On truncation errors due to vertical differences in various numerical prediction models

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ABSTRACT

Some estimates of truncation errors due to vertical finite differences are given. The truncation errors amount to at most 10 % in the phase-speed of the basic mode in a model atmosphere in which the vertical shear of the basic, zonal flow vanishes. An increase of the vertical resolution results in rapidly decreasing truncation errors in this case.

In a model atmosphere, in which the vertical windshear is different from zero, and the static stability parameter is a function of pressure, we find small truncation errors in the initial displacement of the wave, but large errors in the initial amplification.

The initial development of a baroclinic wave is evaluated in three different baroclinic models. The models differ only in the vertical variation of the static stability parameter which is a simple function of pressure in the first model, a constant in the next and zero in the third model. It is shown that the vertical variation of static stability is important for the amplification and the slope of the wave in the vertical direction.

The initial change of the kinetic energy in the models is computed as a function of wave-length. It is shown that the change of kinetic energy is larger in the models where the static stability parameter is constant or zero than in the models where it varies with pressure.

The last section contains some comments on the upper boundary condition. It is shown that the conditions $\omega = 0$ and $\phi < \infty$ (ϕ = geopotential) at the upper boundary are equivalent in certain problems.

1. Introduction

The complicated non-linear nature of the dynamic equations used in short-range numerical prediction of the atmosphere has forced us to replace the differential equations by a corresponding set of difference equations which are solved numerically. Such a procedure always involves truncation errors, because the difference quotient is only an approximation to the differential quotient. Several investigations have been made of truncation errors due to the finite differences taken in the horizontal planes and in time, but very little is known about the truncation errors due to the vertical resolution in the numerical prediction models. One of the reasons for this situation is that it is very difficult to obtain exact wave-solutions of the dynamic equations, even when they are linearized, in a model atmosphere where the meteorological quantities (the three-dimensional velocity, pressure, density and temperature) are allowed to change as continuous functions of the vertical co-ordinate. This is especially true if we want to include the case where the phase-speed of the waves can be complex, and the wave-solutions, therefore, can represent amplifying or damping waves. In the most general case treated so far (CHARNEY, 1947), it was only possible to find the transition between neutral and amplifying waves, but the amplification rate, depending on the imaginary part of the complex phase speed, could not be found, because it would require a knowledge of the zero-points of the confluent hypergeometric function in the complex plane.

An exact determination of any truncation error requires a good, if not exact knowledge of the solution to the differential equation. In view of the situation mentioned in the preceding paragraph, it is obvious that we must restrict ourselves to simple cases where we know the exact solution. In the first paragraphs of this paper we shall find the truncation error in atmospheric models, where by assumption the wind does not change with pressure. We shall, more specifically, find the truncation error in the phase speed of harmonic waves. It will be assumed that we have no truncation error in the horizontal planes in order to isolate the truncation error caused by the finite differences taken along the vertical coordinate axis. The assumption that the wind is constant with respect to the vertical coordinate prevents the appearance of amplifying (or damping) waves as solutions to the linearized equations. It is therefore not necessary to work in the complex domain.

In later sections of this paper we shall investigate the error in the initial displacements and amplification of baroclinic waves due to vertical finite differences. We are forced to restrict the investigation to the initial tendencies due to the fact that we are unable to solve for the complex phase speed in the general case where we allow a vertical shear in the horizontal wind and a deviation from neutral equilibrium in the vertical variation of the temperature field. However, in the special case where the vertical temperature gradient in the basic state is equal to the adiabatic lapse-rate ($\partial\theta/\partial p = 0$), in which case we have the so-called advective model, it is possible to solve exactly for the complex phase-speed. We can therefore in this simple case say more about the truncation error due to the vertical resolution.

In a final paragraph some comments are offered on the influence of the upper boundary condition on the wave-solutions of certain simplified models of the atmosphere.

2. The wave equation

The estimates of truncation errors due to finite differences in the vertical direction will in this paper be restricted to the most simple quasi-nondivergent models. The basic equations for these models can be written in the following form

$$\frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\nabla(\zeta + f) = f_0\frac{\partial\omega}{\partial p} \tag{2.1}$$

$$f_0\left[\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial p}\right) + \mathbf{v}\cdot\nabla\left(\frac{\partial\psi}{\partial p}\right)\right] + \sigma\omega = 0. \tag{2.2}$$

These equations contain only two unknowns, the streamfunction ψ and the vertical velocity, $\omega = dp/dt$, where p is pressure and t is time. The rest of the variables can be expressed in the streamfunction because the relative vorticity, $\zeta = \nabla^2\psi$, where ∇^2 is the horizontal Laplace operator, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\mathbf{v} = \mathbf{k}x\nabla\psi$, where $\mathbf{k}$ is a vertical unit vector. The measure of static stability is denoted by $\sigma = \sigma(p) = -(\alpha/\theta).\ \partial\theta/\partial p$, where α is specific volume, and θ potential temperature; f is the Coriolis parameter (f_0 a standard value). In (2.2) we have made the further approximation that

$$\frac{\partial\phi}{\partial p} = f_0\frac{\partial\psi}{\partial p}, \tag{2.3}$$

where ϕ is the geopotential. The nature of this approximation is discussed by PHILLIPS (1958).

In all our calculations we shall use the boundary conditions that $\omega = 0$ for $p = 0$ and $p = p_0 = 1000$ mb. Some comments on these boundary conditions will be given toward the end of the paper.

We are going to consider wave-solutions of the equations (2.1) and (2.2) in a linearized form. The solutions will be assumed to have the form

$$\psi = \Psi(p)e^{ik(x-ct)}, \quad \omega = \Omega(p)e^{ik(x-ct)}. \tag{2.4}$$

It is thus seen that we assume the perturbations to be independent of the meridional coordinate. The linearized form of the equations (2.1) and (2.2) becomes then:

$$\frac{\partial\nabla^2\psi}{\partial t} + U\frac{\partial\nabla^2\psi}{\partial x} + \beta\frac{\partial\psi}{\partial x} = f_0\frac{\partial\omega}{\partial p}, \tag{2.5}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial p}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial p}\right) - \frac{dU}{dp}\frac{\partial\psi}{\partial x} + \frac{\sigma}{f_0}\omega = 0, \tag{2.6}$$

where $U = U(p)$ is the zonal windspeed in the basic current and $\beta = df/dy$.

When we substitute the perturbations (2.4) in (2.5) and (2.6) we obtain the following set of equations for the amplitudes, $\Psi(p)$ and $\Omega(p)$, of the streamfunction and the vertical velocity, respectively:

$$ik\cdot[k^2(c-U)+\beta]\,\Psi = f_0\frac{d\Omega}{dp}, \tag{2.7}$$

$$ik\left[(c-U)\frac{d\Psi}{dp} + \frac{dU}{dp}\Psi(p)\right] = \frac{\sigma}{f_0}\Omega. \tag{2.8}$$

It is a straightforward procedure to eliminate either Ψ or Ω from the two equations, (2.7) and (2.8), and obtain a single equation, which must be solved subject to the boundary conditions stated above.

The determination of the truncation errors due to finite differences will in the following sections proceed in the following way. We will select cases where it is possible to solve the system (2.7) and (2.8) exactly. We obtain in those cases an explicit expression for the phase speed, c, and will also know the functions, $\Psi = \Psi(p)$ and $\Omega = \Omega(p)$. The equations (2.7) and (2.8) or equations equivalent to them will next be solved approximating derivatives with respect to the vertical coordinate by finite differences. We shall usually compare the phase speeds obtained in this way with those obtained in the continuous case and measure the truncation error by the difference.

3. The case, $U = U_{00} = \text{const} > 0$, $\sigma = \sigma_{00} = \text{const} > 0$

As the first case we have selected an extremely simple example, where all the calculations can be carried through using only elementary functions. We shall assume that the basic zonal wind has no vertical shear, and that the stability parameter, σ, is a constant.

This case is not very realistic from the atmospheric point of view, but will illustrate the general procedure. Due to the fact that the boundary conditions are expressed in the vertical velocity ($\Omega = 0$ for $p = 0$ and $p = p_0$) it is an advantage to eliminate Ψ from the equations (2.7) and (2.8). In our particular case we get the following equation for $\Omega = \Omega(p)$:

$$\frac{d^2\Omega}{dp^2} + q\Omega = 0, \tag{3.1}$$

where
$$q = -\frac{\sigma_{00}}{f_0^2} k^2 \frac{c - U_{00} + \beta/k^2}{c - U_{00}}. \tag{3.2}$$

We shall first consider the case where

$$q > 0, \tag{3.3}$$

which obviously corresponds to the following inequality for the phase speed

$$U_{00} - \beta/k^2 < c < U_{00}. \tag{3.4}$$

These waves are long internal waves with nodal surfaces (see Charney, 1947, footnote, p. 148).

The complete solution to (3.1) is obviously

$$\Omega = A \sin(\sqrt{q} \cdot p) + B \cos(\sqrt{q} \cdot p), \tag{3.5}$$

where A and B are constants of integration.

The boundary condition $\Omega = 0$ for $p = 0$ gives $B = 0$, and $\Omega = 0$ for $p = p_0$ gives

$$\sqrt{q} \cdot p_0 = m\pi \quad (m = 0, 1, 2, \ldots). \tag{3.6}$$

Substituting the expression (3.2) for q in (3.6) we find:

$$c = U_{00} - \frac{\beta/k^2}{1 + m^2 \dfrac{\pi^2}{p_0^2} \cdot \dfrac{f_0^2}{\sigma_{00}} \cdot \dfrac{1}{k^2}}. \tag{3.7}$$

The vertical velocity is obviously

$$\Omega = A \sin\left(m\pi \frac{p}{p_0}\right) \tag{3.8}$$

while the streamfunction Ψ can be found from (2.7) to be

$$\Psi = \frac{1}{ik}\frac{f_0}{\beta} \cdot m\pi \cdot \frac{1 + m^2 \dfrac{\pi^2}{p_0^2}\dfrac{f_0^2}{\sigma_{00}}\dfrac{1}{k^2}}{m^2 \dfrac{\pi^2}{p_0^2}\dfrac{f_0}{\sigma_{00}}\dfrac{1}{k^2}} \cdot \frac{p}{p_0} \cdot A \cdot \cos\left(m\pi \frac{p}{p_0}\right). \tag{3.9}$$

Waves very similar to these waves have been investigated in details by Arnason (1961), and we shall not discuss the distributions of Ω and Ψ further, but restrict our discussions to the phase-speed c. This quantity depends upon the values of the zonal wind speed U_{00}, the Coriolis parameter f_0, the Rossy parameter β and the static stability parameter σ_{00}. In addition, c is a function of m, determining the vertical modes, and the horizontal wavelength $L = 2\pi/k$. The values of $(U_{00} - c)$ in m sec^{-1}, determined from (3.7) are given in the first column of Table 1 for the following values of the other parameters: $f_0 = 10^{-4}$ sec^{-1}, $\beta = 16 \times 10^{-12}$ m^{-1} sec^{-1}, $\sigma_{00} = 4$ MTS units and $L = 6000$ km.

Our next purpose is to find the phase speed, when we use the very same model, but now approximate vertical derivatives by finite differences. Our basic equation is again (3.1). Consider a division of the interval $0 \leqslant p \leqslant p_0$ in N layers each corresponding to a pressure interval $P = p_0/N$. When we apply the equation (3.1) at the boundary between the layers we obtain, when the interfaces are numbered $i = 1, 2, \ldots (N-1)$, and we use the ordinary approximation to the second derivative

$$\frac{\Omega_{i+1} + \Omega_{i-1} - 2\Omega_i}{P^2} + q\Omega_i = 0. \qquad (3.10)$$

$i = 0$ and $i = N$ will correspond to the surfaces $p = 0$ and $p = p_0$, respectively. It should be remembered that $\Omega_0 = \Omega_N = 0$ according to the boundary conditions. (3.10) may also be written in the form:

$$\Omega_{i-1} + (qP^2 - 2)\Omega_i + \Omega_{i+1} = 0. \qquad (3.11)$$

If this set of $(N-1)$ linear, homogeneous equations shall have non-trivial solutions the determinant must vanish. This condition gives the following equation for the phase speed c:

$$\begin{bmatrix} (qP^2-2) & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & (qP^2-2) & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & (qP^2-2) & 1 & \ldots & 0 & 0 & 0 \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & 1 & (qP^2-2) & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & (qP^2-2) \end{bmatrix} = 0. \qquad (3.12)$$

The equation (3.12) has been solved for $N = 2, 3, \ldots, 7$ by solving first for the quantity qP^2. In each case we know the value of P and we have the relation (3.2) between q and c. If $N = 2$ we get naturally only one value of $(U_{00} - c)$ from (3.12). Each time we increase N by 1, we get an additional value of $(U_{00} - c)$ from the roots in (3.12). The values are tabulated in Table 1, using again as an example $f_0 = 10^{-4}$ sec^{-1}, $\beta = 16 \times 10^{-12}$ m^{-1} sec^{-1}, $\sigma_{00} = 4$ MTS units and $L = 6000$ km.

Table 2 is constructed as Table 1 using the same parameters except L which is now 28,000 km, corresponding to one wave around the hemisphere in middle latitudes.

It is seen from the two tables that the solutions from the finite difference formulation seem to converge toward the solution in the continuous case as the vertical resolution is increased. This is quite apparent especially from the first rows in the two tables.

The absolute error in the phase-speed is greatest for the lowest vertical resolution as seen from Tables 1 and 2. For $L = 6000$ km we find for $N = 2$ an absolute error of 0.7 m sec^{-1} in the phase-speed. This error corresponds to an error of 61 km per 24 hours in the displacement of the wave, or in other words a rather small fraction of the gridsize usually used in short range numerical prediction. For $L = 28{,}000$ km

TABLE 1. *Values of $(U_{00}-C)$ for constant static stability and vanishing vertical windshear.*

Parameters: $f_0 = 10^{-4}$ sec^{-1}, $\beta = 16 \times 10^{-12}$ m^{-1} sec^{-1} $\sigma_{00} = 4$ MTS units and $L = 6000$ km. The first column contains the values of $(U_{00} - c)$ in the continuous case. The remaining columns give the corresponding values for resolutions $N = 2, 3 \ldots 7$.

N	∞	2	3	4	5	6	7
$U_{00} - c_1$	4.43	5.14	4.76	4.63	4.57	4.54	4.52
$U_{00} - c_2$	1.44		2.04	1.76	1.64	1.58	1.55
$U_{00} - c_3$	0.68			1.08	0.92	0.84	0.79
$U_{00} - c_4$	0.39				0.67	0.57	0.52
$U_{00} - c_5$	0.25					0.46	0.39
$U_{00} - c_6$	0.18						0.34

TABLE 2.

As Table 1 except $L = 28{,}000$ km.

N	∞	2	3	4	5	6	7
$U_{00} - c_1$	6.27	7.80	6.95	6.68	6.56	6.50	6.46
$U_{00} - c_2$	1.59		2.35	1.99	1.84	1.77	1.73
$U_{00} - c_3$	0.71			1.17	0.97	0.87	0.84
$U_{00} - c_4$	0.40				0.71	0.59	0.53
$U_{00} - c_5$	0.26					0.48	0.40
$U_{00} - c_6$	0.18						0.34

we find an absolute error of 1.5 m sec^{-1} corresponding to 132 km per 24 hours in the displacement. These errors become smaller when the vertical resolution is increased.

In (3.2) it was assumed that $q > 0$ leading to the phase-speeds described by (3.7). We still have to find the phase-speed in the case where

$$q \leqslant 0. \tag{3.13}$$

In this case we find that the complete solution to (3.1) can be written

$$\Omega = Ae^{(-q)^{\frac{1}{2}}p} + Be^{-(-q)^{\frac{1}{2}}p}. \tag{3.14}$$

The upper boundary condition gives

$$A + B = 0, \tag{3.15}$$

while the lower boundary condition gives the condition

$$Ae^{(-q)^{\frac{1}{2}}p_0} + Be^{-(-q)^{\frac{1}{2}}p_0} = 0. \tag{3.16}$$

To satisfy (3.15) and (3.15) simultaneously it is required that

$$\sinh\left((-q)^{\frac{1}{2}}\,p_0\right) = 0, \tag{3.17}$$

which is only possible if $q = 0$ or

$$c = U_{00} - \beta/k^2. \tag{3.18}$$

This case is formally included in the solution (3.7) by allowing m to attain the value zero and needs therefore no further consideration.

The case considered in this section corresponds in the case $N = 2$ to a two-parameter model in cases where the vertical shear of the horizontal wind is small. Our estimates of the truncation error shows that we may have errors in the phase speeds of the order of magnitude of 10%. The magnitude of the truncation error decreases rather rapidly as the number of gridpoints in the vertical direction is increased.

4. The case $U = U_{00} > \text{const} > 0$, $\sigma = \sigma(p)$

The procedure used in the preceding section can also be carried through in the case where U is a constant, but where σ is a prescribed simple function of pressure. We shall assume that

$$\sigma = \sigma_0\left(\frac{p_0}{p}\right). \tag{4.1}$$

This distribution has been used by WELANDER (1961) in a different problem, but he has shown that there is a good agreement between the observed distribution of σ (corresponding to mean conditions for the winter season) and the distribution given by (4.1).

From (2.7) and (2.8) we can in this case derive the following equation for Ω:

$$(c - U_{00})\frac{d^2\Omega}{dp^2} - \sigma_0\left(\frac{p_0}{p}\right)\frac{k^2}{f_0^2}\left(c - U_{00} + \frac{\beta}{k^2}\right)\Omega = 0. \tag{4.2}$$

In order to bring (4.2) into a standard form we start by introducing a change in variable

$$p_* = \frac{p}{p_0}. \tag{4.3}$$

The boundary condition in the new variable is

$$\Omega = 0 \quad \text{for} \quad p_* = 0 \quad \text{and} \quad p_* = 1. \tag{4.4}$$

Denoting further

$$c_* = \frac{c}{U_{00}} \tag{4.5}$$

we may write (4.2) in the form

$$(1 - c_*)\,p_*\frac{d^2\Omega}{dp_*^2} + q_*\,(1 - \delta\,(1 - c_*))\,\Omega = 0, \tag{4.6}$$

where $$q_* = \frac{\sigma_0 p_0^2}{f_0^2}\frac{\beta}{U_{00}}, \quad \delta = \frac{k^2 U_{00}}{\beta}. \tag{4.7}$$

We proceed to introouce a second, new, independent variable defined by the relation

$$\xi = \frac{1 - \delta\,(1 - c_*)}{(1 - c_*)}\,p_* \tag{4.8}$$

(4.6) can in this variable be written

$$\xi\frac{d^2\Omega}{d\xi^2} + q^2\,\Omega = 0. \tag{4.9}$$

The boundary conditions expressed in ξ are

$$\Omega = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \frac{1 - \delta\,(1 - c_*)}{1 - c_*}. \tag{4.10}$$

(4.9) is of a form treated by Kamke (see p. 440). The solution can be written

$$\Omega(\xi) = C_1\,\xi^{\frac{1}{2}}\,J_1(2q_*^{\frac{1}{2}}\,\xi^{\frac{1}{2}}) + C_2\,\xi^{\frac{1}{2}}\,Y_1(2q_*^{\frac{1}{2}}\,\xi^{\frac{1}{2}}), \tag{4.11}$$

where C_1 and C_2 are constants of integration, and J_1 and Y_1 are the Bessel functions of the first and second kind, respectively, and of the first order.

When $\xi=0$ we find $J_1(0)=0$, while

$$Y_1(2p_*^{\frac{1}{2}}\xi^{\frac{1}{2}})\approx-\frac{1}{\pi}\cdot q_*^{-\frac{1}{2}}\xi^{-\frac{1}{2}} \qquad (4.12)$$

for small values of ξ. It follows therefore that the second term in (4.11) will approach a constant $(-\pi^{-1}q_*^{-\frac{1}{2}})$ for $\xi\to0$ from which it follows that $C_2=0$. The solution (4.11) can therefore be written

$$\Omega(\mu)=C_1\frac{1}{2q_*^{\frac{1}{2}}}\mu J_1(\mu), \qquad (4.13)$$

where

$$\mu=2q_*^{\frac{1}{2}}\xi^{\frac{1}{2}}. \qquad (4.14)$$

The values of the phase-speed c, can now be determined in the following way. It is obvious that the boundary condition $\Omega=0$, $\xi=0$ is satisfied automatically. The zero-points of the Bessel function of the first kind determine those values of μ for which $J_1(\mu)=0$. To each of these values we find one value of c. Denoting the zero-points of J_1 by μ_n $(n=1, 2, 3, \ldots)$ we find

$$\xi_n=\frac{\mu_n^2}{4q_*},\quad U_{00}-c_n=\frac{\beta/k^2}{1+\dfrac{1}{U_{00}}\dfrac{\beta}{k^2}\xi_n}. \qquad (4.15)$$

The values of c_n are given in the first column of Table 3 for $n=1$, 2, 3 and 4 and for $f_0=10^{-4}$ sec^{-1}, $\beta=16\times10^{-12}$ m^{-1} sec^{-1}, $\sigma_0=2$ MTS units, $U_{00}=10$ m sec^{-1} and $L=2\pi/k=6000$ km, while Table 4 contains the values for $L=28{,}000$ km.

We proceed next to find the values of c corresponding to finite difference models. The basic equation is naturally again (4.2), which we in this case prefer to write in the form:

$$\frac{d^2\Omega}{dp^2}+r(p)\,\Omega=0, \qquad (4.17)$$

where
$$r(p)=-\sigma_0\cdot\left(\frac{p_0}{p}\right)\cdot\frac{k^2}{f_0^2}\,\frac{c-U_{00}+\beta/k^2}{c-U_{00}}. \qquad (4.18)$$

TABLE 3. *Values of c for variable static stability.*

Parameters: $f_0=10^{-4}$ sec^{-1}, $U_{00}=10$ m sec^{-1}, $\beta=16\times10^{-12}$ m^{-1} sec^{-1}, $\sigma_0=2$ MTS units and $L=6000$ km. The first column contains the values of c in the continuous case. The remaining columns give the corresponding values for resolutions $N=2$, 3, 4 and 5.

N	∞	2	3	4	5
c_1	4.57	4.86	4.69	4.64	4.61
c_2	7.80		8.08	7.93	7.88
c_3	8.86			9.04	8.96
c_4	9.31				9.44

Using the same subdivision as before we can write the finite difference form of (4.17) as

$$\Omega_{i-1}+(r_iP^2-2)\,\Omega_i+\Omega_{i+1}=0, \qquad (4.19)$$

where r_i is the value of r for the ith level. We may write r_iP^2 in the form:

$$r_iP^2=\frac{1}{iN}r_0 \qquad (4.20)$$

with
$$r_0=-\sigma_0^2P_0^2\frac{k^2}{f_0^2}\cdot\frac{c-U_{00}+\beta/k^2}{c-U_{00}}, \qquad (4.21)$$

where $i=1, 2, \ldots, N-1$ and N is the number of layers in the subdivision. The determinant of the system (4.19) has to vanish in order to have non-trivial solutions. This condition can be expressed:

$$\begin{bmatrix}
\left(\frac{1}{N}r_0-2\right) & 1 & 0 & 0\ \ldots\ 0 & 0 & 0\\
1 & \left(\frac{1}{2N}r_0-2\right) & 1 & 0\ \ldots\ 0 & 0 & 0\\
0 & 1 & \left(\frac{1}{3N}r_0-2\right) & 1\ \ldots\ 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot\ \cdots\ \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot\ \cdots\ \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0\ \ldots\ 1 & \left(\frac{1}{(N-2)N}r_0-2\right) & 1\\
0 & 0 & 0 & 0\ \ldots\ 0 & 1 & \left(\frac{1}{(N-1)N}r_0-2\right)
\end{bmatrix}=0. \qquad (4.22)$$

TABLE 4.

As Table 3 except $L = 28{,}000$ km.

N	∞	2	3	4	5
c_1	1.51	2.20	1.81	1.68	1.62
c_2	7.42		7.76	7.60	7.53
c_3	8.77			8.97	8.89
c_4	9.28				9.42

(4.22) has been solved for $N=2$, 3, 4 and 5. When the solutions for r_0 have been found, the relation (4.21) has been used to compute the values of c. The results are entered in Tables 3 and 4. The parameters, used in the calculations, are the same as given in the calculation of c_n in the continuous case.

The main difference between the results given in Tables 3 and 4 compared with those of Tables 1 and 2 is that the variable static stability has the effect that the waves in the finite difference models move too fast compared with the waves in the continuous case. The order of magnitude of the truncation error is the same and amount in no case to more than 1 m sec^{-1}.

5. Very long waves, $U = U_{00} = \text{const} > 0$, $\sigma = \sigma(p)$

It has recently been suggested (WELANDER, 1961; WIIN-NIELSEN, 1961) that the very long waves in the atmosphere may be described by a system of equations which is obtained by a simplification of the system (2.1) and (2.2). The simplification is introduced in the first of these equations which reduces to the form

$$\mathbf{v} \cdot \nabla f = f_0 \frac{\partial \omega}{\partial p} \tag{5.1}$$

while (2.2) is unchanged.

It is interesting to compare the solution in this case with the solution obtained in the preceding section. We shall proceed to solve the equation corresponding to (4.2) in the case where the vorticity equation is replaced by (5.1). It is easily seen that (4.2) in this case is replaced by

$$(U_{00} - c)\frac{d^2\Omega}{dp^2} + \sigma \frac{\beta}{f_0^2}\Omega = 0. \tag{5.2}$$

Introducing the transformation of variables (4.3) we obtain the equation

$$(1 - c_*)\, p_* \frac{d^2\Omega}{dp_*^2} + q_*\,\Omega = 0. \tag{5.3}$$

In this equation we transform again to a new independent variable defined by

$$\xi_* = \frac{p_*}{1 - c_*} \tag{5.4}$$

(5.3) becomes then

$$\xi_* \frac{d^2\Omega}{d\xi_*^2} + q_*\Omega = 0 \tag{5.5}$$

with the boundary conditions

$$\Omega = 0 \text{ for } \xi_* = 0,\ \xi_* = \frac{1}{1 - c_*}. \tag{5.6}$$

The equation (5.5) is formally equivalent to (4.9) and the solution is therefore

$$\Omega(\mu_*) = C_1^* \frac{1}{2q_*^{\frac{1}{2}}}\mu_* J_1(\mu_*), \tag{5.7}$$

where $$\mu_* = 2q_*^{\frac{1}{2}}\,\xi_*^{\frac{1}{2}}. \tag{5.8}$$

The phase speeds can again be found from the zero-points of the first order Bessel function of the first kind, but now we use the second boundary condition (5.6) to find the phase speeds, which becomes

$$c_n = U_{00} - \frac{U_{00}}{\xi_{*n}}, \tag{5.9}$$

where ξ_{*n} is that value of ξ which corresponds to the nth zero-point μ_{*n} of the Bessel function according to the relation (5.8).

The computed values of c_n are given in Table 5 using the same parameters as before except that the wave-length does not enter into this calculation.

The solutions in the finite difference case will this time be based on the following equation

$$\frac{d^2\Omega}{dp^2} + s(p)\,\Omega = 0, \tag{5.10}$$

where $$s(p) = -\sigma_0 \cdot \left(\frac{p_0}{p}\right) \cdot \frac{\beta}{f_*^2}\frac{1}{C - U_{00}} \tag{5.11}$$

The finite difference form of (5.10) becomes

TABLE 5. *Values of c for very long waves.*
Parameters and arrangement as in Table 3.

N	∞	2	3	4	5
c_1	1.27	2.00	1.63	1.45	1.38
c_2	7.40		7.69	7.58	7.51
c_3	8.76			8.97	8.89
c_4	9.28				9.42

$$\Omega_{i+0} + (s_i P^2 - 2)\,\Omega_i + \Omega_{i+0} = 0, \qquad (5.12)$$

where $s_i P^2$ can be written in the form

$$s_i P^2 = \frac{iN}{1} s_0 \qquad (5.13)$$

with
$$s_0 = -\sigma_0 p_0^2 \frac{\beta}{f_0^2} \frac{1}{C - U_{00}}. \qquad (5.14)$$

The determinant for the determination of c will look like (4.22) with r_0 replaced by s_0. This determinant has been solved for $N = 2$, 3, 4 and 5, and the results are given in Table 5. This table should be compared with Table 4, which is based upon the original vorticity equation. It is found that there is, indeed, very little difference between the two tables. This is to be expected, because Table 4 is based upon a very small value of k, corresponding to $L = 2\pi/k = 28{,}000$ km, while k in the computation leading to Table 5 was zero.

In this context it should not be forgotten that the model treated in Section 4 also allows the phase-speed, c, to be equal to the Rossby speed

$$c = U_{00} - \frac{\beta}{k^2} \qquad (5.15)$$

This value of c (not listed in Tables 3 and 4) corresponds to $\Omega = 0$ everywhere. The value (5.15) is not possible in the model treated in Section 5 as has been shown earlier by WIIN-NIELSEN (1961). The assumption (5.1) eliminates the Rossby waxes. This is therefore the only main difference between the two models. The present calculation shows that the velocity of the higher modes are changed only slightly.

6. The case, $U = U(p)$, $\sigma = \sigma(p)$

The models treated so far are extremely simplified because they do not allow the basic, zonal windspeed to have any vertical shear. On the other hand, if we make the models more realistic and allow dU/dp to be different from zero, we are no longer able to solve the perturbation problem in the same complete sense as we are in the very simple models.

GREEN (1960) has recently extended the baroclinic stability investigations by CHARNEY (1947) and EADY (1949) to more general cases. He uses a technique somewhat similar to the one adopted in the previous sections of this paper. The general frequency equation for quasi-geostrophic flow was solved using finite differences in the vertical direction, but Green restricts his investigation to one mode, presumably the basic one, by the assumption that the differential equation has only one pair of eigen values. This cannot always be the case simply because the more general frequency equation in the case where the vertical windshear becomes very small will reduce in the limit to the equation treated in Section 4, and this equation has infinitely many eigen solutions.

In this section we shall not treat the baroclinic case as an eigen value problem, but rather treat the problem of finding the displacements of the waves as an initial value problem.

We look therefore no longer for solutions corresponding to harmonic waves moving with the same speed at all levels, in which case the mathematical problem is to find the (complex) values of the phase speeds and the corresponding vertical structure of the waves expressed by the vertical distribution of the stream function, the temperature, the vertical velocity and other dynamically important quantities.

When we treat the problem as an initial value problem, it is seen from the equations (2.1) and (2.2) that it is necessary and sufficient to prescribe the values of the stream function, ψ, everywhere initially, simply because the other variable, the vertical velocity, can be eliminated from the two equations. A straightforward method to investigate the importance of the vertical resolution is therefore to integrate the equations (2.1) and (2.2) from given initial conditions using an increasing number of information levels in the vertical direction. It is this idea which is being used in practical numerical forecasting where we recently have seen a tendency to increase the number of information levels in the vertical direction. So far, there has not been a systematic comparison made between time-integrations of the quasi-geostrophic equations (or any meteorological set of equations) where the only difference has been the number of information levels.

In the following we shall make an attempt to make such a comparison. In order to keep the problem so simple that we can treat it without extensive numerical integrations, we shall restrict ourselves to the linearized equations (2.5) and (2.6), and we shall use only very idealized initial conditions.

Let us assume that the stream function and the vertical velocity can be written in the form

$$\psi(x, p, t) = A(p, t) \sin kx + B(p, t) \cos kx, \quad (6.1)$$

$$\omega(x, p, t) = C(p, t) \sin kx + D(p, t) \cos kx. \quad (6.2)$$

It is first of all seen that we have assumed that the stream function (and the vertical velocity) are independent of the meridional direction. We have further restricted ourselves to considerations of only a single wave number, k. According to the earlier remarks we are only allowed to specify the stream function initially. This means that the coefficients $C(p,t)$ and $D(p,t)$ are determined from $A(p,t)$ and $B(p,t)$.

When the expressions (6.1) and (6.2) are substituted in (2.5) and we have equated the coefficients of the sin-terms and cos-terms respectively, we get the following equations:

$$\frac{\partial A}{\partial t} = k\left(U - \frac{\beta}{k^2}\right) B - \frac{f_0}{k^2}\frac{\partial C}{\partial p}, \quad (6.3)$$

$$\frac{\partial B}{\partial t} = -k\left(U - \frac{\beta}{k^2}\right) A - \frac{f_0}{k^2}\frac{\partial D}{\partial p}. \quad (6.4)$$

Using a similar procedure on (2.6) we arrive at the following two equations:

$$\frac{\partial^2 A}{\partial t \partial p} = k\left(U\frac{\partial B}{\partial p} - \frac{dU}{dp}B\right) - \frac{\sigma}{f_0} C, \quad (6.5)$$

$$\frac{\partial^2 B}{\partial t \partial p} = -k\left(U\frac{\partial A}{\partial p} - \frac{dU}{dp}A\right) - \frac{\sigma}{f_0} D. \quad (6.6)$$

The system (6.3)–(6.6) is the prognostic equations of the quasi-nondivergent model corresponding to the specified distributions (6.1) and (6.2). It is convenient to replace the equations (6.5) and (6.6) by a new set of equations, obtained by differentiating (6.3) and (6.4) with respect to pressure and subtracting the resulting equations from (6.5) and (6.6), respectively. We obtain then a set of equations containing no time derivatives. They correspond to the diagnostic equation for the vertical velocity obtained in the more general case. These equations are:

$$\frac{\partial^2 C}{\partial p^2} - \sigma\frac{k^2}{f_0^2} C = \frac{k^3}{f_0}\left(2\frac{dU}{dp}B - \frac{\beta}{k^3}\frac{\partial B}{\partial p}\right), \quad (6.7)$$

$$\frac{\partial^2 D}{\partial p^2} - \sigma\frac{k^2}{f_0^2} D = -\frac{k^3}{f_0}\left(2\frac{dU}{dp}A - \frac{\beta}{k^3}\frac{\partial A}{\partial p}\right). \quad (6.8)$$

The equations (6.7) and (6.8) show clearly that $C(p,t)$ and $D(p,t)$ are determined from prescribed distributions of $A(p,t)$ and $B(p,t)$. In the following we shall prefer to work with the equations (6.3), (6.4), (6.7) and (6.8). For convenience we introduce in these the change of variable

$$p_* = \frac{p}{p_0}. \quad (6.9)$$

If we at the same time assume that

$$U = U_0(1 - p_*), \quad \sigma = \sigma_0 p_*^{-2}, \quad (6.10)$$

which means that we have assumed that the zonal wind varies linearly from zero at $p_* = 1$ to U_0 at $p_* = 0$, and that the static stability parameter varies inversely with pressure (WIIN-NIELSEN, 1959), we find that the four equations take the form:

$$\left.\begin{aligned}
&\frac{\partial A}{\partial t} = kU_0(1 - u_* - p_*)B - \frac{f_0}{k^2 p_0}\frac{\partial C}{\partial p_*},\\
&\frac{\partial B}{\partial t} = -kU_0(1 - u_* - p_*)A - \frac{f_0}{k^2 p_0}\frac{\partial D}{\partial p_*},\\
&p_*^2\frac{\partial^2 C}{\partial p_*^2} - \frac{k^2\sigma_0 p_0^2}{f_0^2}C = -\frac{k^3 p_0 U_0}{f_0}\left(2B + U_*\frac{\partial B}{\partial p_*}\right)p_*^2,\\
&p_*^2\frac{\partial^2 D}{\partial p_*^2} - \frac{k^2\sigma_0 p_0^2}{f_0^2}D = +\frac{k^3 p_0 U_0}{f_0}\left(2A + U_*\frac{\partial A}{\partial p_*}\right)p_*^2.
\end{aligned}\right\} \quad (6.11)$$

In these equations $u_* = \beta/k^2 U_0$.

The system (6.11) is a complete, closed system of equations. From given initial values of $A(p_*,0)$ and $B(p_*,0)$ we can compute $C(p_*,0)$

and $D(p_*,0)$ from the last two equations using the boundary conditions $C=0$ and $D=0$ for $p_*=0$ and $p_*=1$. The values of $C(p_*,0)$ and $D(p_*,0)$ can be used to compute $(\partial A/\partial t)_0$ and $(\partial B/\partial t)_0$. Making a time-extrapolation we obtain new values of $A(p_*,t)$ and $B(p_*,t)$ one time-step later, and we are ready to start a new cycle.

The equations must in the general case be solved numerically. However, some preliminary tendency calculations can be made for simple distributions of A and B at $t=0$. We shall perform such tendency calculations and compute A and B at some later time making use of Taylor expansions of A and B in time. In general, we may write:

$$A(\Delta t)=A_0+(\partial A/\partial t)_0\,\Delta t+\tfrac{1}{2}(\partial^2 A/\partial t^2)_0\,\Delta t^2+\ \dots$$
$$B(\Delta t)=B_0+(\partial B/\partial t)_0\,\Delta t+\tfrac{1}{2}(\partial^2 B/\partial t^2)_0\,\Delta t^2+\ \dots \tag{6.12}$$

when the subscript zero denotes values at the initial time.

For small values of Δt we may with sufficient accuracy truncate the series (6.12) after only a few terms. We shall compute only the three first terms in (6.12). The procedure outlined above contains naturally also a truncation error in time.

We shall restrict the computations in this section to the very simple case, where the initial values of A and B are given by:

$$A(p_*,\,0)=A_0=0,\quad B(p_*,\,0)=B_0=\text{const}>0. \tag{6.13}$$

In this case, we consider perturbations on a baroclinic, zonal current. The perturbations are initially in phase and have the same amplitude at all levels. The first goal is to find how the phase and amplitude will change during a short time-interval Δt. We shall obtain this information by computing the first and second time derivative at the initial time and then use (6.12) to compute $A(\Delta t)$ and $B(\Delta t)$.

Some information can be obtained immediately from an inspection of (6.11). Since $A_0=0$, it follows from the boundary conditions that $D_0=0$ everywhere. We have consequently $(\partial B/\partial t)_0=0$ everywhere. It is also easy to show that $(\partial^2 A/\partial t^2)_0=0$ everywhere. Differentiating the first equation with respect to time we find that

$$\frac{\partial^2 A}{\partial t^2}=kU_0(1-u_*-p_*)\frac{\partial B}{\partial t}-\frac{f_0}{k^2p_0}\frac{\partial}{\partial p_*}\left(\frac{\partial C}{\partial t}\right). \tag{6.14}$$

The first term on the right side is zero according to the argument presented above. We next turn our attention to $\partial C/\partial t$. Differentiating the third equation with respect to time we find

$$P_*^2\frac{\partial^2}{\partial p_*^2}\left(\frac{\partial C}{\partial t}\right)-\frac{k^2\sigma_0p_0^2}{f_0}\left(\frac{\partial C}{\partial t}\right)=$$
$$-\frac{k^3p_0U_0}{f_0}\left(2\frac{\partial B}{\partial t}+u_*\frac{\partial}{\partial p_*}\left(\frac{\partial B}{\partial t}\right)\right)p_*^2. \tag{6.15}$$

The complete right-hand side of this equation is zero initially, because $(\partial B/\partial t)_0=0$. Therefore, we find $(\partial C/\partial t)_0=0$, and it follows that $(\partial^2 A/\partial t^2)_0=0$.

(6.12) can therefore for our simple case be written:

$$\left.\begin{aligned}A(\Delta t)&=(\partial A/\partial t)_0\,\Delta t\\ B(\Delta t)&=B_0+\tfrac{1}{2}(\partial^2 B/\partial t^2)_0\,\Delta^0t^2\end{aligned}\right\} \tag{6.16}$$

and we have only two terms to compute.

We solve first the third equation in (6.11) in order to find C_0. The complete solution is

$$C_0=-\frac{2k^3p_0U_0}{f_0^2(2-a^2)}B_0(p_*^2-p_*^\varepsilon), \tag{6.17}$$

$$\text{where}\quad a^2=\frac{k^2\sigma_0p_0^2}{f_0^2},\ \varepsilon=\tfrac{1}{2}(1+[1+4a^2]^{\frac{1}{2}}), \tag{6.18}$$

and where we already have incorporated the boundary conditions.

We find consequently

$$\left(\frac{\partial A}{\partial t}\right)_0=B_0\left[kU_0(1-u_*-p_*)+\frac{2U_0k}{2-a^2}(2p_*-\varepsilon p_*^{\varepsilon-1})\right]. \tag{6.19}$$

In order to find $(\partial^2 B/\partial t^2)_0$, we need first to find $(\partial D/\partial t)_0$.

Let the latter quantity be designated by $\partial D/\partial t=D_t$: It follows then that

$$p_*^2\frac{\partial^2 D_t}{\partial p_*^2}-a^2D_t=\frac{k^3p_0U_0}{f_0}\left(2\left(\frac{\partial A}{\partial t}\right)+u_*\frac{\partial}{\partial p_*}\left(\frac{\partial A}{\partial t}\right)\right)p_*^2. \tag{6.20}$$

$\partial A/\partial t$ is already known (6.19). When this expression is substituted in (6.20), we can solve the equation, determine the integration constants from the boundary conditions, and write the solution as:

$$D_t = m_2\,(p_*^2 - p_2^{\varepsilon}) + m_3(p_*^3 - p_*^{\varepsilon}) + m_{\varepsilon+1}\,(p_*^{\varepsilon+1} - p_*^{\varepsilon}). \tag{6.21}$$

The constants, m_2, m_3 and $m_{\varepsilon+1}$ are given by the expressions

$$\left.\begin{aligned} m_2 &= \frac{k^4 p_0 U_0^2}{f_0} B_0 \left(2 - 3u_* + \frac{4u_*}{2-a^2}\right)\cdot\frac{1}{2-a^2},\\ m_3 &= \frac{k^4 p_0 U_0^2}{f_0} B_0 \left(\frac{8}{2-a^2} - 2\right)\frac{1}{6-a^2},\\ m_{\varepsilon+1} &= -\frac{k^4 p_0 U_0^2}{f_0} B_0 \frac{4\varepsilon}{2-a^2}\cdot\frac{1}{\varepsilon(\varepsilon+1)-a^2}.\end{aligned}\right\} \tag{6.22}$$

The complete expressions for $A(\Delta t)$ and $B(\Delta t)$, accurate to the order of Δt^2, can be written:

$$A(\Delta t) = B_0\,\Delta t\,kU_0\left[(1-u_*-p_*) + \frac{2}{2-a^2}(2p_* - \varepsilon p_*^{\varepsilon-1})\right] \tag{6.23}$$

and $$B(\Delta t) = B_0 - \tfrac{1}{2}\Delta t^2 B_0 k^2 U_0^2 (1-u_*-p_*) \times\left[(1-u_*-p_*) + \frac{2}{2-a^2}\,(2p_* - \varepsilon p_*^{\varepsilon-1})\right] - \tfrac{1}{2}\Delta t^2 \frac{f_0}{k^2 p_0} \times [m_2(2p_* - \varepsilon p_*^{\varepsilon-1}) + m_3\,(3p_*^2 - \varepsilon p_*^{\varepsilon-1}) + m_{\varepsilon+1}\,(\varepsilon+1)\,p_*^{\varepsilon} - \varepsilon p_*^{\varepsilon-1})]. \tag{6.24}$$

It is obvious that the values of $A(\Delta t)$ and $B(\Delta t)$ can be computed when we specify the parameters going into the solution. A computation has been made for the following values of the parameters: $L = 2\pi/k = 6000$ km, $f_0 = 10^{-4}$ sec^{-0}, $\beta = 16\times10^{-12}$ m^{-1} sec^{-1}, $\sigma_0 = 0.5$ MTS units and $U_0 = 40$ m sec^{-1}.

We have finally set $\Delta t = 12$ hours and $kB_0 = 10$ m sec^{-1} ($B_0 = 9.55\times10^6$ m^2 sec^{-1}).

The results have been expressed in amplitude and phase-angle for the stream function ψ in the following way. At any time we have

$$\psi(x,\,p,\,t) = A\sin kx + B\cos kx = R\cos(kx+\delta), \tag{6.25}$$

$$\text{where}\qquad R = (A^2+B^2)^{\frac{1}{2}},\ \tan\delta = \frac{A}{B}. \tag{6.26}$$

At $t=0$ we have $R_0 = B_0 = $ const. and $\delta_0 = 0$. δ gives therefore the position of the ridge in the waves.

Fig. 1 shows R as a function of pressure for $\Delta t = 12$ hours. The vertical dashed line in the figure gives the initial constant value of $R = B_0$. It is seen that the amplitude has increased at all levels above $p_* \simeq 0.63$, but decreased below this level. The maximum increase in the amplitude takes place close to 200 mb, where the amplitude has increased by a factor 1.3, approximately.

The phase angle of the wave at $\Delta t = 12$ hours is shown in figure 2, where 360 degrees corresponds to a complete wave length. $\delta = 0$ is the initial phase angle. The figure shows that a wave initially in phase and with the same amplitude at all levels will move toward the east at elevations below about 400 mb, retrograde slowly between 150 and 400 mb, and progress fast at levels above 150 mb. It develops therefore initially a westward slope with decreasing pressure at the lower levels, but, at

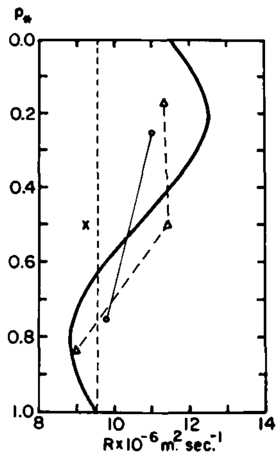

FIG. 1. The continuous curve gives the amplitude as a function of pressure after 12 hours in a baroclinic wave with wave length 6000 km. The initial amplitude, constant with pressure, is indicated by the vertical dashed line. Other parameters: $f_0 = 10^{-4}$sec^{-1}, $\beta = 16\times10^{-12}$m^{-1}sec^{-1}, $\sigma_0 = \frac{1}{2}$ MTS units, $u_0 = 40$ m sec^{-1}, $kB_0 = 10$ m sec^{-1}. The cross, circles and triangles show the amplitude estimated from a one, two and three parameter model, respectively.

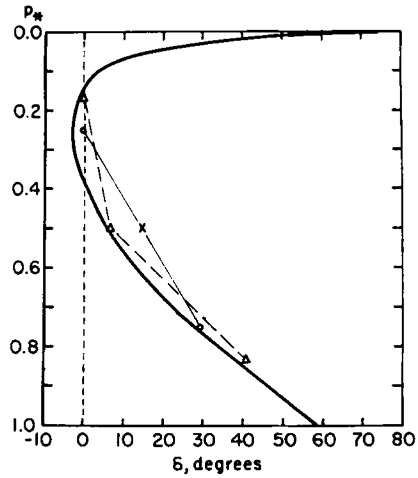

FIG. 2. The phase angle as a function of pressure after 12 hours in a baroclinic wave. The initial phase-angle, constant with pressure, is given by the vertical dashed line. Parameters as in Fig. 1.

least in the present example, the opposite slope at the very high levels. It might be supposed that the large positive wave velocity for small values of the pressure could partly be due to the unrealistic increase of the zonal wind as the pressure becomes small, but it will be shown later that it is caused by the large values of the static stability prescribed for low values of the pressure, according to (6.10).

It is next our purpose to investigate what we will get, if we apply finite differences in the vertical direction, however using the same basic parameters as before. The finite difference case can be carried through for different vertical resolutions. We shall start by a consideration of the most simple case:

a. *The one-level case*

The equations for A and B (applied at the midlevel, where $p_* = \frac{1}{2}$, and denoted by the subscript 1) becomes

$$\left.\begin{aligned}\frac{\partial A_1}{\partial t} &= kU_0\left(\tfrac{1}{2}-u_*\right)B_1\\ \frac{\partial B_1}{\partial t} &= -kU_0\left(\tfrac{1}{2}-u_*\right)A_1\end{aligned}\right\}\qquad(6a.1)$$

because $\partial C/\partial p_*$ and $\partial D/\partial p_*$ vanish at the midlevel, when they are evaluated by centered finite differences. From (6*a*.1) it follows that

$$\left.\begin{aligned}\frac{\partial^2 A_1}{\partial t^2} &= kU_0\left(\tfrac{1}{2}-u_*\right)\frac{\partial B_1}{\partial t},\\ \frac{\partial^2 B_1}{\partial t^2} &= -kU_0\left(\tfrac{1}{2}-u_*\right)\frac{\partial A_0}{\partial t}.\end{aligned}\right\}\qquad(6a.2)$$

Assuming as before that, initially, $A_{1,0}=0$, $B_{1,0}=B_0$, we find $(\partial B_1/\partial t)_0=0$ and $(\partial^2 A_1/\partial t^2)_0=0$ and

$$\left.\begin{aligned}A_1(\Delta t) &= \Delta t\cdot kU_0\,B_0\left(\tfrac{1}{2}-u_*\right),\\ B_1(\Delta t) &= B_0-\tfrac{1}{2}(\Delta t)^2\,B_0\,k^2U_0^2\left(\tfrac{1}{2}-u_*\right)^2.\end{aligned}\right\}\qquad(6a.3)$$

We find that

$$R_1(12)=9.24\times10^6\ \mathrm{m^2\,sec^{-1}},\quad \delta_1(12)=14°7.\qquad(6a.4)$$

Since it is well known that the exact solution in this particular case, as can be seen from the solution to the *differential* equations (6a.1), is a wave moving with the speed $c=\frac{1}{2}U_0-\beta/k^2$ and without any change of amplitude, we have a possibility to find the order of magnitude of the time truncation error which we make using the truncated series (6.12). The error in the amplitude is apparently about 3%

If the wave moved with the exact speed, it would after 12 hours have moved 241.92 km. Since the total wave length, corresponding to 360 degrees, is 6000 km, the phase-angle would be 14.5 degrees, which should be compared with the value 14.7 degrees computed from the approximate method. The error in the phase angle is apparently negligible. Since there is no reason to believe that the time truncation error should have another order of magnitude in the more complicated cases treated later, we can ascribe the major part of the differences we find to the truncation errors in the vertical direction.

The values of $R_1(12)$ and $\delta_1(12)$ are entered in Figs. 1 and 2, respectively, by crosses. We find that the wave in this approximation will move too fast and will not amplify.

b. *The two-level case*

The equations for A and B (applied at the level $p_* = \frac{1}{4}$ and $p_* = \frac{3}{4}$ and with subscripts 1 and 3) become:

$$\left.\begin{aligned}
\frac{\partial A_1}{\partial t} &= kU_0(\tfrac{3}{4}-u_*)\,B_1 - \frac{2f_0}{k^2 p_0}C_2\\
\frac{\partial B_1}{\partial t} &= -kU_0(\tfrac{3}{4}-u_*)\,A_1 - \frac{2f_0}{k^2 p_0}D_2\\
\frac{\partial A_3}{\partial t} &= kU_0(\tfrac{1}{4}-u_*)\,B_3 + \frac{2f_0}{k^2 p_0}C_2\\
\frac{\partial B_3}{\partial t} &= -kU_0(\tfrac{1}{4}-u_*)\,A_3 + \frac{2f_0}{k^2 p_0}D_2
\end{aligned}\right\}\quad (6b.1)$$

The equations for C_2 and D_2 (the vertical velocity amplitudes at the level $p_* = \frac{2}{4}$, subscript 2) are:

$$\left.\begin{aligned}
-(2+a^2)\,C_2 &= -\tfrac{1}{4}\frac{k^3 p_0 U_0}{f_0}(B_1+B_3\\
&\quad + 2u_*\,(B_3-B_1)),\\
-(2+a^2)\,D_2 &= +\tfrac{1}{4}\frac{k^3 p_0 U_0}{f_0}(A_1+A_3\\
&\quad + 2u_*\,(A_3-A_1)).
\end{aligned}\right\}\quad (6b.2)$$

Equations (6b.1) and (6b.2) are now used to obtain $(\partial A_1/\partial t)_0$, $(\partial A_3/\partial t)_0$, $(\partial^2 B_1/\partial t^2)_0$ and $\partial^2 B_3/\partial t^2)_0$, which are necessary to compute $A_1(\Delta t)$, $B_1(\Delta t)$, $A_3(\Delta t)$ and $B_3(\Delta t)$ according to (6.12) in the case where, initially, $A_1 = A_3 = 0$ and $B_1 = B_3 = B_0$. The calculation is straightforward and we use the same procedure as in the earlier sections. Using the same parameters as before we arrive at the results that

$$\left.\begin{aligned}
R_1(12) &= 11.04\times 10^6, & \delta_1(12) &= -\ 0.1,\\
R_3(12) &= \ 9.84\times 10^6, & \delta_3(12) &= +29.6.
\end{aligned}\right\}\quad (6b.3)$$

The results are entered on Figs. 1 and 2 at the proper levels using circles. A straight solid line connects the points.

It is seen that we find only small errors in the displacement of the waves, a result, which is in good agreement with the results obtained in sections 4, 5 and 6, where all the waves were neutral. However, the errors in the amplitudes are much larger. At the lower level we find even an increase in the amplitude, while the continuous model, containing only time truncation errors, which probably are small, predicts a decrease in the amplitude for $p_* = \frac{3}{4}$. At the upper level we find about 50 % error in the change of the amplitude.

We conclude therefore that only small truncation errors are found in the displacements of the wave, while larger errors appear in the amplification of the waves.

c. *The three-level case*

In this case we have six equations for the amplitudes, A and B, obtained by applying the first two equations in the system (6.11) at the levels $p_* = \frac{1}{6}$, $\frac{3}{6}$ and $\frac{5}{6}$. We have four equations for the amplitudes of the vertical velocities, C and D, obtained by applying the last two equations in (6.11) at the levels $p_* = \frac{2}{6}$ and $p_* = \frac{4}{6}$. The complete set of 10 equations will not be reproduced here. It suffices to say that the equations have been solved using the same values of the basic parameters as in the earlier sections. The following values were obtained:

$$\left.\begin{aligned}
R_1(12) &= 11.34\times 10^6\ \mathrm{m^2\ sec^{-1}},\\
&\qquad \delta_1(12) = \ 0.2 \text{ degrees}\\
R_3(12) &= 11.42\times 10^6\ \mathrm{m^2\ sec^{-1}},\\
&\qquad \delta_3(12) = \ 6.7 \text{ degrees}\\
R_5(12) &= \ 9.00\times 10^6\ \mathrm{m^2\ sec^{-1}},\\
&\qquad \delta_5(12) = 40.8 \text{ degrees}
\end{aligned}\right\}\quad (6c.1)$$

These values are introduced on Figure 1 and 2, using triangles connected by dashed lines. It is seen that the truncation errors in the displacements are quite small. The errors in the amplitude changes are decreased compared with the two-level case. At the upper level we find an error in the amplitude change of about 37 % compared with about 50 % in the two-level case, while the error at the lowest level in the three-level case is 21 %. A large error (62 %) is found at the mid-level ($p_* = \frac{1}{2}$).

The case which has been treated in this section is certainly a very special one because the perturbation on the baroclinic zonal current initially has a phase angle and amplitude which is independent of pressure. Atmospheric conditions would be closer simulated if both of these quantities varied in a way close to the observed, mean state of the atmosphere. It is, nevertheless, felt that the present calculations provide a first estimate of the order of magnitude of the truncation errors in the vertical direction in quasi-nondivergent models. It is found that the truncation errors are large in the amplitude changes of growing, baroclinic disturbances, while the truncation errors in the displacements are small. In the finite difference models which have been treated here, we find that there is a tendency for a decrease in the truncation errors, when the vertical resolution is increased. It is therefore a tentative conclusion that we need a rather high vertical resolution to describe the amplification of baroclinic disturbances in an accurate way.

7. Comparison with simpler models

The computations carried out in Section 6 can, of course, be equally well made for somewhat simpler models. In this section we shall make a short comparison between the results obtained for the continuous model in Section 6 and models in which the static stability parameter (σ) is either constant or zero. The comparison will illustrate the importance of the vertical variation of the static stability parameter.

As the most simple case we shall first consider the so-called advective model, where by assumption $\sigma = 0$, i.e. it is assumed that $\partial\theta/\partial p = 0$ everywhere. This model has been analyzed in great detail by HOLMBOE (1958) and it is possible in this case to find exact solutions for the phase speed and amplification of the unstable waves. The first ones to analyze the structure of developing waves in a model where temperature changes are caused by horizontal advection only was ROSSBY (1944) and later on FJØRTOFT (1949).

In our case we shall only make calculations analogous to those in section 6. The linearized equations for the model are the first two equations of (6.11), while the last two equations for the vertical velocity amplitudes will be modified to be:

$$\left.\begin{aligned}\frac{\partial^2 C}{\partial p_*^2} &= -\frac{k^3 p_0 U_0}{f_0}\left(2B + u_*\frac{\partial B}{\partial p_*}\right),\\ \frac{\partial^2 D}{\partial p_*^2} &= +\frac{k^3 p_0 U_0}{f_0}\left(2A + u_*\frac{\partial A}{\partial p_*}\right).\end{aligned}\right\} \tag{7.1}$$

We shall assume as before that $A = 0$ and $B = B_0 = \text{const.}$ initially. The equations (6.16) apply therefore again. The solution can be obtained formally from (6.23) and (6.24) by setting $a^2 = 0$ and $\varepsilon = 1$, and we obtain:

$$A(\Delta t) = \Delta t \cdot B_0 \cdot kU_0\left[(1 - u_* - p_*) + (2p_* - 1)\right], \tag{7.2}$$

$$\begin{aligned}B(\Delta t) = B_0 - \tfrac{1}{2}\Delta t^2 B_0 kU_0(1 - u_* - p_*)\\ \times\left[(1 - u_* - p_*) + (2p_* - 1)\right] - \tfrac{1}{2}\Delta t^2 k^2 U_0^2 B_0\\ \times\left[\tfrac{1}{3}(p_*^2 - 1) - \tfrac{1}{2}u_*(2p_* - 1)\right].\end{aligned} \tag{7.3}$$

We may now compute $A(\Delta t)$ and $B(\Delta t)$ using the same values of the parameters as before except that $\sigma_0 = 0$ in this calculation. The results of this calculation are given in Figs. 3 and 4 expressed in amplitude and phase-angle using (6.26).

Fig. 3 shows that the amplification in the advective model is considerably larger than in the general quasi-nondivergent model. At no

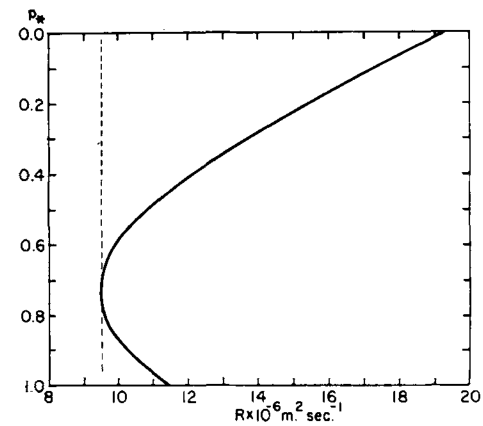

FIG. 3. Amplitude as a function of pressure after 12 hours in a baroclinic wave computed from the advective model. Parameters as in Fig. 1.

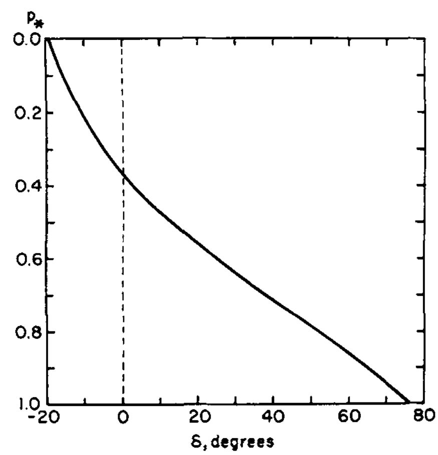

FIG. 4. Phase-angle as a function of pressure after 12 hours in a baroclinic wave computed from the advective model.

level do we find any decrease in the amplitude. It will be shown in the following section of this paper that the results shown in Fig. 3 are not peculiar to the specific example, but that we in general find a greater instability in the advective model.

It is seen from Fig. 4 that the unstable wave in this model moves toward the east in the lower part of the atmosphere, while its retrograde in the upper part. This result is different from the results shown in Fig. 2 for the more general quasi-nondivergent model, where the wave moves eastward initially in the upper part of the atmosphere. The result is, however, in agreement with the results obtained by ROSSBY (1944) and HOLMBOE (1958).

In the remaining part of this section we shall discuss the amplification and displacement of the waves in a quasi-nondivergent model in which we assume that the static stability parameter is constant, $\sigma = \bar{\sigma} = \text{const.}$ The purpose of the investigation is to find out, whether or not the vertical variation of σ is responsible for the difference in slope found in the upper and lower part of the atmosphere in Fig. 2.

The first two equations in (6.11) are unchanged, while the remaining equations are changed to

$$\left.\begin{aligned} \frac{\partial^2 C}{\partial p_*^2} - \bar{a}^2 C &= -\frac{k^3 p_0 U_0}{f_0}\left(2B + u_*\frac{\partial B}{\partial p_*}\right), \\ \frac{\partial^2 D}{\partial p^2} - \bar{a}^2 D &= +\frac{k^3 p_0 U_0}{f_0}\left(2A + u_*\frac{\partial A}{\partial p_*}\right), \end{aligned}\right\} \quad (7.4)$$

where
$$\bar{a}^2 = \frac{k^2 \bar{\sigma} p_0^2}{f_0^2}. \quad (7.5)$$

We consider again the case where $A = 0$ and $B = B_0 = \text{const.}$ at $t = 0$. The formulas (6.16) apply therefore to compute the values of $A(\Delta t)$ and $B(\Delta t)$. In order to compute $(\partial A/\partial t)_0$ we must first compute C_0 by solving the first equation (7.4), which in this case has solutions of the exponential type. After application of the boundary conditions $C = 0$ at $p_* = 0$ and $p_* = 1$, we find that

$$C_0 = \frac{2k^3 p_0 U_0}{\bar{a}^2 f_0} B_0\left(1 - \frac{\sin h(\bar{a}p_*)}{\sin h\,\bar{a}} - \frac{\sin h(\bar{a}(1-p_*))}{\sin h\,\bar{a}}\right) \quad (7.6)$$

and the solution for $A(\Delta t)$ becomes:

$$A(\Delta t) = \Delta t \cdot kU_0 B_0 \cdot \left[(1 - u_* - p_*) + \frac{2}{\bar{a}}\left(\frac{\cos h(\bar{a}p_*)}{\sin h(\bar{a})} - \frac{\cos h(\bar{a}(1-p_*))}{\sin h(\bar{a})}\right)\right]. \quad (7.7)$$

From the second equation in (6.11) we find that

$$\frac{\partial^2 B}{\partial t^2} = -kU_0(1 - u_* - p_*)\frac{\partial A}{\partial t} - \frac{f_0}{k^2 p_0}\frac{\partial}{\partial p_*}\left(\frac{\partial D}{\partial t}\right). \quad (7.8)$$

The differential equation for $\partial D/\partial t = D_t$ is

$$\frac{\partial^2 D_t}{\partial t^2} - \bar{a}^2 D_t = \frac{k^3 p_0 U_0}{f_0}\left(2\frac{\partial A}{\partial t} + u_*\frac{\partial}{\partial p_*}\left(\frac{\partial A}{\partial t}\right)\right). \quad (7.9)$$

$(\partial A/\partial t)_0$ is known from (7.7). Substituting this expression for in (7.9) we can solve for D_t which turns out to be:

$$D_t = -\frac{n_0}{\bar{a}^2} - \frac{n_1}{\bar{a}^2}p_* + \frac{n_0}{\bar{a}^2}\cos h(\bar{a}p_*) + \sin h(\bar{a}p_*) \times \left[-\frac{n_0}{\bar{a}^2}\tan h\left(\frac{\bar{a}}{2}\right) + \frac{n_1}{\bar{a}^2}\frac{1}{\sin h(\bar{a})} - \frac{n_2}{2\bar{a}} - \frac{n_3}{2\bar{a}}\cot h\,\bar{a}\right] + \frac{n_2}{2\bar{a}}p_*\sin h(\bar{a}p_*) + \frac{n_3}{2\bar{a}}\cos h(\bar{a}p_*), \quad (7.10)$$

where

$$\left.\begin{aligned} n_0 &= \frac{k^4 U_0^2 p_0 B_0}{f_0}(2 - 3u_*), \\ n_1 &= -\frac{k^4 U_0 p_0 B_0}{f_0}, \\ n_2 &= \frac{2k^4 U_0^2 p_0 B_0}{f_0 \bar{a}\sin h\,\bar{a}}[2(1 - \cos h\,\bar{a}) + u_*\,\bar{a}\sin h\,\bar{a}], \\ n^3 &= \frac{2k^4 U_0^2 p_0 B_0}{f_0\,\bar{a}\sin h\,\bar{a}}[2\sin h\,\bar{a} + u_*\,\bar{a}(1 - \cos h\,\bar{a})]. \end{aligned}\right\} \quad (7.11)$$

When we substitute from (7.7) and (7.10) into (7.8) we can compute $(\partial^2 B/\partial t^2)_0$ and consequently $B(\Delta t)$. Knowing $A(\Delta t)$ and $B(\Delta t)$ we can compute the amplitude (R) and phase angle (δ) at $t = \Delta t$. The distribution of δ as a function of p_* is shown in Fig. 5. The amplitude (not shown) has a structure very similar to the one shown in Fig. 1.

The distribution in Fig. 5 shows clearly that the change of the vertical slope of the ridge line of the wave in the upper part of the atmosphere

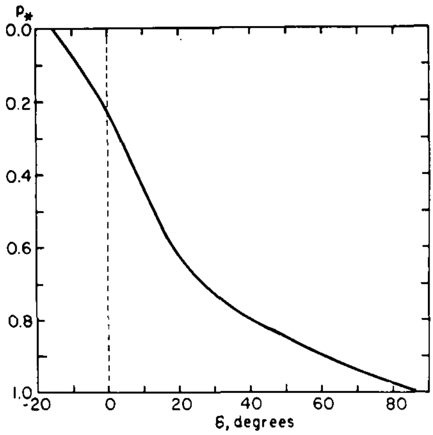

FIG. 5. Phase-angle as a function of pressure after 12 hours in a baroclinic wave computed from a model in which $\sigma = \bar{\sigma} = \text{const.}$

as found in Fig. 2 is due to the increase of the stability parameter (σ) when the pressure becomes small, because the only difference between the calculations leading to Figs. 2 and 5 is that $\sigma = \sigma(p)$ in the first calculation while $\sigma = \bar{\sigma} = \text{const.}$ in the second calculation.

8. The initial change of the kinetic energy

The perturbations considered in this paper are very special, because we have only considered wave-disturbances which initally have the same phase angle and amplitude at all levels. Even so, it is of some interest to investigate the stability of these simple disturbances. We shall do this by computing the initial change of the kinetic energy, because we are unable to compute the amplification rate in the usual way in most cases. Normally, the amplification rate is computed from the imaginary part of the wave speed, and we can not determine this quantity in the most general case without approximations.

The kinetic energy of the disturbance is in our case measured by the following integral:

$$K = \frac{1}{2g}\int_0^{p_0}\int_0^L v^2\,dx\,dp, \tag{8.1}$$

where L is the wave length of the disturbance.

The integral (8.1) can easily be evaluated for the disturbances prescribed in (6.1). We obtain

$$K = \frac{\pi}{2g}\int_0^{p_0}(A^2+B^2)\,dp \tag{8.2}$$

from which it follows that

$$\left(\frac{dK}{dt}\right)_0 = \frac{\pi}{g}\int_0^{p_0}\left(A\frac{\partial A}{\partial t}+B\frac{\partial B}{\partial t}\right)dp. \tag{8.3}$$

We have considered disturbances in the preceding sections in which $A = 0$ and $B = B_0 =$ const. at $t = 0$. It was shown that $(\partial B/\partial t)_0 = 0$. It follows therefore that $(dK/dt)_0$ also vanishes. In order to find the initial change of the kinetic energy in our particular case, it is therefore necessary to consider the second derivative of K with respect to time. We find from (8.3)

$$\left(\frac{d^2K}{dt^2}\right)_0 = \frac{\pi}{g}\int_0^{p_0}\left(A\frac{\partial^2 A}{\partial t^2}+B\frac{\partial^2 B}{\partial t^2} + \left(\frac{\partial A}{\partial t}\right)^2 + \left(\frac{\partial B}{\partial t}\right)^2\right)dp. \tag{8.4}$$

The first and the last term in the integrand vanish for the disturbances mentioned above. We have therefore

$$\left(\frac{d^2K}{dt^2}\right)_0 = \frac{\pi}{g}\int_0^{p_0}\left(B\frac{\partial^2 B}{\partial t^2} + \left(\frac{\partial A}{\partial t}\right)^2\right)dp. \tag{8.5}$$

If the integral (8.5) is positive, we have an initial growth of the kinetic energy, while $(d^2K/dt^2)_0 < 0$ indicate a decrease.

The computations made in Sections 6 and 7 can obviously be used to evaluate the initial change of the kinetic energy for the different model: the quasi-nondivergent model with $\sigma = \sigma(p)$, the same model with $\sigma = \bar{\sigma}$ and finally $\sigma = 0$. In each case it is sufficient to substitute the expressions for $(\partial A/\partial t)_0$ and $(\partial^2 B/\partial t^2)_0$ in (8.5) and evaluate the integral. It is, in fact, seen from the second equation in (6.11) that

$$\left(B\frac{\partial^2 B}{\partial t^2}\right)_0 = -B_0\left[kU_0(1-u_*-p_*)\left(\frac{\partial A}{\partial t}\right)_0 + \frac{f_0}{k^2 p_0}\frac{\partial}{\partial p_*}\left(\frac{\partial D}{\partial t}\right)_0\right]. \tag{8.6}$$

The last terms in the bracket gives no contribut ionto the integral because $(\partial D/\partial t)_0$ for $p_* = 0$ and $p_* = 1$. It is therefore only necessary to compute C_0 and thereby $(\partial A/\partial t)_0$.

In case of the quasi-nondivergent model with $\sigma = \sigma(p_*) = \sigma_0/p_*^2$ we obtain from (8.5)

$$\left(\frac{d^2K}{dt^2}\right)_0 = \frac{\pi p_0}{g} U_0^2 B_0^2 k^2 \left[\frac{2}{2-a^2}\left(\frac{\varepsilon}{\varepsilon+1}-\frac{2}{3}\right) + \frac{4}{(2-a^2)^2}\left(\frac{4}{3}+\frac{\varepsilon^2}{2\varepsilon-1}-\frac{4\varepsilon}{\varepsilon+1}\right)\right]. \tag{8.7}$$

a^2 and ε are defined in (6.18). In Fig. 6 we have plotted (8.7) as a function of wave length using $f_0 = 10^{-4}$ sec^{-1}, $\sigma_0 = \frac{1}{2}$ MTS units $U_0 = 40$ m sec^{-1} and $B_0 = 9.55 \times 10^6$ m^2 sec^{-1}. This curve is the solid curve in Fig. 6. It is seen that the waves are amplifying for $L \geqslant 2500$ km. The maximum change of the kinetic energy is found for $L \simeq 6000$ km. A decrease of kinetic energy is found for $L < 2500$ km.

The result displayed in Fig. 6 (solid curve) is in qualitative agreement with the baroclinic instability theory for the quasi-nondivergent model. The main difference is that we find a decrease of the kinetic energy for sufficiently small wave lengths for the special disturbances considered here. The results by CHARNEY (1947) show no tendency for stable conditions where the wave length becomes small. Our result is, however, in agreement with the result of FJØRTOFT (1949). We emphasize again that our result only is valid for disturbances which have the same phase-angle and amplitude at all levels at two, and that it therefore is not directly comparable with the results obtained by CHARNEY and FJØRTOFT (loc. cit.). Their results are derived under the assumption that the disturbance moves with the same speed at all levels.

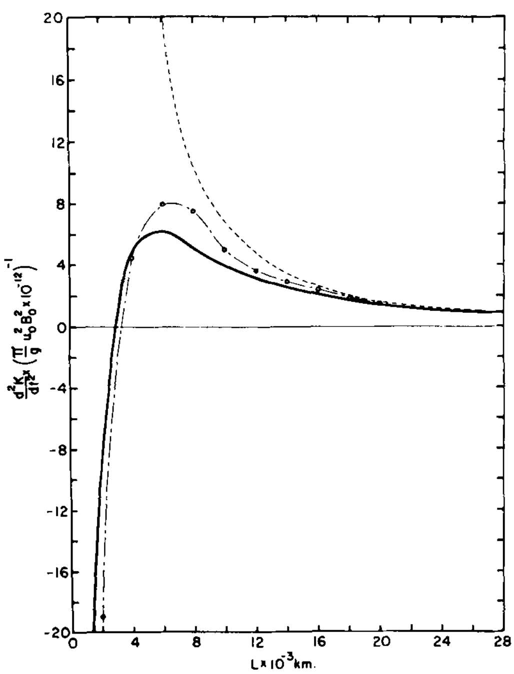

FIG. 6. The initial change of kinetic energy of the perturbation as a function of wave number. The solid curve for a model, where $\sigma = \sigma(p)$, the dashed curve for a model with $\sigma = 0$, and dashed-dotted curve for a model with $\sigma = \bar{\sigma} = $ const.

If we use the assumption $\sigma = 0$ (Section 7) we obtain from (8.5)

$$\left(\frac{d^2K}{dt^2}\right)_0 = \frac{1}{6}\frac{\pi p_0}{g} U_0^2 B_0^2 k^2. \tag{8.8}$$

The curve corresponding to (8.8) is shown in Fig. 6 as the dashed curve. We find a good agreement with the more general quasi-nondivergent model for large wave lengths, but a large increase of the change of kinetic energy for smaller wave lengths. This result is well known from earlier investigations of the advective model.

For the model where we assume that $\sigma = \bar{\sigma} =$ const, we finally obtain

$$\left(\frac{d^2K}{dt^2}\right)_0 = \frac{\pi p_0}{g} U_0^2 B_0^2 \left\{ 2k^2 \frac{1}{\bar{a} \sinh \bar{a}} \times \left(\frac{2 \sinh^2\left(\frac{\bar{a}}{2}\right)}{\left(\frac{\bar{a}}{2}\right)^2} - \frac{\sinh \bar{a}}{\bar{a}} - \frac{\tanh \frac{\bar{a}}{2}}{\frac{\bar{a}}{2}} \right) \right\}. \tag{8.9}$$

The curve corresponding to (8.9) is shown in Fig. 6 as a function of wave length (the dashed dotted curve). In general we find a good qualitative agreement with the corresponding curve for the more general quasi-nondivergent model. It should be noted that the model with $\sigma = \bar{\sigma}$ shows somewhat larger increase in the kinetic energy for intermediate wave lengths than the model $\sigma = \sigma(p)$. This result depends naturally somewhat on the value ascribed to $\bar{\sigma}$. In the calculation in Section 7 and also in the evaluation of (8.9) we have used a value of $\bar{\sigma} = 2.56$ MTS units, which is close to the value adopted in short range prediction. We conclude therefore that models based upon the assumption

$\sigma = \bar{\sigma}$ will show too great amplification measured by the kinetic energy change compared with models where $\sigma = \sigma(p)$ has a more realistic distribution.

9. Some comments on the upper boundary condition

It will be noted that all the solutions in this paper are obtained using the upper boundary condition $\omega = dp/dt = 0$ for $p = 0$. Some criticism of this boundary condition has recently been given. (WELANDER, 1961). It was claimed by Welander (without proof), that the boundary condition mentioned above allows the amplitude of the geopotential to become infinite as the pressure approaches zero, and that $\omega = 0$ for $p = 0$ is a relatively weak condition. The condition that the geopotential should remain finite for $p = 0$ was suggested as an alternative and stronger condition.

In view of Welander's statements it is worthwhile to investigate the upper boundary condition in some detail. A possible method to investigate the influence of two different boundary conditions is, of course, to solve the basic equations using first one and then the other boundary condition and finally compare the two solutions. This procedure is, of course, not always possible in reality, because it depends upon our ability to solve a problem with two different sets of boundary conditions. However, in the particular case discussed by Welander we are able to do this as we shall show below.

The linearized equations for the very long waves in the atmosphere have been proposed to be

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial \phi}{dp}\right) - \frac{dU}{dp}\frac{\partial \phi}{\partial x} + \sigma\omega = 0, \quad \beta\frac{\partial \phi}{\partial x} = f^2\frac{\partial \omega}{\partial p}. \tag{9.1}$$

The notations are the same as earlier in this paper. We introduce wave disturbances of the form

$$\left.\begin{aligned} \phi &= \Phi(p)\, e^{ik(x-ct)}, \\ \omega &= \Omega(p)\, e^{ik(x-ct)} \end{aligned}\right\} \tag{9.2}$$

and obtain the equations

$$\left.\begin{aligned} \sigma\Omega &= ik\left[(c-U)\frac{d\Phi}{dp} + \frac{dU}{dp}\Phi\right], \\ f^2\frac{d\Omega}{dp} &= ik\beta\Phi. \end{aligned}\right\} \tag{9.3}$$

If we first eliminate Ω from the system (9.3), we obtain the equation for the geopotential originally solved by Welander using the boundary condition

$$\Phi < \infty,\ p = 0, \quad \Omega = 0,\ p = p_0. \tag{9.4}$$

Introducing the transformation

$$p_* = \frac{p}{p_0},\ \xi = \frac{P_*}{1 - c_*} \tag{9.5}$$

and assuming the distributions (6.10) for U and σ with $c_* = c/U_0$, we can write the equation in the form:

$$(1-\xi)\,\xi\frac{d^2\Phi}{d\xi^2} + (1-\xi)\frac{d\Phi}{d\xi} + (1+q)\,\Phi = 0, \tag{9,6}$$

where

$$q = \frac{\beta}{f^2}\frac{\sigma_0 p_0^2}{U_0}. \tag{9.7}$$

(9.6) is a special case of the hypergeometeric equation. According to ERDELYI (1953, p. 75) we can write the complete solution to (9.6) in the form

$$\Phi(\xi) = A\Phi_1(\xi) + B\Phi_2(\xi), \tag{9.8}$$

where

$$\left.\begin{aligned} \Phi_0(\xi) &= F(r, -r;\ 1;\ \xi), \\ \Phi_2(\xi) &= (1-\xi)\,F(1-r,\ 1+r;\ 2;\ 1-\xi). \end{aligned}\right\} \tag{9.9}$$

$$r^2 = 1 + q. \tag{9.10}$$

$F(a, b; c; \xi)$ is the notation for the hypergeometric function.

The integration constant B can be determined from the boundary condition that $\Phi < \infty$ for $\xi \to 0$. In fact, $B = 0$ because $\Phi_2 \to \infty$ for $\xi \to 0$ (see ERDELYI, 1953, p. 104, eq. (46)).

The complete solution is therefore

$$\Phi(\xi) = A\Phi_1(\xi). \tag{9.11}$$

The lower boundary condition can be used to determine the values of C^* as done by Welander.

We shall next solve the system (9.3) by elimination of Φ and making use of the boundary condition:

$$\Omega = 0, \quad p = 0, \quad p = p_0. \tag{9.12}$$

The equation for Ω becomes, using the same transformations as before,

$$(1-\xi)\,\xi\frac{d^2\Omega}{d\xi^2} + \xi\frac{d\Omega}{d\xi} + q\Omega = 0. \tag{9.13}$$

(9.13) is another special case of the hypergeometric equation. It has the solution (Erdelyi, 1953, p. 75).

$$\Omega(\xi) = A^*\Omega_1(\xi) + B^*\Omega_2(\xi), \tag{9.14}$$

where

$$\left.\begin{aligned}\Omega_1(\xi) &= \xi\, F(r, -r;\ 2;\ \xi),\\ \Omega_2(\xi) &= (1-\xi)^2 F(1-r,\ 1+r;\ 3;\ 1-\xi).\end{aligned}\right\} \tag{9.15}$$

We shall now use the boundary condition (9.12). For $\xi \to 0$ we find $\Omega_1 \to 0$, while

$$\Omega_2(0) = \frac{\Gamma(3)\,\Gamma(1)}{\Gamma(2+r)\,\Gamma(2-r)}, \tag{9.16}$$

where Γ is the gammafunction. The value (9.16) is finite, but different from zero if $r < 2$. It follows therfore that $B^* = 0$, and the solution corresponding to the boundary condition (9.12) is

$$\Omega(\xi) = A^*\,\Omega_1(\xi). \tag{9.17}$$

We shall next show that the solution (9.17) leads to the same solution for Φ as we have obtained in (9.11) using the boundary condition (9.4). We use the second equation in (9.3) for this purpose and obtain

$$\Phi = \frac{1}{ik}\frac{f^2}{\beta}\frac{d\Omega}{dp} = \frac{1}{ik}\frac{f^2}{\beta}\frac{1}{p_0(1-c_*)}\frac{d\Omega}{d\xi}, \tag{9.18}$$

but we have

$$\frac{d\Omega}{d\xi} = A^*\frac{d}{d\xi}(\xi\, F(r_1 - r;\ 2;\ \xi)) = A^*\, F(r_1 - r;\ 1,\ \xi). \tag{9.19}$$

The last relation follows from Erdelyi (1953, p. 102, eq. (22)). It is therefore seen that the two sets of boundary conditions (9.4) and (9.12) lead to exactly the same solutions. We can therefore state that the boundary conditions (9.4) and (9.12) are completely equivalent in this particular case.

It is beyond the scope of this paper to find the general condition under which the two different sets of boundary conditions lead to the same solutions, but it is strongly suspected that they will be equivalent if the temperature and its first derivatives in time and space are finite at the upper boundary.

10. Summary and conclusions

Some estimates of truncation error due to vertical finite differences have been obtained. In the case of no vertical windshear, we find errors in the phase speeds of the order of 10% in the basic mode for the smallest vertical resolution. The truncation error in the phase-speed decreases rapidly, when the resolution is increased. The truncation errors are of the same order of magnitude, but of a different sign if the vertical stability parameter is allowed to change with respect to pressure, but the vertical windshear still vanishes.

In the case where we have a vertical windshear different from zero and the stability (σ) varies with pressure, we find only small truncation errors in the displacement of the wave, but large errors in the amplification. For the smallest vertical resolution the errors may amount to 50% but seems to decrease somewhat when the resolution is increased. The conclusion is therefore that a rather high vertical resolution is needed in order to predict amplification with small truncation errors.

Section 7 contains a comparison between the development of baroclinic waves in quasi-nondivergent models where $\sigma = \sigma(p)$, $\sigma = \bar{\sigma} =$ const. and $\sigma = 0$. It is found that the increase of the static stability parameter in the stratosphere must be extremely important for the slope of the waves in this part of the atmosphere.

The final section contains some comments on the upper boundary condition. It is especially shown that the two boundary conditions $\omega = 0$ for $p = 0$ and $\varphi < \infty$ for $p = 0$ are equivalent for a model proposed for very long waves in the atmosphere.

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