Establishment of a scheme stable in initial data to compute meteorological elements by means of solving thermohydrodynamic equations

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ABSTRACT

The stability of prognostic schemes obtained by means of the solution of primitive equations is studied. Three possible schemes are examined among which only one scheme is stable in Richtmyer's sense. The relationship between time step and grid step depends essentially on the least characteristic number λ_1 of a matrix A as follows: $\frac{\Delta t}{\Delta s} \leqslant \sqrt{6 \cdot \lambda_1}$. The order of the matrix A is equal to the number of levels considered in the prognostic sheme.

1. To determine a would-be atmospheric pressure and wind field in a baroclinic atmosphere for short-range forecast of these fields under the hypothesis of quasi-static atmosphere and adiabatic processes, let us proceed from the following set of thermohydrodynamic equations (1)

$$u_t - lv + H_x = -F^{(u)}, \quad u_x + v_y + \bar{\tau}_\zeta = 0,$$

$$v_t + lu + H_y = -F^{(v)}, \quad \bar{\tau} = \frac{\zeta}{RT}[H_t + M(H) - gw],$$

$$\zeta^2 H_{\zeta t} + c^2 \bar{\tau} = -\zeta^2 M(H_\zeta), \tag{1.1}$$

where $F^{(u,v)}$ and $M(H, H_\zeta)$ are the operators expressed by

$$F^{()} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \bar{\tau}\frac{\partial}{\partial \zeta}, \quad M(\ ) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y};$$

H, geopotential; $f_s = \partial f/\partial s$, ($f = u, v, H, \bar{\tau}$; $s = x, y, \zeta, t$); $\zeta = p/p_0$, reduced pressure; p, atmospheric pressure; p_0, mean sea-level atmospheric pressure; u, v, w, wind speed vector components; $c^2 = \alpha RT$, $\alpha = (\gamma_a - \gamma) R/g$; R, gas constant; T, temperature, γ_a, adiabatic gradient, and γ, vertical temperature gradient; l, Coriolis parameter; g, acceleration due to gravity.

Boundary conditions for (1.1) over ζ will be as follows:

$$\text{at } \zeta = 0, \ \zeta w = 0; \quad \text{at } \zeta = 1, \ w = 0. \tag{1.2}$$

The relationship between functions H and $\bar{\tau}$ (see (1.1)) will be used to establish boundary conditions for determining functions H and $\bar{\tau}$.

An additional assumption that the right-hand terms of these equations defined by the non-linear operators $F^{(\ )}$ and $M(\ )$ may be substituted by their values at the initial moment of time, and consequently will be known functions of coordinates x, y, ζ, will determine the functions u, v, τ and H by means of solving (1.1).

Transformations of (1.1) will be made in the following way. Transferring H_x and H_y to the right-hand side and considering them to be known, we shall determine the functions u and v from the first two equations of motion. Then u and v may be expressed by[1]

$$\left.\begin{aligned} u(x,y,\zeta,t) &= u_c(t,t_1) + v_s(t,t_1) \\ &\quad - \int_{t_1}^{t} [H_x \cos l(t-t_0) + H_y \sin l(t-t_0) \\ &\quad + F_c^{(u)}(t,t_0) + F_s^{(v)}(t,t_0)]\, dt_0 = K^{(u)}, \\ v(x,y,\zeta,t) &= v_c(t,t_1) - u_s(t,t_1) \\ &\quad - \int_{t_1}^{t} [H_y \cos l(t-t_0) - H_x \sin l(t-t_0) \\ &\quad + F_c^{(v)}(t,t_0) + F_s^{(u)}(t,t_0)]\, dt_0 = K^{(v)}, \end{aligned}\right\} \tag{1.3}$$

[1] Terms with $\beta = dl/dy$ under the integral in (1.3) are omitted. β may be taken into account by differentiating the functions $u_{c,s}$, $v_{c,s}$, $F^{(u)}_{c,s}$ and $F^{(v)}_{c,s}$.

where

$$\xi_{c,s}(t, t_1) = \xi(x, y, \zeta, t_1) \begin{cases} \sin l\,(t-t_1) \\ \cos l\,(t-t_1) \end{cases}; \quad \xi = u, v, F^{(u)}, F^{(v)}.$$

Horizontal divergence of the wind field

$$D = u_x + v_y$$

may be computed by means of (1.3) as follows:

$$D(x, y, \zeta, t) = D_c(t, t_1) + \Omega_s(t, t_1) - B(t, t_1) - \int_{t_1}^{t} \Delta H \cos l\,(t-t_0)\,dt_0, \qquad (1.4)$$

where

$$D_c = u_{cx} + v_{cy}; \quad \Omega_s = v_{sx} - u_{sy}; \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};$$

$$B(t, t_1) = \int_{t_1}^{t} [\bar{D}(t, t_0, F_c) + \bar{\Omega}(t, t_0, F_s)]\,dt_0;$$

$$\bar{D} = F_{cx}^{(u)} + F_{cy}^{(v)}; \quad \bar{\Omega} = F_{sx}^{(v)} - F_{sy}^{(u)}.$$

Using the value of function D and applying the equation of continuity, the function $\bar{\tau}$ will be removed from the thermodynamic equation and one integro-differential equation for geopotential H will be formed. As a result of this transformation we have

$$\frac{\partial}{\partial \zeta}\frac{\zeta^2}{c^2}\frac{\partial^2 H}{\partial \zeta \partial t} + \int_{t_1}^{t} \Delta H \cos l\,(t-t_0)\,dt_0 = D_c - \Omega_s - B - \frac{\partial}{\partial \zeta}\frac{\zeta^2}{c^2} M(H_\zeta). \qquad (1.5)$$

For convenience the following function may be introduced

$$\eta = H(x, y, \zeta, t) - H_1(x, y, \zeta, t_1), \qquad (1.6)$$

so that at $\quad t = t_1, \quad \eta \equiv 0.$

We shall integrate (1.5) over t in the interval (t_1, t_2). Then, using Dirichlet's formula for changing the order of integration

$$\int_{t_1}^{t_2} dt' \int_{t_1}^{t'} \phi(t_0) \begin{Bmatrix} \sin l\,(t'-t_0) \\ \cos l\,(t'-t_0) \end{Bmatrix} dt_0 = \frac{1}{l}\int_{t_1}^{t_2} \phi(t_0) \begin{Bmatrix} 1 - \cos l\,(t_2-t_0) \\ \sin l\,(t_2-t_0) \end{Bmatrix} dt_0,$$

the function η will be expressed by the following equation

$$\frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial \eta}{\partial \zeta} + \frac{1}{l}\int_{t_1}^{t_2} \Delta \eta \sin l\,(t_2 - t_0)\,dt_0 = q(x, y, \zeta, t_1, t_2), \qquad (1.7)$$

$$q = D_s + \left(\Omega - \frac{1}{l^2}\Delta H_1\right)\left[1 - \cos l\,(t_2 - t_1)\right] - B_1(t_2, t_1) - (t_2 - t_1)\frac{\partial}{\partial \zeta} K(\zeta) M(H_\zeta),$$

$$B_1(t_2, t_1) = \frac{1}{l}\int_{t_1}^{t_2} [\bar{D}(t', t_1, F_s) + \bar{\Omega}(t_1, t_1, F_c) - \bar{\Omega}(t', t_1, F_c)]\,dt'; \quad K(\zeta) = \zeta^2/c^2.$$

The boundary conditions for (1.7) along the variable ζ are formulated as follows:

$$\left.\begin{aligned} &\text{at } \zeta = 0 \\ &\zeta^{1+\theta}\frac{\partial \eta}{\partial \zeta} = 0 \quad 0 < \theta < \alpha, \\ &\text{at } \zeta = 1 \\ &\zeta\frac{\partial \eta}{\partial \zeta} + \alpha\eta = q_{n+1} \\ &\qquad = -(t_2 - t_1)\,[M(H_\zeta) + \alpha M(H)]_{\zeta=1}. \end{aligned}\right\} \qquad (1.8)$$

If c^2 is a function of ζ (horizontal changes of the parameter are neglected) then the method of substituting the operator

$$R = \frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial}{\partial \zeta}$$

by a finite-differential analogue at spectrum points $0 \leqslant \zeta_i \leqslant 1$[1] will be used in solving the initial-value problem (1.7), (1.8). ζ_i corresponds here to the ith standard pressure surface. Instead of (1.7), the equivalent set of integro-differential equations relative to the functions $\eta_i = \eta(x, y, \zeta_i, t)$ is obtained in this case. Solution of this set may conveniently be obtained by a method of canonizing such sets of equations, based on computation of eigenvectors of the matrix $A = \|a_{ij}\|^n$. The elements of this matrix are determined by a finite-differential analogue of the operator (NEMCHINOV, 1960; 1961).

$$R(\eta) = \left[\frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial \eta}{\partial \zeta}\right]_{\zeta = \zeta_i} = a_{ii+1} \cdot \eta_{i+1} + a_{ii}\eta_i + a_{ii-1}\eta_{i-1}$$

$$a_{ii+1} + a_{ii} + a_{ii-1} = 0. \qquad (1.9)$$

[1] Other methods for solving this initial-value problem are given in NEMCHINOV (1961*b*).

Then instead of (1.7) there will be a set of equations

$$a_{ii+1}\eta_{i+1}+a_{ii}\eta_i+a_{ii-1}\eta_{i-1}+\frac{1}{l}\int_{t_2}^{t_1}\Delta\eta_i \sin l\,(t_2-t_0)\,dt_0=q_i \quad (i=1,2,\ldots,n), \tag{1.10}$$

where the functions η_0 and η_{n+1} are deleted from (1.10) by boundary conditions of the form:

$$\left.\begin{aligned} &\text{at}\quad \zeta=\zeta_{\frac{1}{2}}=0 \quad \zeta^{1+\theta}\frac{\partial\eta}{\partial\zeta}=0 \quad \text{i.e. } a_{10}\equiv 0.\\ &\text{at}\quad \zeta=\zeta_{n+1}=1 \quad \zeta\frac{\partial\eta}{\partial\zeta}+\alpha\eta=q_{n+1}.\end{aligned}\right\} \tag{1.11}$$

If matrices are introduced

$$\psi=\begin{vmatrix}\eta_1\\ \eta_2\\ \vdots\\ \vdots\\ \vdots\\ \eta_n\end{vmatrix};\quad A=\begin{vmatrix}a_{11}\,a_{12} & 0 & 0\ldots & \ldots & 0\\ a_{21}\,a_{22} & a_{23} & 0\ldots & \ldots & 0\\ 0\quad a_{32} & a_{33} & a_{34}\ldots & 0 & 0\\ \vdots & & & & \\ \vdots & & & & \\ 0 & \ldots\ldots & \ldots\ldots & & a_{nn-1}\,a_{nn}\end{vmatrix};$$

$$Q=\begin{vmatrix}q_1\\ q_2\\ \vdots\\ \vdots\\ \vdots\\ q_n+bq_{n+1}\end{vmatrix}$$

where $b=\zeta_{n+1}-1$, then (1.10) with boundary conditions of (1.11) may be written in a matrix form

$$A\psi+\frac{1}{l}\int_{t_1}^{t_2}\Delta\psi\sin l\,(t_2-t_0)\,dt_0=Q. \tag{1.13}$$

It is possible to show that the matrix A is of a prime structure i.e. all eigenvalues $\lambda_j(j=1,2,\ldots,n)$ are prime and furthermore are real and negative (if $K(\zeta)>0$; see NEMCHINOV, 1960; and FADDEEV, 1960). But then matrix A will be expressed as

$$A=B^{-1}LB,\quad L=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}, \tag{1.14}$$

where L is a diagonal matrix with elements λ_j and matrix B is a non-singular[1] square matrix of the nth order. Here the kth matrix-row B is an eigenvector $b^{(k)}=(b_1^{(k)},b_2^{(k)},\ldots,b_n^{(k)})$ of matrix A, the $b_j^{(k)}$ components of which are defined as algebraic cofactors of any determinant-column

$$\det|A-\lambda_k E|=0\quad (k=1,2,\ldots n) \tag{1.15}$$

Vectors $b^{(k)}$, therefore, will form one set of eigenvectors of matrix A. Another set of eigenvectors of this matrix $\boldsymbol{b}^{(\nu)}=(\boldsymbol{b}_1^{(\nu)},\boldsymbol{b}_2^{(\nu)},\ldots\boldsymbol{b}_n^{(\nu)})$ consists of the B^{-1} matrix columns. The components $\boldsymbol{b}_j^{(\nu)}$ are defined here as algebraic cofactors of any column of the same determinant (1.15) at $k=\nu$.[2] Since the sets of eigenvectors $b_j^{(k)}$ and $\boldsymbol{b}_j^{(\nu)}$ are orthogonal

$$\sum_j b_j^{(k)}\boldsymbol{b}_j^{(\nu)}=\begin{cases}0, & k\neq\nu\\ d_k^2, & k=\nu\end{cases} \tag{1.16}$$

it is convenient to normalize them. Then the components $c_j^{(k)}$ and $\boldsymbol{c}_j^{(\nu)}$ from the equalities

$$c_j^{(k)}=b_j^{(k)}/d_k,\quad \boldsymbol{c}_j^{(\nu)}=\boldsymbol{b}_j^{(\nu)}/d_k$$

should be used instead of components $b_j^{(k)}$ and $\boldsymbol{b}_j^{(\nu)}$.

Using matrix A as (1.14), we multiply equation (1.13) from the left by matrix B and introducing the notations $\bar{\psi}=B\psi$ and $\bar{Q}=BQ$ for the vector $\bar{\psi}$, we obtain the equation

$$L\bar{\psi}+\frac{1}{l}\int_{t_1}^{t_2}\Delta\bar{\psi}\sin l\,(t_2-t_0)\,dt_0=\bar{Q}, \tag{1.17}$$

which is the canonic form of (1.10). The appropriate method for solving this equation may be found for every particular case.

When the function $\bar{\psi}$ is found, the unknown vector ψ is determined by $\psi=B^{-1}\bar{\psi}$.

Since equation (1.17) is hyperbolic, it is at initial time sufficient to know the unknown functions only within the cone of characteristics. But since the module of the least characteristic number of matrix A, $|\lambda_1|$, is sufficiently small and becomes less as n increases (n is the matrix

[1] Non-singularity of the matrix, i.e. $\det B\neq 0$, is explained by a prime structure of matrix A.

[2] This method for constructing the matrix B^{-1} is applied only in the case when the matrix A is asymmetric, i.e. if, when approximating the operator R by means of initial-difference analogue, the interval (0, 1) is devided into inequal segments by points ζ_i. This is the case when (1.9) is solved using standard pressure levels. If the interval (0, 1) is devided into equal segments by points ζ_i, then A will be a symmetric matrix. Then $A=A'$, where the prime denotes transposition, and consequently $b^{(k)}\equiv\boldsymbol{b}^{(k)}$. It then follows that $B^{-1}=B'$.

order), for one of these equations of (1.17), corresponding to λ_1, the radius of the circle with given initial values is sufficiently large, and as large as $|\lambda_1|$ is small (or n is large).

Then, apparently, it is of advantage to substitute equation (1.17) by some other elliptic equation with a Green's function, which vanishes as r increases, but theoretically is not identically zero along the plane. This can be done if, for example, the vector $\bar{\psi}$ under the integral in (1.17) is given in a form of an interpolated Newton's polynomial in the variable t with the initial point t_2.[1] Then, considering that

$$\bar{\psi}_{t-t} = \bar{\psi}_1 \equiv 0 \quad \text{and} \quad \bar{\psi}_{t-t_2} = \bar{\psi}_2 \qquad (1.18)$$

and according to (1.16)

$$\int_{t_1}^{t_2} \bar{\psi} \sin l\,(t_2 - t_0)\,dt_0$$

$$\approx (t_2 - t_1) \int_0^1 (1-\mu)\,\bar{\psi}_2 \sin \varepsilon\mu d\mu = \frac{\varepsilon - \sin\varepsilon}{l\varepsilon}\,\bar{\psi}_2$$

$$\varepsilon = \Delta t l = l\,(t_2 - t_1) \qquad (1.19)$$

Therefore, taking (1.19) into account, equation (1.17) may be substituted by a new elliptical equation, a Helmholtz equation

$$L\bar{\psi}_2 + K_1(\varepsilon)\Delta\bar{\psi}_2 = \bar{Q}, \qquad (1.20)$$

where $K_1(\varepsilon) = (\varepsilon - \sin\varepsilon)/(l^2 \varepsilon L^2)$ and L is a characteristic length scale.

The Green's function of this equation is a Bessel function of imaginary argument $K_0(ar)$, where

$$a^2(\lambda_j) = \left|\frac{\lambda_j}{K_1(\varepsilon)}\right|, \quad j = 1, 2, \ldots, r. \qquad (1.21\,\text{a})$$

Since the asymptotic representation for $K_0(ar)$ for $r \to \infty$ is $\exp(-ar)/\sqrt{ar}$ then obviously, by an appropriate fitting of $\varepsilon = \varepsilon_1$, it will be possible to find the $a(\lambda_j)$ that will satisfy the inequality $a(\lambda_j) > 1$ for all $j = 1, 2, 3, \ldots, n$. The condition that Green's function should converge rapidly to zero for $r \to \infty$ will be satisfied in this case. ε_1 is obviously determined when λ_1 is given.[2]

[1] There is some other way given in NEMCHINOV (1961*b*). It is possible to multiply equation (1.17) by $\sin l(t_2 - t_1)$ and to integrate it with respect to t over the interval (t_1, t_3). In this case, the derived equation will be constructed for function ψ_{av}, which corresponds to time t_{av}, $(t_1 < t_{av} < t_3)$.

[2] For example, at $n = 4$ ($\zeta_1 = 0.3$, $\zeta_2 = 0.5$, $\zeta_3 = 0.7$, $\zeta_4 = 0.85$, $\zeta_5 = 1$). $\lambda_1 \approx -10^{-5}$, and $\lambda_5 \approx -0.6 \times 10^{-2}$. Then at $L = 800$ km and $a(\lambda_1) = \sqrt{2}$, $k_1(\varepsilon_1) = 0.5 \times 10^{-5}$. Accordingly $\varepsilon_1 = 0.43$ which corresponds to $\Delta t = 3580$ sec, or $\Delta t \approx 1$ hr. Then $a^2(\lambda_5) = a^2(\lambda_1)(\lambda_5/\lambda_1) = 1.2 \times 10^3$. Thus practically, the Laplace operator for λ_5 in (2.4) may be neglected.

It is interesting to note the physical interpretation of $a(\lambda_1)$, λ_1 and ε_1.

Since the components φ_j of the vector $\bar{\psi}_2$ are scalar products of two vectors

$$\varphi_j = c^{(j)}\psi, \quad j = 1, 2, \ldots, r, \qquad (1.21\,\text{b})$$

where $c^{(j)}$ is the row-vector of the eigenvector of matrix A for a characteristic number λ_j, and ψ is the column-vector with components corresponding to pressure values at other pressure surfaces; then, due to variation of its components, the vector $\bar{\psi}_2$ describes the variation of pressure in a baroclinic model of the atmosphere. Each component may be interpreted physically.

It should be noted that among all vectors $c^{(j)}$ there is one and only one vector $c^{(1)}$ which has components $c_k^{(1)} > 0$ for all $k = 1, 2, \ldots, n$ (GUNTMACHER, 1953). Thus for the matrix A this vector corresponds to the eigenvalue $\lambda_1 = \lambda_{\max}$ (all $\lambda_j < 0$). All other vectors $c^{(j)}$ have components with both positive and negative signs, and there is no such vector that has all components negative, as otherwise multiplication by -1[1] will give the vector positive components, which is impossible.

Since $|c_k^{(1)}| < 1$; $k = 1, 2, \ldots, n$ and due to orthonormalization the module of the vectors $|c^{(j)}| = 1$, the scalar φ_1 can be interpreted as mean atmospheric pressure.

Variations in the scalar φ_1 can be interpreted either as pressure variations in any barotropic model corresponding to a baroclinic model of the atmosphere, or as barotropic pressure variations in a given baroclinic model of the atmosphere. Since the vector $c^{(1)}$ is unique the scalar φ_1 with its properties is also unique. This means that we cannot have correspondence between two barotropic models for a given baroclinic model of the atmosphere, and that there does not exist two or more different values for the φ_j that could be interpreted as barotropic pressure variations in a baroclinic model of the atmosphere.

Due to the uniqueness of the vector $c^{(1)}$ all other components of $c^{(j)}$, $j = 2, 3, \ldots, n$ have φ_j with both positive and negative signs. They

[1] Eigenvectors of any matrix are defined, as is known, to the nearest arbitrary factor.

can be interpreted as the sum of the thicknesses between pressure surfaces, or as the sum of mean temperatures for different layer combinations in the baroclinic model of the atmosphere. Variations in these components of the vector $\bar{\psi}_2$ are, therefore, to be considered as baroclinic pressure variations in a baroclinic model of the atmosphere.

Now it is clear that λ_1, ε_1 and $a^2(\lambda_1)$ can be interpreted as parameters in some way dependent on barotropic pressure variations in a baroclinic atmosphere.

2. The problem of determining the future state or pressure and wind fields in a baroclinic atmosphere is solved by means of the following equation:

$$\left.\begin{aligned} L\bar{\psi}_2 + K_1(\varepsilon)\Delta\bar{\psi}_2 &= \bar{Q} \\ u_2 &= K^{(u)} \\ v_2 &= K^{(v)} \\ \bar{\tau}_2 &= \int_0^{\zeta} [K_x^{(u)} + K_y^{(v)}]\, d\zeta' \end{aligned}\right\} \qquad (2.1)$$

If the right-hand terms of these equations are considered as known functions, determined at initial time t_1, then the set (2.1) becomes linear and is easy to solve. If, on the other hand, H_x and H_y in the functions $K^{(u)}$ and $K^{(v)}$, and the nonlinear operators $D(t,t_1,F_s)$ and $\bar{\Omega}(t,t_1,F_c)$ in Q are evaluated at time t_2, then the solution of this set becomes rather complicated. There must, however, exist some method to determine the unknown functions for all possible solutions of (2.1). There are two possible ways of obtaining a solution: either to reduce the set of equations to a new set of independent equations and to solve each of them separately, or to introduce a consistent order for approximate computation of the functions. The first way is rather complicated and we shall not use it here. The second way is more simple for practical forecasting of meteorological elements. In this case it is apparently possible to give three different versions. In scheme A the future (prognostic) value of the geopotential H_2 is determined on the basis of the initial wind field and pressure field by means of the first equation in (2.1). These computed values of the geopotential together with the wind at initial time make it possible to determine the functions u_2, v_2 and $\bar{\tau}_2$ from the three last equations. Thus, in this case the pressure for a future moment of time should in some instance be consistent with the computed future state of the wind field. In scheme B the opposite sequence is performed. Finally, the future potential field and the wind field in scheme C are computed from initial data separately, and the new values so determined make it possible to continue such computations cyclically.

Advantages and disadvantages of these schemes can be found out only when put into practical use. However, before the computations have started, each scheme can be investigated with respect to stability of computation by using initial data. The results so obtained may be valid in the solutions of many problems connected with difficulties in realization of certain computations. Such problems arise in particular when setting limits of variation values Δs—grid step of horizontal coordinates at finite-differential approximation of derivatives in x and y, and Δt—the interval of an elementary time step and the ratio of these increments i.e. types of function $\Delta t = f(\Delta s)$ at Δt, $\Delta s \to 0$.

Since the nonlinear operators $F^{(u)}$, $F^{(v)}$, $\bar{D}(t,t_1,F_s)$, $\bar{\Omega}(t,t_1,F_c)$ are included in Q, $K^{(u)}$, $K^{(v)}$ then due to the complexity of (2.1), the study of stability of schemes A, B and C can in general only be made for a linearized set of equations corresponding to (2.1).

3. We shall select a certain main flow along x-axis with constant velocity $U_{\max}$. We shall linearize (2.1) in accordance with this flow. Since $\partial U_{\max}/\partial\zeta = 0$, then terms with function $\bar{\tau}$ in the first three equations of the set will vanish, being of second order. The first three linearized equations of (2.1) will constitute a closed system. The function $\bar{\tau}$ can be determined separately from the fourth equation of (2.1). Then, when the equations for u_2, v_2 and $\bar{\psi}_2$ are linearized, it will be possible to study the stability of computation for every characteristic number λ_j. We introduce new vectors

$$\boldsymbol{H}_{1,2}^{(j)},\ \boldsymbol{u}_{1,2}^{(j)},\ \boldsymbol{v}_{1,2}^{(j)}$$

according to the formulas

$$\boldsymbol{s}_{1,2}^{(j)} = \sum_{k=1}^{n} c_k^{(j)} s_{1,2}^{(k)};\ (s = u, v, H); \qquad j = 1, 2, \ldots n \qquad (3.1)$$

(where $H_{1,2}^{(k)}$ is the geopotential value of the kth isobaric surface; $u_{1,2}^{(k)}$ and $v_{1,2}^{(k)}$ are the velocities along the same surface; $c_k^{(j)}$ the coordinates of the eigenvector $c^{(j)}$ for a characteristic number λ_j and indices 1 and 2 correspond to the initial and the following moment of time). Instead of (2.1) we may now take a new equivalent set of equations

$$\left.\begin{aligned} &\lambda_j[\boldsymbol{H}_2^{(j)}-\boldsymbol{H}_1^{(j)}] \\ &\quad+\frac{k_1(\varepsilon)}{4l^2(\Delta s)^2}(\delta_{xx}+\delta_{yy})[\boldsymbol{H}_2^{(j)}-\boldsymbol{H}_1^{(j)}]=q^{(j)}; \\ &\qquad\qquad\qquad j=1,2,\ldots,r. \\ &\boldsymbol{u}_2^{(j)}=\boldsymbol{k}_j^{(u)} \\ &\boldsymbol{v}_2^{(j)}=\boldsymbol{k}_j^{(v)} \end{aligned}\right\} \tag{3.2}$$

where

$$\boldsymbol{k}_j^{(u)}=\cos\varepsilon\,\boldsymbol{u}_1^{(j)}-\sin\varepsilon\,\boldsymbol{v}_1^{(j)}-\frac{\sin\varepsilon}{2l\Delta s}[\delta_x\boldsymbol{H}_{1,2}^{(j)}+U_{\max}\delta_x u_1^{(j)}]$$
$$-\frac{1-\cos\varepsilon}{2l\Delta s}[\delta_y\boldsymbol{H}_{1,2}^{(j)}+U_{\max}\delta_y\boldsymbol{u}_1^{(j)}],$$

$$\boldsymbol{k}_j^{(v)}=\cos\varepsilon\,\boldsymbol{v}_1^{(j)}-\sin\varepsilon\,\boldsymbol{u}_1^{(j)}-\frac{\sin\varepsilon}{2l\Delta s}[\delta_y\boldsymbol{H}_{1,2}^{(j)}+U_{\max}\delta_y\boldsymbol{u}_1^{(j)}]$$
$$-\frac{1-\cos\varepsilon}{2l\Delta s}[\delta_x\boldsymbol{H}_{1,2}^{(j)}+U_{\max}\delta_x\boldsymbol{v}_1^{(j)}];$$

$$q^{(j)}=\frac{\sin\varepsilon}{2l\Delta s}[\delta_x\boldsymbol{u}_1^{(j)}+\delta_y\boldsymbol{v}_1^{(j)}]+\frac{1-\cos\varepsilon}{2l\Delta s}[\delta_x\boldsymbol{v}_1^{(j)}-\delta_y\boldsymbol{u}_1^{(j)}]$$
$$-\frac{U_{\max}(1-\cos\varepsilon)}{4l^2(\Delta s)^2}[\delta_{xx}\boldsymbol{u}_{1,2}^{(j)}+\delta_{yx}\boldsymbol{v}_{1,2}^{(j)}]$$
$$-\frac{U_{\max}(\varepsilon-\sin\varepsilon)}{4l^2(\Delta s)^2}[\delta_{xx}\boldsymbol{v}_{1,2}^{(j)}-\delta_{yx}\boldsymbol{u}_{1,2}^{(j)}]$$
$$-\frac{1-\cos\varepsilon}{4l^2(\Delta s)^2}(\delta_{xx}+\delta_{yy})H_1^{(j)}-\frac{\varepsilon\lambda_j U_{\max}}{2l\Delta s}\delta_x\boldsymbol{H}_1^{(j)}.$$

The symbols δ_x, δ_y, δ_{yx}, δ_{yy}, δ_{xx} denote finite differences of first and second order respectively.

We shall expand the functions $\boldsymbol{H}_{1,2}^{(j)}$, $\boldsymbol{u}_{1,2}^{(j)}$, $\boldsymbol{v}_{1,2}^{(j)}$ in Fourier series:

$$\left.\begin{aligned} &\boldsymbol{s}_{1,2}^{(j)}=\sum_m s_{1,2}^{(j)}(m)\,e^{im(x+y)}, \\ &\boldsymbol{s}=\boldsymbol{u},\boldsymbol{v},\boldsymbol{H};\ s=u,v,U_{\max},H. \end{aligned}\right\} \tag{3.3}$$

The selection of common wave numbers m for the variables x and y is not of great importance and is used only for the sake of simplicity.

If we now introduce vectors

$$\Phi_{1,2}^{(j)}(m)=\begin{vmatrix} u_{1,2}^{(j)}(m) \\ v_{1,2}^{(j)}(m) \\ H_{1,2}^{(j)}(m) \end{vmatrix} \tag{3.4}$$

then, when the computation of the finite differences δ_x, δ_y and so on, is made, the system (3.2) may, taking (3.3) into account, be expressed in matrix form

$$G_{2,p}^{(j)}(\Delta t,m)\,\Phi_2^{(j)}(m)=G_{1,p}^{(j)}(\Delta t,m)\,\Phi_1^{(j)}(m) \tag{3.5}$$

or since $\det G_{2,p}^{(j)}\neq 0$ (in the opposite case (3.2) cannot be solved), then

$$\Phi_2^{(j)}(m)=G_p^{(j)}(\Delta t,m)\,\Phi_1^{(j)}(m) \tag{3.6}$$

where $\quad G_p^{(j)}(\Delta t,m)=[G_{2,p}^{(j)}]^{-1}G_{1,p}^{(j)}$.

The matrix $G_p^{(j)}$ is called the amplification matrix (Richtmyer, 1960), index p may take the meaning of symbols A, B and C corresponding to the schemes[1] discussed above. It is known that the problem of computational stability in realization of the schemes A, B and C depends on the character of the eigenvalues of matrix $G_p^{(j)}$. These eigenvalues are derived by the parameters Δt, Δs, $\Delta t=f(\Delta s)$ λ_j and p.

If we now introduce the notations[2]

$$\left.\begin{aligned} &\lim_{\Delta t,\Delta s\to 0} U_{\max}\frac{\Delta t}{\Delta s}=\sigma=\mathrm{const}>0 \\ &\varepsilon^2 q_1(\varepsilon)=\varepsilon-\sin\varepsilon;\ \varepsilon^2 q_2(\varepsilon)=1-\cos\varepsilon \\ &-b=\tilde{\boldsymbol{\lambda}}_1 U_{\max}^2+2\sigma^2 q_1\sin^2 m\Delta s \\ &h=\sigma^2(q_1+q_2)\sin^2 m\Delta s \\ &c=\sigma^2(q_2-q_1)\sin^2 m\Delta s \\ &\boldsymbol{h}=i\sigma(1-\varepsilon q_1-\varepsilon q_2)\sin m\Delta s \\ &\boldsymbol{c}=i\sigma(1-\varepsilon q_1+\varepsilon q_2)\sin m\Delta s \\ &\mu=\cos\varepsilon-i\sigma\varepsilon q_2\sin m\Delta s \\ &\varepsilon\nu=\varepsilon\sin\varepsilon-i\sigma\sin\varepsilon\sin m\Delta s \\ &d=b+2\sigma^2 q_2\sin^2 m\Delta s+i\tilde{\boldsymbol{\lambda}}_j U_{\max}^2 \end{aligned}\right\} \tag{3.7}$$

then for scheme A

[1] If below only one symbol p is used, then all arguments and conclusions are valid for any of the symbols A, B or C. In other words, the result will be satisfactory for all three computation schemes.

[2] $\boldsymbol{\lambda}_j=-\lambda_j$; as all $\lambda_j<0$ (Nemchinov, 1960), then $\boldsymbol{\lambda}_j>0$.

$$G^{(j)}_{2,\mathrm{A}} = \begin{vmatrix} 1 & 0 & \mathbf{c} \\ 0 & 1 & \mathbf{h} \\ 0 & 0 & b \end{vmatrix}; \quad G^{(j)}_{1,\mathrm{A}} = \begin{vmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ h_0 & c_0 & d \end{vmatrix}; \quad (3.8)$$

$$G^{(j)}_{\mathrm{A}} = \frac{1}{b}\begin{vmatrix} \mu b - \mathbf{c} h_0 & \nu - \mathbf{c} c_0 & -\mathbf{c} d \\ \nu b - \mathbf{h} h_0 & \mu b - \mathbf{h} c_0 & -\mathbf{h} d \\ h_0 & c_0 & d \end{vmatrix}$$

$$h_0 = h + \mathbf{h}; \quad c_0 = c + \mathbf{c}$$

for scheme B

$$G^{(j)}_{2,\mathrm{B}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -c & b \end{vmatrix}, \quad G^{(j)}_{1,\mathrm{B}} = \begin{vmatrix} \mu & \nu & -\mathbf{c} \\ -\nu & \mu & -\mathbf{h} \\ \mathbf{h} & \mathbf{c} & d \end{vmatrix};$$

$$G^{(j)}_{\mathrm{B}} = \begin{vmatrix} \mu & \nu & -\mathbf{c} \\ -\nu & \mu & -\mathbf{h} \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (3.9)$$

$$bc_1 = \mathbf{h} + \mu h - \nu c; \quad bc_2 = \mathbf{c} + \mu c + \nu h; \quad bc_3 = d - c\mathbf{h} - \mathbf{c}h,$$

for scheme C

$$G^{(j)}_{2,\mathrm{C}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{vmatrix}; \quad G^{(j)}_{1,\mathrm{C}} = \begin{vmatrix} \mu & \nu & -\mathbf{c} \\ -\nu & \mu & -\mathbf{h} \\ h_0 & c_0 & d \end{vmatrix};$$

$$G^{(j)}_{\mathrm{C}} = \begin{vmatrix} \mu & \nu & -\mathbf{c} \\ -\nu & \mu & -\mathbf{h} \\ \frac{h_0}{b} & \frac{c_0}{b} & \frac{d}{b} \end{vmatrix} \qquad (3.10)$$

Thus, all amplification matrices $G^{(j)}_{\mathrm{A}}$, $G^{(j)}_{\mathrm{B}}$ and $G^{(j)}_{\mathrm{C}}$ are of the same structure.

The following statement is a more general condition for computational stability of (3.6): If $G^{(j)*}_p$ is a hermitian-conjugate matrix to matrix $G^{(j)}_p$ (i.e. complex conjugate and transpose), then the necessary and sufficient stability conditions of the algorithm (3.6) is the inequality

$$|\beta(r)| \leqslant 1 + O(\Delta t) \qquad (3.11)$$

for $O < \Delta t < \tau^*$, $r = 1, 2, 3, \ldots$ and for all m where $\beta(r)$ are the characteristic numbers of the matrix $G^{(j)*}_p G^{(j)}_p$. This condition is satisfied if the matrix $G^{(j)}_p$ satisfies the Lipschitz condition at $\Delta t = 0$ in a sense (RICHTMYER, 1960) that

$$G^{(j)}_p(\Delta t, m) = G^{(j)}_p(0, m) + O(\Delta t), \quad \Delta t \to 0, \qquad (3.12)$$

provided that the norm of the matrix $\|G^{(j)}_p(0, m)\| \leqslant 1$. The norm of the matrix here is supposed to be the L-norm. It defines the norm of the matrix as $\|L\| = \max \|G(m)\|$ for all m. The norm itself is given by

$$\|G(m)\| = \max_{|V|=1} |G(m)V| = \max_{|V| \neq 0} \frac{|G(m)V|}{|V|}, \qquad (3.13)$$

where the value $|V|$ of the n-dimensional space vector V is expressed by $(|V_1|^2 + |V_2|^2 + \ldots |V_n|^2)^{\frac{1}{2}}$.

The analysis of the norm of the matrix $G^{(j)}_p(0, m)$ shows that its value is larger than 1 for all σ values. It turns out that this fact is stipulated by the existence of linearized terms in the set of nonlinear equations (1.1). If we suppose that the order of these terms along Δt is less than the order of the linear terms, then in principle it is only possible to construct a stable scheme for the linearized set (3.2) in case A. Schemes B and C remain unstable. The stability of scheme A is given by the values of the parameter σ from

$$\sigma^2 = k_0(\xi_j)\, U^2_{\max} \lambda_j \quad \text{or} \quad \sigma_1 = \frac{\Delta t}{\Delta s} \leqslant \sqrt{k_0 \lambda_j}. \qquad (3.14)$$

The value of the constant k_0 depends on the consistency conditions for the order of horizontal divergence introduced by the vector $\mathbf{D}^{(j)}_1 = (\sigma/2)[\delta_x \mathbf{u}^{(j)}_1 + \delta_y \mathbf{v}^{(j)}_1]$ into the equation for a function $\mathbf{H}^{(j)}_2$ with other terms of the equation. It results, for example, for the fact that at $\lambda = \lambda_1$ the vector $\mathbf{D}^{(1)}_1$ corresponds to barotropic pressure variations for which the integral over the whole atmosphere of the horizontal divergence of the wind is very small. The value of the vector $\mathbf{D}^{(j)}_1$ at $\lambda = \lambda_1$ may therefore be defined as $\mathbf{D}^{(1)}_1 = \xi_1 \chi(\lambda_1)$ where $|\chi(\lambda_1)| \to |\lambda_1| \to \varepsilon_0 > 0$ at $n \to \infty$. ε_0 may be as small as possible but not zero, if $\alpha \neq 0$, and $\varepsilon_0 \equiv 0$, if $\alpha = 0$ (NEMCHINOV, 1960). Generalizing this, let us assume that $\mathbf{D}^{(j)}_1 = \xi_j \chi(\lambda_j)$. It is then evident that the consistency condition for the terms of order $\mathbf{H}^{(j)}_2$ in equation (3.2) requires that $\xi_j \chi(\lambda_j) \simeq b$.

As $\sqrt{\lambda_j}$ is the upper bound for σ_1, σ_1 is determined only by means of λ_j. As $\lambda_n/\lambda_1 \gg 1$ (ill-conditionedness of matrix A) and as at $n \to \infty$ this expression tends to infinity, it is not possible to give a universally suitable value for σ_1 that could be of practical importance at any j mentioned above. Therefore, when computations are made for a multi-level model of the atmosphere it is necessary to estimate its smallest possible value, and if this is inadmissible from the point of view of machine time budget, one must reduce the number of levels to be forecast. The inequality $\sigma_1 = (\Delta t/\Delta s) \leqslant \sqrt{6\lambda_1}$ may be presented as the optimum value for the smallest characteristic module number λ_1. Then, for example, for a four-level model (see the previous note on page 6), $\lambda_1 = 10^{-5}$ for $\Delta s =$ 500 km we find $\Delta t = 3878$ sec, and for $\Delta s = 250$ km, $\Delta t = 1939$. In this case $|D_1^{(1)}| \simeq \xi_1 \lambda_1 U_{\max}^2$ where $0.3 \leqslant \xi_1 \leqslant 1$.

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