hand, ignore the possibility of the relatively steady updraught driven by a continuing source of heat below cloud base or within the cloud, which seems to be implied by the existence of the flat bases which are often observed. In fact it seems that an improved model of cloud growth should include features from both these types of theory.

A "starting plume" model has recently been proposed (TURNER, 1962) which takes a first step towards this integration. This concentrates on the advancing front of a plume which is being established *ab initio* in uniform surroundings by suddenly starting to supply buoyant fluid from a steady source. At the front of the advancing fluid is a cap which looks very like a thermal, but whose total buoyancy is increasing because buoyant fluid is being added to it from the rear. Behind the cap at any instant, the flow is like that in a steady plume, until it overtakes the cap and becomes incorporated in it.

Though this model is strictly applicable only to dry convection or to small clouds, it does suggest that the flow everywhere except right at the top of a cloud can be regarded as quasisteady. Further consideration with this model in mind also indicates in which cases it is likely that either the bubble or jet model separately could be used to describe clouds as a whole. For small clouds, whose depth is about equal to their diameter, the mixing at the top will be dominant, and the "thermal" model will be appropriate. For intermediate sizes, mixing both at the top and at the sides must be considered. For large clouds with strong updraughts, much taller than they are wide, mixing over the top will be significant only for a small fraction of the depth of the cloud, and so the plume or jet model will be the more appropriate one for these large clouds. An important implication of the quasi-steady "starting plume" model is that, provided the cloud is deep, updraughts in the body of the cloud (i.e. everywhere except very near the top) can be calculated using a steady-state model even at intermediate stages of its development, despite the fact that the top of the cloud is developing in time. Thus although a thunderstorm cell as described by Byers & Braham grows continuously throughout the cumulus stage, so that it is not really a steady-state phenomenon, it seems likely that over the greater part of its depth at any moment it may be fairly well represented by the steady-state jet model.

STOMMEL's original treatment (1947) of the jet model was empirical, and aimed at finding out what entrainment rates would produce the height distributions of temperature observed in trade-wind cumuli by BUNKER *et al.* (1949). A theoretical approach to the problem was foreshadowed by SCHMIDT (1947), who showed that the resistance due to entrainment is exactly like that experienced by a jet in a liquid. This was first applied in detail by MORTON (1957), who used the dynamical principle (which has been suggested by dimensional arguments and supported by experiment in other contexts) that the inflow velocity from an environment at rest at the edge of the plume or jet is proportional to the upward velocity at any level.

The dimensional arguments which originally suggested this simple entrainment law in the case of dry convection from a maintained source (BATCHELOR, 1954) do not, of course, apply in the case of cloudy atmospheric convection, and there is only indirect experimental evidence on this point. The extension of the law to the case under consideration amounts to assuming that the structure and intensity of turbulence at the shear face (which obviously must determine the rate at which environmental air is entrained) is determined solely by the local velocity difference between the updraught and the environment and is effectively independent of the previous history of the rising air. This seems to be a reasonably plausible first approach to the problem. A second possibility which is neglected is that the turbulence within clouds may derive a significant proportion of its energy from temperature and density differences between neighbouring parcels of air. Even though the updraught may be uniform, such temperature differences must arise when dry environmental air is entrained piecemeal into a cloud. During the process of mixing between such dry air parcels and the cloudy air, some liquid water must be evaporated with consequent cooling of the mixing region. There is, of course, no corresponding effect of this kind in the case of dry convection.

Using this general entrainment principle as a starting point, all the properties of the plume may be evaluated up to condensation level and above. Morton found that the behaviour depends markedly on the condensation level, and is quite different according as this occurs above or below a critical height which lies some distance below the level where the stability of the environment would have brought the plume to rest in the absence of condensation. This theory also shows that the proportional increase of mass of a cloud due to entrainment depends markedly on its size, in consonance with the observations of RIEHL (1954) and MALKUS (1960). Morton did not carry his calculations far beyond cloud base, and in fact specifically restricted his equations to small total heights by making various simplifying approximations appropriate to such cases. The main purpose of this paper is to derive and solve equations which depend on this entrainment law, but which are valid over large height intervals.

THE NATURE OF THE ENTRAINMENT PROCESS

On the view taken here, entrainment occurs only by the continuous involvement of stationary environmental air in the peripheral eddies of the updraught. This results in the turbulent region growing at the expense of the non-turbulent, since all the air in a small region of mixing necessarily becomes turbulent and rapidly comes to share the upward momentum and other properties of the updraught. In the case of laboratory jets, it is observed (see, for example, TOWNSEND, 1956) that at any instant the edge of the jet is a sharp boundary between stationary environmental air and air with properties characteristic of the jet. The Gaussian profiles which are usually observed relative to fixed axes are good time-average representations of the horizontal variations, but they are attributable to the lateral fluctuations of the jet or plume boundary. The mean profile relative to the plume axis is likely to be more nearly rectangular in form.

Only when the environment is turbulent to a degree comparable with the updraught will there be an interchange of matter in both directions. Thus near the ground, where the shear-induced turbulence may be dominant, the motion of buoyant elements rising from a heated surface may be regarded almost as part of the general turbulence, and in this case, transfer will occur both into and out of the buoyant elements. At cloud levels, the environmental turbulence in shallow layers of strong wind shear may be comparable with that in the updraught, but in general this is not so; the association between visible cloud and turbulence as experienced by an aircraft in a convective layer is very close indeed.

This treatment will be restricted to the case of no general wind shear, so that the updraught will remain vertical, and *a fortiori* the environment non-turbulent. The notion of "detrainment", or loss of material from the updraught will not therefore be introduced. "Dynamic entrainment", which has sometimes been regarded as the net entrainment, or the difference between entrainment and detrainment, also seems to be out of context here. It has been defined as the entrainment required by continuity of mass, assuming that the cloud shape is prescribed in advance. For example, if the cloud is assumed to be cylindrical, the acceleration caused by buoyancy may require an inflow through the cylinder. Since in fact a cloud is not restrained by walls, this concept does not describe a physical reality; it was indeed used first by HOUGHTON & CRAMER (1951) as a heuristic device intended to build into the model the observed fact that convective clouds often have an approximately cylindrical shape over a large fraction of their depth. On the view adopted here, however, the imposition of a prescribed shape on the updraught amounts to a redundant constraint, for it is sufficient to specify the entrainment which, physically, is a consequence only of the shear between the upcurrent and the environment.

In the model used here, the variation with height of the cross-section of the updraught follows from the dynamics of the system. Moreover, as in the bubble model, the overall rate of dilution decreases as the size increases, and the heights attained by clouds consequently depend on their diameters. The constant α relating inflow velocity to updraught speed can be taken with reasonable accuracy from laboratory measurements on the spread of jets, so that a major arbitrariness of former treatments is mitigated. This being granted, the theory involves no disposable parameters. For practical purposes the numerical value of α is not very critical, since it will be shown that a change of α is very nearly equivalent to an inverse change in cloud radius.

Some predictions may be made on the basis of earlier calculations of MORTON (1957) and of MORTON, TAYLOR & TURNER (1956). Since buoyancy is released as a result of condensation, with consequent acceleration, the normal spread of a buoyant plume will not always occur; in the lower part of the updraught one must expect a reduced rate of spreading, or even a reduction of cross-section with height. Near cloud top in the upper troposphere the release of buoyancy by condensation becomes small, and since the ambient lapse rate is always less than the dry adiabatic, the updraught must be expected to become negatively buoyant, decelerate, and spread sideways. This simple picture of a cumulo-nimbus updraught is in consonance with the observed "hour-glass" shapes of these clouds as described in "Thunderstorm Rainfall", Hydrometeorological Report No. 5 of the U.S. WEATHER BUREAU (1947), where a similar interpretation in terms of accelerating and decelerating currents is given.

THE CONDITIONS AT CLOUD BASE

It seems clear that thunderstorms derive their energy almost entirely from the latent heats of condensation and of freezing, and are relatively independent of local sources of heat at the surface. Hence an adequate treatment of the turbulent, buoyant, condensing plume should include most of the essential physics of the problem. The treatment given here consists of integrating the governing equations with height as the independent variable, the boundary conditions being given by the properties of the updraught at cloud base.

The updraught at this level is specified by its radius, velocity and temperature. In consonance with the view that thunderstorm updraughts do not depend on surface heat sources, but once triggered merely release the energy stored in the environment, the updraught will be assumed to have, at cloud-base level, the same virtual temperature as the environment (when the environment is unsaturated, this indeed implies that the updraught is slightly cooler). This assumption seems to be the only appropriate one when considering these extensive and continuing currents of air. A positive virtual temperature anomaly could arise only if the source region for the updraught were some distance below cloud base in a super-adiabatic layer—an unlikely possibility, especially in the late afternoon or evening or, as often happens, when general cloudiness has for some time prevented direct insolation of the ground over which new cells of the storm are forming. In this respect, the plume model differs essentially from the bubble or thermal model which, as remarked earlier, seems more appropriate for fair weather cumulus.

In those cases where the environment is considered to be unsaturated, it is implied that, although the updraught at cloud-base level is at the same virtual temperature as the surroundings, its moisture content is significantly higher; this would be so if the updraught originated in a well-mixed moist layer, the top of which lay a short distance below cloud base, and if this layer were surmounted by a somewhat drier and stabler region through which the clouds grew. This is consistent with some detailed soundings below and between rather smaller convective clouds (BUNKER *et al.*, 1949; JAMES, 1953) which have indicated that just below the bases of cumuli there is frequently a stable layer within which the mixing ratio of water vapour decreases rapidly with height.

Given a sounding and cloud-base level the problem is then completely specified by the radius and velocity of the updraught at cloud base, since it is taken to be saturated at this level, and at the same virtual temperature as the environment. It will in fact be shown that cloud growth is determined primarily by only one variable—the mass flux transported by the updraught. No way has been found, however, to relate the flux of the updraught at cloud base to the pre-existing state of the atmosphere, or to the energy realized in the cloud above, and it has been necessary to choose values of the mass flux suggested by observation. It is assumed that the environment remains unchanged, and that the compensating downflow occurs at a great distance.

3. The equations

A "top-hat" profile will be assumed, that is, one in which all properties are constant across the updraught. This has been assumed in all previous treatments of cloudy convection, and it is somewhat more convenient for calculations than, for example, a Gaussian profile. In addition, there is the experimental evidence mentioned earlier that it is a more realistic description of the properties relative to the plume axis at any instant. Direct support for this idea has been obtained from measurements in clouds; WARNER (1955), for example, has shown that the liquid water concentration in moderate-sized cumuli often rises sharply from zero at the edge of the cloud, rather than rising and falling gradually. All condensed materials are supposed to have negligible terminal velocities, so that they move with the air. It is further assumed that no pressure disturbance occurs in the cloud; in other words, that the isobaric surfaces are horizontal planes. Plausible values for cloud buoyancy—suggested by observation or by the computations of several authors—indicate that in the absence of vertical accelerations, the pressure deficit below a thunderstorm would be of the order of 5–10 mb. Typical pressure falls observed by BYERS & BRAHAM (1949) are of the order of 1 mb. On these grounds it seems likely that the hydrostatic assumption made here is not a serious cause of error.

CONTINUITY

The equation of continuity (in a medium of varying density) for a plume which is adding mass at each level at a rate proportional to the local upward velocity and to the surface area is

$$\frac{d}{dz}(b^2 u \varrho) = 2b\alpha u \varrho_0, \tag{1}$$

where b, u, ϱ are the radius, velocity, and density in the plume, z is the height, ϱ_0 is the density in the environment and α is the entrainment constant.

After a careful survey of the literature, MORTON (1959) concluded that the most accurate value of α could be obtained from experiments on jets in air, and adopted the value (appropriate to top-hat profiles) of $\alpha = 0.116$, which was deduced from published experimental results available at that time. More recently, RICOU & SPALDING (1961) have suggested why previous measurements of entrainment into jets might be inadequate, and have described a new method which measures the amount of entrained air directly; they arrive at a value equivalent to $\alpha = 0.080$. They also report a few experiments with buoyant plumes which suggest that the rate of entrainment is higher than it is in jets.

In this paper we shall take $\alpha = 0.10$, a value which is likely to be within say 15% of the true value, and therefore it is specified much more closely than has usually been the case in cloud-mixing models. The most important point, however, is not the accuracy of this estimate but the fact that it has been made independently of measurements in clouds. As we mentioned earlier, the value of α used is not critical for the following reason. If we define the quantity $M = b^2 u \varrho$ proportional to the mass flux at height z, then the continuity equation (1) may be written approximately as

$$\frac{1}{M}\frac{dM}{dz} \approx \frac{2\alpha}{b},$$

neglecting the difference in density between the updraught and its environment. Thus the proportional rate of entrainment per unit height, and therefore all the properties of the cloud which depend on mixing, such as the velocity, buoyancy and liquid water concentration, will be the same functions of height, provided the ratio α/b is kept fixed, and changes of α may be compensated by changes in b. This result will be demonstrated directly in a later section.

MOMENTUM

Let σ be the concentration of liquid water, gm/gm. The equation expressing the increase of momentum flux due to the buoyancy forces is

$$\frac{d}{dz}(Mu) = gb^2(\varrho_0 - \varrho - \varrho\sigma), \tag{2}$$

where g is the acceleration due to gravity.

BUOYANCY

The variation of the temperature of the updraught follows from the conservation of energy. Neglecting the internal energy of condensed material, and the difference between the specific heats of air and water vapour, this gives, in the case of an all-water cloud

$$\frac{dT'}{dz} = (T_0' - T')\frac{1}{M}\frac{dM}{dz} - \frac{\Gamma\varrho_0}{\varrho} - \frac{L}{c_p}\frac{1}{M}\left(\frac{dMq}{dz} - q_0\frac{dM}{dz}\right) + \frac{1}{\gamma}\left\{\frac{d}{dz}(\delta T) + (\delta T - \delta T_0)\frac{1}{M}\frac{dM}{dz}\right\}, \tag{3}$$

where
T', T_0' are the virtual temperatures inside and outside the cloud,
δT and δT_0 are the differences between temperature and virtual temperature,
Γ is the dry adiabatic lapse rate,
L is the latent heat of vaporization,
q, q_0 are the mixing ratios of water inside and outside the cloud,
c_p and γ refer to air.

The introduction of freezing processes will be discussed later. With the approximations mentioned above, the energy equation is more accurately expressed if the temperature variation of L is ignored, the value appropriate to some mean temperature being inserted. However, in the later manipulation of these equations, L reappears via the Clausius–Clapeyron relationship, and here it is treated as a function of temperature and written $L(T)$.

CONTINUITY OF WATER SUBSTANCE

Finally, the continuity of water in the vapour and condensed forms leads to the equation:

$$\frac{dM\sigma}{dz} = q_0\frac{dM}{dz} - \frac{d}{dz}(Mq), \tag{4}$$

using the previous notation.

These four equations may be brought to a more suitable form for computation by using the Clausius–Clapeyron equation

$$\frac{dT}{dz} = \frac{ART^2}{\varepsilon L(T)e}\frac{de}{dz}, \tag{5}$$

where
T is the absolute temperature,
R is the gas constant for dry air, per gram,
A is the reciprocal of the mechanical equivalent of heat,
e is the vapour pressure,
ε is the ratio of the density of water vapour to that of air,
p is the pressure,
and $L(T)$ is now evaluated as a function of temperature. Virtual temperature is related to temperature as follows:

$$T' = T(1 + \lambda q), \tag{6}$$

with

$$\lambda = \left(\frac{1}{\varepsilon} - 1\right). \tag{7}$$

The final form of equations used for computation follows after some algebra. They are:

$$\frac{dM}{dz} = \frac{2\alpha p^{\frac{1}{2}}}{R^{\frac{1}{2}} T_0'}(MuT')^{\frac{1}{2}}, \tag{8}$$

$$\frac{du}{dz} = \frac{g}{u}\left(\frac{T'}{T_0'} - 1 - \sigma\right) - \frac{u}{M}\frac{dM}{dz}, \tag{9}$$

$$\frac{dT'}{dz} = \frac{\left[\frac{D}{M}\frac{dM}{dz}\left\{(T_0' - T') - \frac{L}{c_p}(q - q_0) + \frac{(\delta T - \delta T_0)}{\gamma}\right\} - \frac{D\Gamma T'}{T_0'} - \frac{L}{c_p}\frac{gq}{RT_0'}(1 + \lambda q) + \frac{\lambda}{\gamma}\frac{gqT'}{RT_0'}\right]}{\left[D + \frac{L}{c_p}\frac{q\varepsilon L(T)}{ART^2} - \frac{\lambda q}{\gamma}\left(\frac{\varepsilon L(T)}{ART} + 1\right)\right]} \tag{10}$$

$$\frac{d\sigma}{dz} = (q_0 - q - \sigma)\frac{1}{M}\frac{dM}{dz} - \frac{q}{D}\left\{\frac{\varepsilon L}{ART^2}\frac{dT'}{dz} + \frac{g}{RT_0'}(1 + \lambda q)\right\}, \tag{11}$$

where

$$D = 1 + \lambda q + \frac{q\varepsilon\lambda L(T)}{ART}. \tag{12}$$

The basic dependent variables are thus mass flux, velocity, virtual temperature and liquid water concentration.

INTEGRATION OF EQUATIONS

The equations have been integrated using a standard Runge–Kutta routine on an electronic computer, with 50 m steps for the first few km and then 100 m steps. The usual check—integrating the equations with increments only half as big—indicated that the errors were small, hardly ever exceeding one unit in the fourth significant digit.

For linear lapse rates of virtual temperature, which will be the only cases discussed here, the pressure at height z can be calculated from the relation

$$\frac{T_0' - \beta z}{T_0'} = \left(\frac{p}{p_0}\right)^{R\beta/g}, \tag{13}$$

where the virtual temperature decreases uniformly with height at rate β. For more complicated temperature profiles, the hydrostatic equation must be integrated.

The equations (10) and (11) assume a knowledge of T, T_0 and q, q_0, and the calculation of either temperature or mixing ratio from virtual temperature requires a use of the other quantity. The Clausius-Clapeyron relation might have been used again here, but it is more accurate to employ an empirical expression such as the following, derived from that given by TETENS (1930)

$$e_s(T) = \exp\left[\frac{19.079T - 4782.9}{T - 35.9}\right] \text{mb}, \tag{14}$$

where T is the absolute temperature and $e_s(T)$ is the saturation vapour pressure. An iterative procedure, using (6) and (14), was devised to give a fast convergence to the required values.

THE INTRODUCTION OF ICE

It is generally believed that the formation of ice in large clouds has a significant effect on their dynamics. A few cloud-seeding experiments seem to have resulted in a cumulus growing rapidly into a cumulo-nimbus, apparently as a result of the release of latent heat of freezing. It seems desirable therefore to take freezing into account in computing buoyancy of the updraught. There are, however, virtually no measurements relating to the way in which freezing proceeds in an updraught. General experience indicates that ice begins to appear at about the $-15°$C level and at around $-40°$C the cloud is wholly glaciated. Several major problems arise if an attempt is made to compute the freezing process; it is not generally agreed whether freezing nuclei operate by causing the freezing of droplets in which they have become immersed, or by sublimation; observations of spectra of ice-forming nuclei indicate that their occurrence is highly variable; the growth of ice crystals depends on their crystal habit as well as their size; it depends also on the ambient vapour pressure which will not be even approximately equal to the saturation vapour pressure over water, except at the beginning of the glaciation process.

In view of these difficulties, which seem at present quite insuperable, a very simple model of glaciation was used: the proportion of ice to total condensed material is assumed to vary linearly with temperature from zero at some initiating temperature ($-15°$C usually) to unity at $-40°$C. The vapour pressure in the cloud was assumed to be the mean of that over water and ice, weighted by their respective mass concentrations.

Thus, writing σ_w for the concentration of water (gg^{-1}) and σ_i for that of ice,

$$\sigma_w + \sigma_i = \sigma,$$

where $\sigma_w = f(T)$, f being linear, and

$$e_m = \frac{e_w \sigma_w + e_i \sigma_i}{\sigma},$$

where e_w and e_i are the saturation vapour pressures over water and ice respectively; and e_m the ambient vapour pressure in the mixed cloud.

The introduction of freezing modifies only the energy equation (10) as follows:

$$\frac{dT'}{dz} = \frac{\begin{aligned}&\left[\frac{\mathcal{D}}{M}\frac{dM}{dz}\left[(T_0' - T') + \frac{(\delta T - \delta T_0)}{\gamma} + (q_0 - q_m)\left\{\frac{L + L_f(1 - f(T))}{c_p}\right\}\right]\right. \\ &\quad - \Gamma\mathcal{D}\frac{T'}{T_0'} + \frac{g q_m \lambda T}{\gamma R T_0'} + \frac{L_f}{c_p}\sigma\frac{df(T)}{dT}\frac{g q_m \lambda T}{R T_0'} \\ &\quad \left. - \left\{\frac{L + L_f(1 - f(T))}{c_p}\right\}(1 + \lambda q_m)\frac{g q_m}{R T_0'}\right]\end{aligned}}{\begin{aligned}&\left[\mathcal{D} - \frac{1}{\gamma}(\mathcal{D} - 1) + \frac{\sigma L_f}{c_p}\frac{df(T)}{dT}\right. \\ &\quad \left. + \frac{1}{c_p \lambda T}[L + L_f(1 - f(T))]\{\mathcal{D} - (1 + \lambda q_m)\}\right]\end{aligned}}, \tag{15}$$

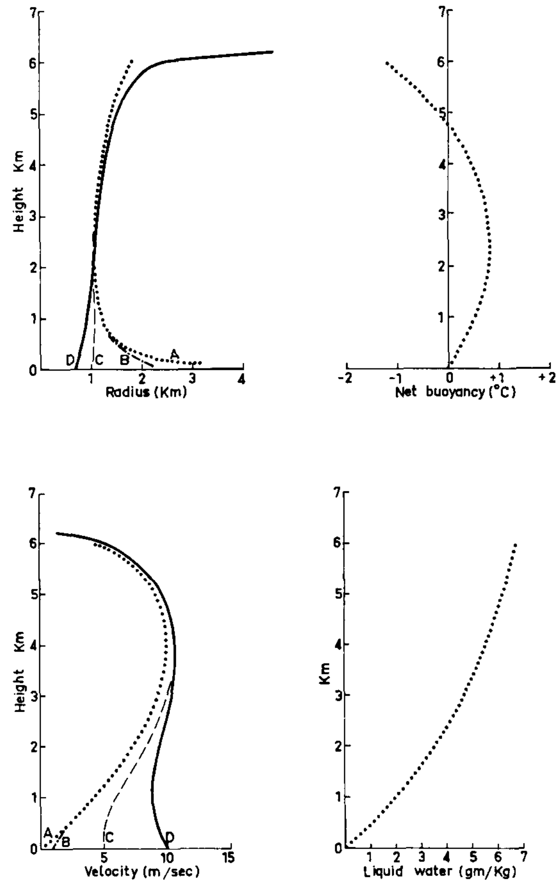

FIG. 1. Variation with height of the properties of cloud updraughts having the same mass flux at cloud base (900 mb) but different initial velocities. Environment saturated, lapse rate of virtual temperature 6°C km^{-1}, initial virtual temperature difference zero. The mass flux corresponds to a radius of 1 km with a velocity of 5 m/sec and the initial velocities are: (A) (dotted line) 20 cm/sec; (B) (dot-dash) 1 m/sec; (C) (dashed) 5 m/sec; (D) (full line) 10 m/sec. Only one curve is shown for buoyancy and liquid water, since they are sensibly the same for all four cases.

where

$$\mathcal{D} = 1 + \lambda q_m + \frac{\varepsilon\lambda}{ART}\{q_m L(T) + q_i L_f(T)(1 - f(T))\} + (q_w - q_i)\lambda T \frac{df(T)}{dT},$$

$q_m = \varepsilon e_m/p$ and L_f is the latent heat of freezing. $\mathcal{D}$ reduces to D and equation (15) to equation (10) if $f(T) = 1$, $f'(T) = 0$, i.e. at temperatures above that at which ice is assumed to appear.

4. Results

The crucial test of any theory of convection is to see whether it correctly predicts the heights reached by clouds in a number of different situations. Other properties predicted theoretically, such as updraught speeds and the concentration of condensed material, would serve equally well if they were observed, but in general they are not. Indeed, one strong reason for attempting to construct a valid model is the hope that it may be possible to use theoretically predicted distributions of vertical velocity and concentration of condensed material, checked where possible against observations, to discuss the growth of precipitation particles. The aim of the present paper is not, however, to attempt an observational check on the entraining plume model, but merely to describe it and to illustrate

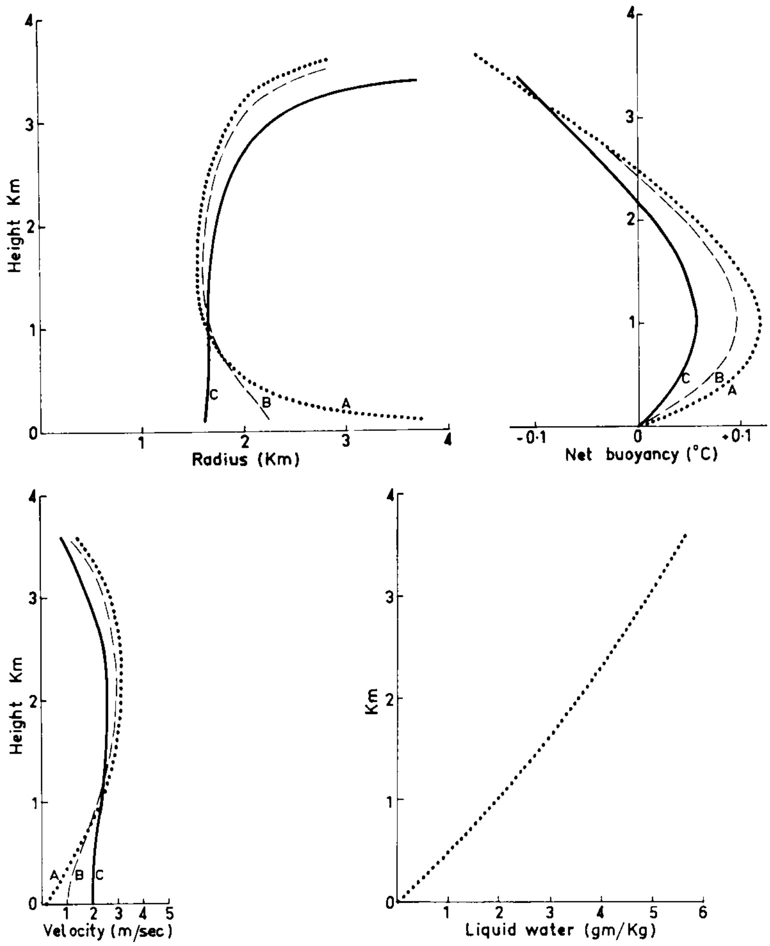

FIG. 2. As for Fig. 1 with the more stable conditions: lapse rate of virtual temperature 5.6°C km^{-1}, 70% relative humidity, pressure at cloud base 850 mb. The initial velocities are: (*A*) (dotted line) 20 cm/sec; (*B*) (dashes) 1 m/sec; (*C*) (full line) 2 m/sec. The liquid water concentration is nearly the same in all three cases.

its consequences in some simple hypothetical atmospheres, with a linear variation of virtual temperatures with height, and constant relative humidity—in most cases, 100%.

The results which seem to possess some degree of generality are:

(*a*) The most important initial condition determining the height reached by a cloud (with a given base temperature) is the mass flux transported through the cloud base level by the updraught.

(*b*) The model reproduces the anvil top of cumulo-nimbi and in those cases when the assumed velocity at cloud base is not too high, the "hour-glass" shape mentioned in Section 2 above.

The first result is illustrated in Fig. 1, which shows the variation of radius, updraught speed, liquid water and net buoyancy with height for four different initial velocities (20 (A), 100 (B), 500 (C), and 1000 (D) cm sec^{-1}), the initial mass flux being the same in all cases, and equal to that corresponding to a radius of 1 km with the velocity of 5 m/sec. The lapse rate of virtual temperature was taken as 6°C km^{-1}, and the environment was saturated. Clearly at a level of 3 km above cloud base and above, there is virtually no difference between these four cases. The net buoyancy and liquid water content curves are sensibly the same. The cloud base virtual temperature was 20°C in all cases. In view of the importance of the latent heat of

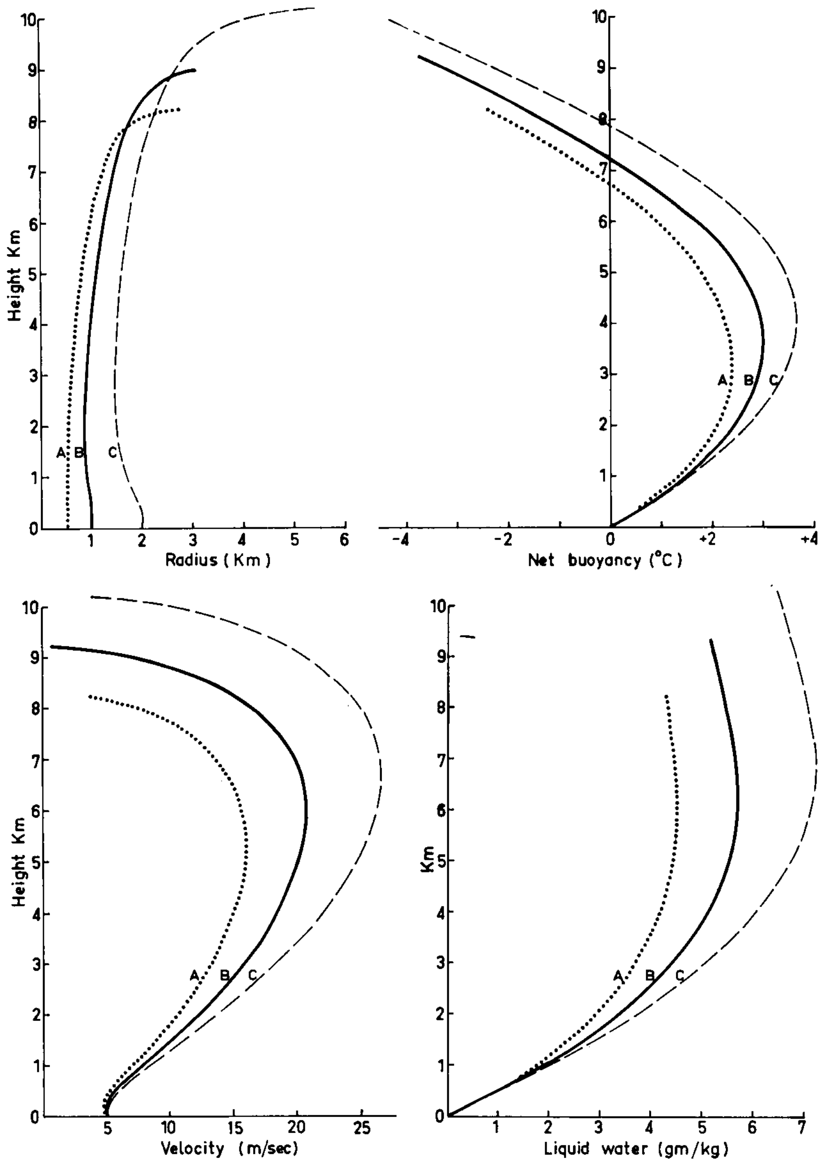

FIG. 3. The effect of changes of mass flux on the properties of updraughts. In all cases the initial velocity is 5 m/sec but the initial radii are: (*A*) (dotted) 0.5 km; (*B*) (full line) 1 km; (*C*) (dashed) 2 km. Environment saturated, lapse rate of virtual temperature 7°C km^{-1}, initial virtual temperature difference zero, cloud base 900 mb.

condensable water, the final height reached would presumably be quite dependent on the base temperature chosen.

The initially fast rising current (case D) has a smaller initial radius and consequently is more diluted by entrainment than the slower current (case A); apparently this is sufficient almost to cancel its initial advantage by the 3 km level. It will be seen that case C shows a slight initial reduction of updraught speed. With higher initial speeds (case D) this becomes more marked, with a steady increase in radius from the base upwards as the current decelerates. This is not only contrary to observation of cloud shapes, but is unrealistic physically. The source of energy in the system lies above the cloud base, and one must presume that the air entering the cloud is drawn up by the release of buoyancy above. While therefore the situation illustrated by cases A and B is physically plausible, no source of kinetic energy is apparent which would give rise to the high velocity at cloud base shown in case D.

Fig. 2 illustrates the same result in a stabler situation with a lapse rate of virtual temperature of 5.6°C km^{-1} near the ground and a relative humidity of 70% (a case taken from observation). Here, however, initial updraught speeds exceeding 2 m sec^{-1} are physically unrealistic, and give rise to a cloud shape which is not observed. In Figs. 1 and 2, the extreme cases computed have been carried through to the level where the updraught velocity falls towards zero and the cloud begins to spread rapidly (without introducing ice). When the cloud wall deflects markedly from the vertical near either the bottom or the top, the assumptions concerning entrainment are no longer quite valid, since horizontal momentum has been neglected, and since hydrostatic instability will in both cases accentuate mixing: the cloud stream is less dense than the environment near the bottom, where it underlies it, and more dense near the top, where it overlies it.

Fig. 3 illustrates the effect of changing the mass flux, again without introducing ice. In all cases, the initial velocity is taken as 5 m sec^{-1}; the initial radius is 0.5 km (A), 1 km (B) and 2 km (C), the mass flux varying by a factor of 16 from case A to case C. The lapse rate of virtual temperature was 7°C km^{-1}, and the environment saturated. The maximum speeds, buoyancy and liquid water contents are notably different in the three cases. The heights at which the final spreading occurs also differ appreciably, although they are less different than might have been expected. This insensitivity reflects the fact that a constant lapse rate of virtual temperature (and even more so, of temperature) is a very poor representation of the actual atmosphere in which the lapse rate normally increases markedly with height, more or less paralleling the variation of the wet adiabatic lapse rate. Thus, although a virtual-temperature lapse rate of 7°C km^{-1} is a reasonable average over a depth of say 10 km for thunderstorm situations, it results in marked stability in the upper troposphere, so that all computed clouds on reaching these levels rapidly lose buoyancy and momentum.

This effect is illustrated also by Fig. 4, where the full curves reproduce case B of Fig. 3, while the dashed curves show the result of decreasing environmental humidity from 100% to 50%, and the dotted curves the result of introducing ice formation as described above. The effects on cloud height are not great, and again this is probably due to the excessive stability aloft of the assumed atmosphere. A comparison of Fig. 1 with the full curve in Fig. 4 shows that a small change in lapse rate has a much bigger effect on the final height achieved than any of the other considerable changes in properties which we have assumed

In order to show the effect of changes in α, we repeated the calculation for case B of Fig. 3, but with $\alpha = 0.05$ instead of 0.1. This is a much bigger change than any experimental uncertainty in α. The velocity, buoyancy and liquid water curves were found to follow very closely the corresponding results for case C in Fig. 3, i.e. for an updraught having twice the initial radius. The radius (not drawn) remains about half that of 3(C), which illustrates the general result referred to in section 3; changes in α are equivalent to inverse changes in cloud radius.

The curves showing the variation of radius with height in the figures all show how the model reproduces the anvil-shaped top of cumulo-nimbi. Those in which the initial velocity at cloud base has been taken as less than the "equilibrium" value show the complete "hour-glass" shape mentioned earlier. Inspection of those curves shows too that the internal properties found are not inconsistent with our rather inadequate observational knowledge of these clouds. In some ten cases studied, where the cloud depth reaches 6 km or more, using plausible properties for the initial updraught and the environment (not all of these are reproduced here), the maximum values of updraught speed ranged from 10 to 26 m sec^{-1}, of liquid water content from 4.5 to 6.7 g kg^{-1}, of net buoyancy from 0.9 to 3.6°C, and of temperature excess from 2.0 to 4.6°C. The net buoyancy is less than the temperature excess because the weight of suspended condensed material dominates the virtual temperature correction in the upper part of the cloud.

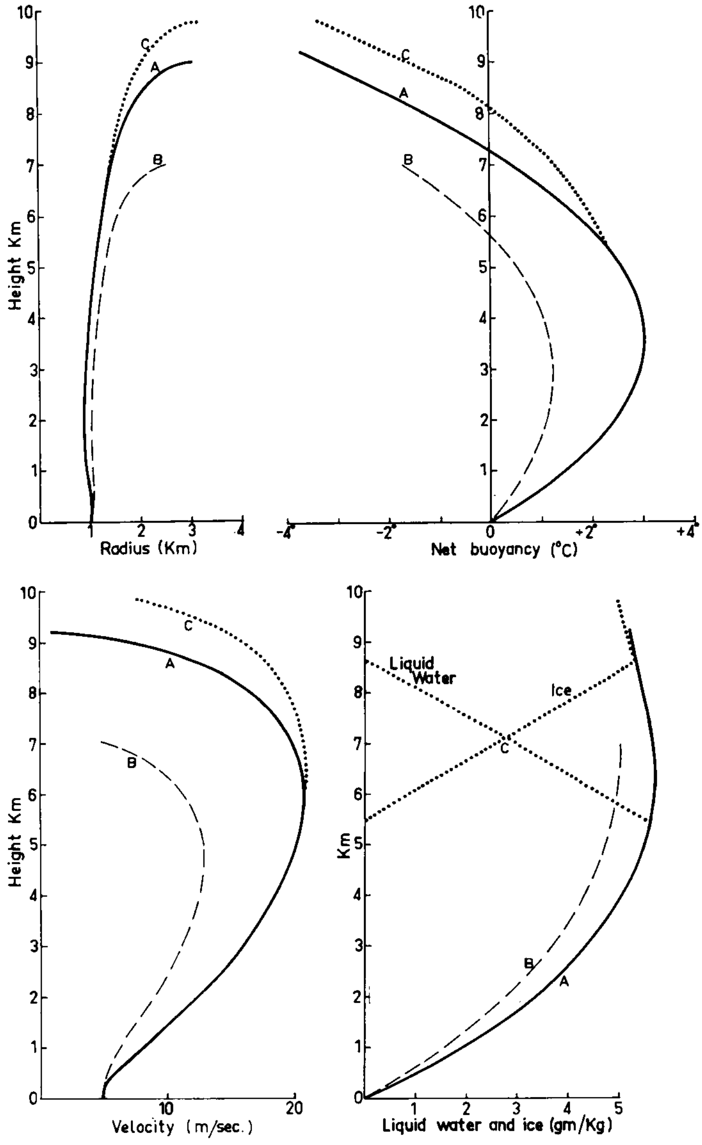

FIG. 4. Showing the effect of low relative humidity and the introduction of ice on the behaviour of an updraught. (A) (full curve) reproduces the case (B) of Fig. 3; (B) (dashed) the relative humidity has been changed to 50 %, keeping everything else constant; (C) (dotted) the formation of ice at temperatures below $-15°C$ has been taken into account as described in the text.

5. Conclusion

This model of a cumulo-nimbus updraught, which rests on a rather firmer physical basis than some earlier treatments, yields results which seem reasonably consistent with observation, especially in regard to cloud shape. In a given atmosphere, it depicts cloud growth as depending primarily on the mass flux at the condensation level. However, a crucial test of this or any model can be made only by applying it to real situations in which adequate cloud observations have been made. Because of this, these results should be regarded merely as illustrating the application of a physical entrainment law to large clouds, and no attempt has been made to cover a wide range of temperature and moisture conditions using linear virtual temperature profiles, which are inherently unrealistic.

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