Spectral considerations in two-dimensional incompressible flow

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(Manuscript received February 20, 1962)

ABSTRACT

The spectrum of two-dimensional incompressible motion is studied with particular attention paid to non-linear interactions. Conditions for the absence of direct interactions between high and low wave numbers are developed. The non-linear tendency of an initial spectrum is computed to illustrate these points. The importance of the angular distribution of the spectrum is illustrated in a spectral stability problem for an isotropic stream function spectrum. Finally, a special inertial equilibrium spectrum is derived exhibiting the properties of an energy barrier in wave number space.

Introduction and general considerations

We propose to study the two-dimensional, incompressible, vorticity equation for a flat geometry, i.e. the "beta"-plane. That this equation is relevant for an understanding of meteorological processes has been clearly demonstrated by CHARNEY (1948) and BURGER (1958). In particular we wish to examine in some detail the non-linear aspects of the equation in spectral form.

If (u,v) are the zonal and meridional velocity components, and ξ is the relative vorticity, then the equations for conservation of vorticity and mass in the "beta"-plane are:

$$\xi_t + u\xi_x + v\xi_y = \nu\Delta\xi - v\beta, \tag{1}$$

$$u_x + v_y = 0. \tag{2}$$

Here x, y, t, subscripts denote differentiation.

We introduce a stream function $\psi(x,y)$ defined by:

$$u = -\psi_y, \tag{3a}$$

$$v = \psi_x \tag{3b}$$

to satisfy (2). Equation (1) becomes:

$$\Delta\psi_t + \psi_x\Delta\psi_y - \psi_y\Delta\psi_x = \nu\Delta\Delta\psi - \psi_x\beta. \tag{4}$$

The non-linear terms appearing in equation (4) are due solely to the advection of vorticity. The restriction of two dimensional flow has eliminated the mechanism of vorticity change due to vortex tube stretching. The absence of this mechanism along with two strong integral constraints on the energy spectrum (BATCHELOR, 1956) indicate that the differences between two- and three-dimensional non-linear flows are very great.

To investigate this further we expand $\psi(x,y)$ in the following manner:

$$\psi(\mathbf{r}) = \sum_{\mathbf{k}} \psi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{5}$$

where we must make the restriction $\psi_{\mathbf{k}}^* = \psi_{-\mathbf{k}}$ to insure the reality of $\psi(\mathbf{r})$. ($\psi_{\mathbf{k}}^*$ is the complex conjugate of $\psi_{\mathbf{k}}$.)

The sum in (5) is a two-dimensional sum over all $\mathbf{k}$ vectors in the $\mathbf{k}$ plane. The representation in (5) may be thought of as a *formal* way of writing a two-dimensional Fourier integral or sum. If the region in (x,y) space is unbounded $\mathbf{k}$ will have all possible values. If the region is finite, boundary conditions, as yet unspecified, will allow in general only certain $\mathbf{k}$ values.

Substitution of (5) into (4) leads to the form:

$$-\sum_{\mathbf{k}} k^2 \dot{\psi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{\mathbf{k}',\mathbf{k}''} (k_x' k_y'' - k_y' k_x'') k''^2 \psi_{\mathbf{k}'}\psi_{\mathbf{k}''} e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}} = \sum_{\mathbf{k}} (\nu k^4 - i\beta k_x)\, \psi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{6}$$

The orthogonality properties of the exponential

[1] This work has in part been supported by the Office of Naval Research (Contract number N 62558-3033).

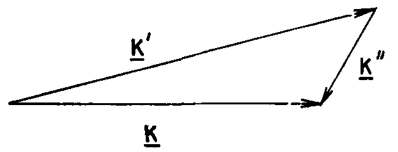

FIG. 1. An interaction picture for the production or annihilation of $\psi_{\mathbf{k}}$ by $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}''}$.

$e^{i\mathbf{k}\cdot\mathbf{r}}$ in the infinite plane or in a finite sub-plane (with the use of the proper boundary conditions) allow us to write (6) as:

$$k^2\dot{\psi}_{\mathbf{k}} - \sum_{\mathbf{k}',\mathbf{k}''}{}' (k'_x k''_y - k'_y k''_x) k''^2 \psi_{\mathbf{k}'}\psi_{\mathbf{k}''} = -\left(\nu k^2 - \frac{\beta i k_x}{k^2}\right)\psi_{\mathbf{k}}, \quad (7)$$

where

$$\sum_{\mathbf{k}',\mathbf{k}''}{}' \equiv \sum_{\mathbf{k}',\mathbf{k}''} \quad \text{such that} \quad \mathbf{k}'+\mathbf{k}''=\mathbf{k}.$$

To write (7) in a more symmetric form we interchange $\mathbf{k}'$ and $\mathbf{k}''$ in the sum and add the resulting equation to (7) leading finally to:

$$\dot{\psi}_{\mathbf{k}} + \sum_{\mathbf{k}',\mathbf{k}''}{}' \psi_{\mathbf{k}'}\psi_{\mathbf{k}''} \frac{(k''^2 - k'^2)(\mathbf{k}''\times\mathbf{k}')}{2k^2} = -\nu k^2\psi_{\mathbf{k}} + \frac{i\beta k_x}{k^2}\psi_{\mathbf{k}}. \quad (8)$$

Here $\mathbf{k}''\times\mathbf{k}'$ is shorthand for $(k''_x k'_y - k''_y k'_x)$.

The non-linear interactions between $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}''}$ appearing in (8) for $\psi_{\mathbf{k}}$ are modulated or biased by the function

$$F^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''} = \frac{(k''^2 - k'^2)(\mathbf{k}''\times\mathbf{k}')}{2k^2} \quad (9)$$

$F^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}$ is symmetric in $\mathbf{k}''$, $\mathbf{k}'$. It is clear that the interaction biased by $F^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}$ is triangular in nature as indicated in Fig. 1, i.e. $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}''}$ interact according to (8) to produce or annihilate $\psi_{\mathbf{k}}$. From (9) we observe that this interaction is empty (in the sense it does not contribute to $\dot{\psi}_{\mathbf{k}}$) when $\mathbf{k}'$ or $\mathbf{k}''$ are co-linear vectors, or when they are equal in magnitude. All isosceles interactions of $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}''}$ into $\psi_{\mathbf{k}}$ are empty.

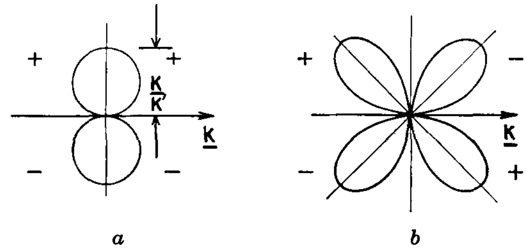

FIG. 2a. Plot of $k/k' \sin\theta_{k'k}$.
b. Plot of $-\sin 2\theta_{k'k}$.

It is clear from Fig. 1 that given $\mathbf{k}$ and $\mathbf{k}'$, $\mathbf{k}''$ is determined. Let us take advantage of this in (8) and write the equation explicitly in terms of only $\mathbf{k}$ and $\mathbf{k}'$ in the following manner:

$$\dot{\psi}_{\mathbf{k}} + \tfrac{1}{2}\sum_{\mathbf{k}'} \psi_{\mathbf{k}'}\psi_{\mathbf{k}-\mathbf{k}'} \frac{(k^2 - 2\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}\times\mathbf{k}')}{k^2} = -\nu k^2\psi_{\mathbf{k}} + \frac{ik_x}{k^2}\beta\psi_{\mathbf{k}}. \quad (10)$$

In this form we define the bias function

$$F(\mathbf{k}',\mathbf{k}) = \frac{(k^2 - 2\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}\times\mathbf{k}')}{k^2}. \quad (11\text{ a})$$

If the angle between $\mathbf{k}'$ and $\mathbf{k}$ is called $\theta_{k'k}$ (see Fig. 1) then $F(\mathbf{k}',\mathbf{k})$ may be written as:

$$F(\mathbf{k}',\mathbf{k}) = kk' \sin\theta_{k'k} - k'^2 \sin 2\theta_{k'k}. \quad (11\text{ b})$$

A polar plot of the function $F(\mathbf{k}',\mathbf{k})/k'^2$ as a function of $\theta_{k'k}$ will be the superposed plots of Figs. (2a and b). The maximum value of the function graphed in 2a is k/k' while in 2b it is unity. The plus and minus signs refer to the relevant sign of the functions in each quadrant.

It is clear that for $k < k'$ the second pattern (2b) dominates the bias function while for $k > k'$ the first pattern (2a) dominates.

If we consider the interaction diagram in Fig. 3 (where $\mathbf{k}''$ is not explicitly drawn) it is clear from 2a that for this case, since $k' < k$ the maximum value of the bias function occurs when $\mathbf{k}'$ is perpendicular to $\mathbf{k}$. However, if we

FIG. 3. Interaction diagram for $k' \ll k$.

interchange the roles of $\mathbf{k}$ and $\mathbf{k}'$ in Fig. 3 (i.e. $\mathbf{k}$ entering in an interaction into $\mathbf{k}'$) then $\mathbf{k}$ is oriented to provide an empty interaction into $\mathbf{k}'$ if $\theta_{k'k} = \frac{1}{2}\pi$ as seen from Fig. 2b.

These angular patterns clearly demonstrate the asymmetry between interactions of the short and long wave parts of the spectrum. We may also note that given $\mathbf{k}$ and $\mathbf{k}'$, the angular extremums of the bias function are easily found to be governed by the equation

$$(\operatorname{Cos}\theta_{k'k})_{mF(\mathbf{k}',\mathbf{k})} = \frac{k}{8k'} \pm \sqrt{\left(\frac{k}{8k'}\right)^2 + \frac{1}{2}}. \quad (12)$$

This is simply an analytic expression for the information shown in Fig. 2. However, for only $\mathbf{k}$ given there is no absolute maximization of the bias function. This is a reflection of the fact that the interactions, while modulated by $F(\mathbf{k}', \mathbf{k})$ are of course dependent on the distribution of the set $\psi_{\mathbf{k}}$ themselves.

Interactions between long and short waves

The asymmetry shown in the angular patterns of interactions between long and short waves suggests that we look in more detail at the relation between the long wave and short wave regions of the spectrum. For this it is more illustrative to consider the spectral energy equation. The amount of energy $\varepsilon_{\mathbf{k}}$ at any wave number $\mathbf{k}$ is

$$\varepsilon_{\mathbf{k}} = \frac{k^2}{2}\psi_{\mathbf{k}}\psi_{-\mathbf{k}}. \quad (13)$$

$\varepsilon_{\mathbf{k}}$ summed over $\mathbf{k}$ will be equal to the kinetic energy in the physical region considered. To focus our attention on the non linear interactions we will henceforth ignore the effects of viscosity and β. Using (13) and (10) the equation for $\varepsilon_{\mathbf{k}}$ may be written as:

$$\frac{1}{2k^2}\frac{\partial\varepsilon_{\mathbf{k}}}{\partial t} = -\tfrac{1}{2}\sum_{\mathbf{k}'}\psi_{\mathbf{k}'}\psi_{\mathbf{k}-\mathbf{k}'}\psi_{\mathbf{k}}F(\mathbf{k}',\mathbf{k}) - \tfrac{1}{2}\cdot\sum_{\mathbf{k}'}\psi_{\mathbf{k}'}\psi_{\mathbf{k}-\mathbf{k}'}\psi_{\mathbf{k}}F(\mathbf{k}',-\mathbf{k}). \quad (14)$$

Consider the contribution to $\partial\varepsilon_{\mathbf{k}}/\partial t$ from $\psi_{\mathbf{k}'}$ such that $k' \gg k$. To do this we restrict our sum in (14) to those $\mathbf{k}'$ for which $k' \gg k$. We also make the following approximations.

(a) For k small and $k' \gg k$ we include only the pattern (2b) in the bias function $F(\mathbf{k}',\mathbf{k})$.

(b) we write $\psi_{\mathbf{k}-\mathbf{k}'} \sim \psi_{-\mathbf{k}'}$; $\psi_{-\mathbf{k}-\mathbf{k}'} \sim \psi_{-\mathbf{k}'}$.

These approximations then allow us to write:

$$\left(\frac{\partial\varepsilon_{\mathbf{k}}}{\partial t}\right)_{\substack{\text{due to}\\ k'\gg k}} = \sum_{\substack{\mathbf{k}'\\ \{k'\gg k\}}} |\psi_{\mathbf{k}'}|^2(\mathbf{k}\cdot\mathbf{k}') \cdot (\mathbf{k}\times\mathbf{k}')(\psi_{\mathbf{k}}+\psi_{-\mathbf{k}}). \quad (15)$$

The short wave effect on the energy at long wave lengths then depends on the density of energy at high wave numbers and on the parity of the spectrum at low wave numbers. If $|\psi_{\mathbf{k}'}|^2$ (i.e. the energy at large wave numbers) is isotropic then the sum is seen to vanish. Also if $\psi_{\mathbf{k}}$ is of odd parity ($\psi_{\mathbf{k}} = -\psi_{-\mathbf{k}}$) the sum also vanishes. *An odd parity stream function will not accept energy from high wave numbers.*

Therefore *if the high wave number region of the energy spectrum is isotropic the low wave numbers will tend to interact among themselves indifferent to the detailed nature of the short wave spectrum.*

Now let us consider a partial sum for which k' small and $k' \ll k$. This time we will use only that portion of the bias function that corresponds to Fig. (2a), and also approximate $\psi_{\mathbf{k}-\mathbf{k}'} \sim \psi_{\mathbf{k}}$. If we do this we find that

$$\frac{1}{2k^2}\left(\frac{\partial\varepsilon_{\mathbf{k}}}{\partial t}\right)_{\text{due to } k'\ll k} = -\sum_{\substack{\mathbf{k}'\\ \{k'\ll k\}}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}}\psi_{-\mathbf{k}}(\mathbf{k}\times\mathbf{k}') - \sum_{\substack{\mathbf{k}'\\ \{k'\ll k\}}}\psi_{\mathbf{k}'}\psi_{-\mathbf{k}}\psi_{\mathbf{k}}(-\mathbf{k}\times\mathbf{k}') = 0. \quad (16)$$

The effect of the long waves on the energy stored in short waves is to this approximation identically zero, again illustrating the asymmetry between the two types of interactions—short wave to long wave and long wave to short.

The high wave number region of the spectrum therefore exists more or less in a state of equilibrium with respect to the low wave numbers while the low wave numbers will also exist in a state of internal equilibrium isolated from the high end of the spectrum if at large wave numbers the energy is isotropic.

We may speculate that if an anisotropy is introduced in the high end of the spectrum the change in the low wave number part of the spectrum indicated by (15) will receive the necessary energy from the high wave numbers until the latter again become isotropic, whereupon each end of the spectrum will return to a condition of relative isolation and equilibrium. By (15) we see that in order for this to happen the low end of the spectrum must be in a proper orientation to receive the energy. An odd parity $\psi_{\mathbf{k}}$ would not be able in this approximation to receive the necessary energy.

We have not considered the action of the intermediate range of wave numbers that may alter to some extent the above conclusions. However, the work of Von Karman (1948) shows how by misrepresenting interactions of the "in-between" spectrum valuable information may be obtained. These precedents allow us to rely with some confidence on the above speculations about the general nature of the spectral dynamics.

A tendency computation

To illustrate less abstractly the interaction dynamics let us have recourse to calculating the tendency of $\psi_{\mathbf{k}}$ in equation (10) in the absence of viscosity and the beta term. If $\psi_{\mathbf{k}}^{(0)}$ is the initial spectrum of ψ then the tendency of $\psi_{\mathbf{k}}$ denoted by $\dfrac{\partial \psi_{\mathbf{k}}^{(1)}}{\partial t}$ will be given by:

$$\frac{\partial \psi_{\mathbf{k}}^{(1)}}{\partial t} + \tfrac{1}{2}\sum_{\mathbf{k}'} \psi_{\mathbf{k}'}^{(0)} \psi_{\mathbf{k}-\mathbf{k}'}^{(0)} F(\mathbf{k}',\mathbf{k}) = 0. \tag{17}$$

For $\psi_{\mathbf{k}}^{(0)}$ we choose the function:

$$\psi_{\mathbf{k}}^{(0)} = A \exp\left(-\left(\frac{\mathbf{k}-\mathbf{k}_0}{\boldsymbol{\sigma}}\right)^2\right) + A^* \exp\left(-\left(\frac{\mathbf{k}+\mathbf{k}_0}{\boldsymbol{\sigma}}\right)^2\right),$$

where $\dfrac{\mathbf{k}}{\boldsymbol{\sigma}} \equiv \left(\dfrac{k_x}{\sigma_x}, \dfrac{k_y}{\sigma_y}\right) \quad \boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y).$

This gives us a spectrum of $\psi_{\mathbf{k}}$ centered about $\mathbf{k}_0$ and represents in physical (x,y) space a ψ-function that is periodic in $\mathbf{k}_0 \cdot \mathbf{r}$ and is modulated by a two way Gaussian or vice versa. We are therefore investigating (depending on the relative sizes of the parameters $\mathbf{k}_0, \boldsymbol{\sigma}$) either a periodic ψ whose period is uncertain about $\mathbf{k}_0$ or a general flow whose stream lines are elliptic, modulated by a trignometric function. That is to say

$$\psi^{(0)}(x,y) \sim A e^{-\sigma_x^2 x^2 - \sigma_y^2 y^2} \begin{Bmatrix} \cos \mathbf{k}_0 \cdot \mathbf{r} \\ \sin \mathbf{k}_0 \cdot \mathbf{r} \end{Bmatrix}.$$

We wish to examine the tendency of this initial distribution of $\psi_{\mathbf{k}}$ in $\mathbf{k}$ space.

To do the sum required by (17) we will make the assumption that we are concerned with a finite area whose sides are given by L_x and L_y. The previously unmentioned boundary conditions are taken to be periodicity in x and y with fundamental periods L_x and L_y. This restricts $\mathbf{k}$ to values such that

$$(k_x, k_y) = 2\pi \left(\frac{m}{L_x}, \frac{n}{L_y}\right); \tag{18}$$

m and n are integers.

The possible $\mathbf{k}$ values are then given by a denumerably infinite number of points forming a lattice in $\mathbf{k}$ space. If we assume that L_x and L_y are very large, the points in the $\mathbf{k}$ space lattice (separated by distances of the order $1/L_x$ or $1/L_y$) crowd together and the space is densely filled with points. We then consider that

$$\sum_{\mathbf{k}'} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \varrho(k_x, k_y),$$

where $\varrho(k_x, k_y)$ is the density of points in $\mathbf{k}$ space and is equal to $L_x L_y/(2\pi)^2$. The $\psi_{\mathbf{k}}$ becomes a continuous function of the argument (k_x, k_y) and the tendency in (17) is finally to be computed by the formula

$$\frac{\partial \psi_{\mathbf{k}}^{(1)}}{\partial t} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \left(\frac{L_x L_y}{(2\pi)^2}\right) \cdot \psi^{(0)}(\mathbf{k}')\, \psi^{(0)}(\mathbf{k}-\mathbf{k}')\, \frac{F(\mathbf{k}',\mathbf{k})}{2}. \tag{19}$$

If we allowed L_x and L_y to become strictly infinite we could have considered the original representation (5) of ψ as a Fourier integral (in which case the allowed $\mathbf{k}$ would strictly fill the $\mathbf{k}$ plane) to avoid such a discussion, but it is more physical to consider a large but bounded domain (L_x, L_y).

After writing the function $F(\mathbf{k}',\mathbf{k})$ in terms of cartesian co-ordinates the necessary quadratures are just those required from kinetic gas theory. The quadrature details are hence tedious but relatively elementary. The result is:

$$\frac{\partial\psi_{\mathbf{k}}^{(1)}}{\partial t} = -\frac{A^2 L_x L_y \sigma_x \sigma_y}{(2\pi)^2}\left[\frac{\pi k_x k_y(\sigma_x^2-\sigma_y^2)}{4k^2} \times \left\{\exp\left(-\frac{1}{2}\left(\frac{\mathbf{k}-2\mathbf{k}_0}{\boldsymbol{\sigma}}\right)^2\right) + \frac{A^{*2}}{A^2} \times \exp\left(-\frac{1}{2}\left(\frac{\mathbf{k}+2\mathbf{k}_0}{\boldsymbol{\sigma}}\right)^2\right) + \frac{2A^*}{A}\exp\left(-\frac{1}{2}\left(\frac{\mathbf{k}}{\boldsymbol{\sigma}}\right)^2\right)\right\}\right] - \frac{A^* A L_x L_y \sigma_x \sigma_y}{(2\pi)^2} \times \left[2\pi\frac{(\mathbf{k}_0\cdot\mathbf{k})(\mathbf{k}_0\times\mathbf{k})}{k^2}\exp\left(-\frac{1}{2}\left(\frac{\mathbf{k}}{\boldsymbol{\sigma}}\right)^2\right)\right] \quad (20)$$

The tendency of $\psi_{\mathbf{k}}$ is composed of two terms. The first term on the right-hand side of (20) illustrates the non-linear tendency due to interactions caused by the non isotropy of the Gaussian spread about $\pm\mathbf{k}_0$. This term vanishes for $\sigma_x = \sigma_y$, and is responsible for $\psi_{\mathbf{k}}$ production peaked at $\mathbf{k}=0,\ 2\mathbf{k}_0,\ -2\mathbf{k}_0$. If σ_x and σ_y were zero, $\psi_{\mathbf{k}}^{(0)}$ would correspond to a delta function $\delta(\mathbf{k}\pm\mathbf{k}_0)$. Production of $\psi_{\mathbf{k}}$ at $\mathbf{k} = \pm 2\mathbf{k}_0$ would in that case imply a co-linear interaction which would have zero effect because of the vanishing of the bias function for such interactions. We may conclude that such spreading of the original wave packet by non-linear effects depends heavily on the lack of sharpness of the original distribution. A similar argument applies for the second term. This second term, however, is responsible for only production of $\psi_{\mathbf{k}}$ centered around zero, i.e. low wave number production. The mechanism is clearly seen to be the interaction pictured in Fig. 4.

The production at low $\mathbf{k}$ by this mechanism exactly follows the maxima and minima of the function $F(\mathbf{k}_0,\mathbf{k})$, $k_0 \gg k$.

The production of higher wave numbers than $\mathbf{k}_0$ is therefore seen to depend strongly on the spread of the initial field while production of low wave numbers depends strongly on the mean initial field.

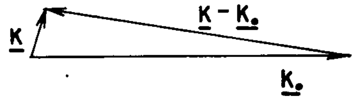

FIG. 4. Production of $\psi_{\mathbf{k}}$ due to interactions of $\psi_{\mathbf{k}_0}$ and $\psi_{\mathbf{k}-\mathbf{k}_0}$ for $k_0 \gg k$.

We also note that for $k_0^2 \gg \sigma_x^2 - \sigma_y^2$ the second term dominates the first on the right hand side of (20) so that if $\psi_{\mathbf{k}}^{(0)}$ is considered an excitation at high wave numbers the immediate effect is largely a production at *only* low wave numbers, illustrating the speculative point made earlier. Indeed by examining the form of $\psi_{\mathbf{k}}^{(0)}$ for $k_0 \gg k$ we see that $\psi_{\mathbf{k}}$ is to a good approximation constant, and therefore almost totally even in $\mathbf{k}$ illustrating the acceptance of an even parity low wave number $\psi_{\mathbf{k}}$.

In this early stage of non-linear interplay it is interesting to note that the new peaks in the $\psi_{\mathbf{k}}$ spectrum at $\mathbf{k}=0,\ \pm 2\mathbf{k}_0$ are less sharply peaked than the initial field, the spread of uncertainty increasing by $\sqrt{2}$. Calculations (not given here) of the second derivative of $\psi_{\mathbf{k}}(t=0)$ indicate new peaks at $\mathbf{k} = \pm\mathbf{k}_0,\ \pm 3\mathbf{k}_0$ with an uncertainty increase on the initial field by $\sqrt{3}$. Clearly we are observing a non-linear diffusion in $\mathbf{k}$ space.

Isotropic spectra

If $\mathbf{k}_0=0$ and $\sigma_x^2=\sigma_y^2$ then $\partial\psi_{\mathbf{k}}^{(1)}/\partial t = 0$. The spectrum is then isotropic and is itself a solution of the Spectral equation (10) in the absence of viscosity and the β effect, illustrating the general principle that if $\psi_{\mathbf{k}}$ is a function of k only, then the spectrum in the absence of viscosity and the β effect is steady. The proof is trivial. If $\psi(\mathbf{k})$ depends only on k then $\psi(\mathbf{r})$ depends only on r and the flow is one dimensional. For such flows the advective terms vanish and so, therefore do the corresponding terms in the spectral equation. To the above restrictions $\dot{\psi}_{\mathbf{k}}=0$.

An interesting question is whether or not such spectra are stable.

Let $f(k)$ be an isotropic spectrum of $\psi_{\mathbf{k}}$ and consider a small perturbation $g(k,\theta,t)$. Here $f(k)$ is not time dependent, in accordance with our above remarks.

The perturbation equation for $g(k,\theta,t)$ becomes:

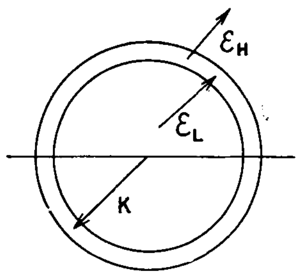

FIG. 5. Energy balance for a region a distance k from the origin in $\mathbf{k}$ space.

$$\dot{g}(\mathbf{k},t)+\tfrac{1}{2}\sum_{\mathbf{k}'} f(|\mathbf{k}-\mathbf{k}'|)\,g(\mathbf{k}')\,F\,(\mathbf{k}',\mathbf{k})$$

$$+\tfrac{1}{2}\sum_{\mathbf{k}'} f(\mathbf{k}')g(\mathbf{k}-\mathbf{k}')F\,(\mathbf{k}',\mathbf{k})=0. \qquad (21)$$

Using the relationship

$$F(\mathbf{k}-\mathbf{k}',\mathbf{k})=F(\mathbf{k}',\mathbf{k}), \qquad (22)$$

equation (21) may be written as

$$\dot{g}(\mathbf{k},t)+\sum_{\mathbf{m}'} g(\mathbf{m}')f(|\mathbf{k}-\mathbf{m}'|)\,F(\mathbf{m}',\mathbf{k})=0. \qquad (23)$$

Let $\quad g(k,\theta,t)=\sum_n G_n(k,t)\,\sin n\theta_k$

$$+K_n(k,t)\,\cos n\theta_k \qquad (n=0,1,2,\ldots).$$

Substitution of this form into equation (23) yields (after utilizing the orthogonality of the trigonometric functions)

$$\dot{G}_n=-\sum_{\mathbf{m}'}{}' K_n(m',t)\,\sin n\theta_{\mathbf{m}'\mathbf{k}} f(|\mathbf{k}-\mathbf{m}'|)\,F(\mathbf{m}',\mathbf{k}) \qquad (24\text{ a})$$

$$\dot{K}_n=\sum_{\mathbf{m}'} G_n(m',t)\,\sin n\theta_{\mathbf{m}'\mathbf{k}} f(|\mathbf{k}-\mathbf{m}'|)\,F'(\mathbf{m}',\mathbf{k}) \qquad (24\text{ b})$$

We may write

$$\begin{Bmatrix} K_n(t,k)=\mathcal{K}_n(k)\,e^{iwt} \\ G_n(t,k)=\mathcal{G}_n(k)\,e^{iwt} \end{Bmatrix}$$

leading to the following set:

$$iw\mathcal{G}_n(k)=-\sum_{\mathbf{m}'}\mathcal{K}_n(m')\sin n\theta_{m'k} f(|k-m'|)\,F(\mathbf{m}',\mathbf{k}) \qquad (25\text{ a})$$

$$iw\mathcal{K}_n=\sum_{\mathbf{m}'}\mathcal{G}_n(m')\ \sin\ n\theta_{m'k} f(|k-m'|)\,F(\mathbf{m}',\mathbf{k}) \qquad (25\text{ b})$$

Consider the case where

$$\begin{Bmatrix} \mathcal{K}_n(m') \\ \mathcal{G}_n(m') \end{Bmatrix}=\mathcal{H}(m')\begin{Bmatrix} \varkappa(n) \\ j(n) \end{Bmatrix},$$

i.e. the case where the angular distribution of the perturbation is *similar for all values of k* the wave vector length. For such perturbations the equation for ω becomes:

$$\omega^2=\frac{R^2(m)}{\mathcal{H}^2(m)}, \qquad (26)$$

where

$$R(k)=\sum_{m'}\mathcal{H}(m')\ \sin\ n\theta_{m'k} f(|\mathbf{k}-\mathbf{m}'|)F(\mathbf{m}',\mathbf{k}).$$

Both $R(m)$ and $\mathcal{H}(m)$ are real. To determine them explicitly involves solving a complicated integral equation for $\mathcal{H}(m)$ However, the most important fact is that since $R(m)$ and $\mathcal{H}(m)$ are necessarily real by virtue of the reality condition ($\psi^*_{\mathbf{k}}=\psi_{-\mathbf{k}}$) it follows that

$$\omega^2>0 \qquad (27)$$

and the spectrum is stable to these disturbances. A necessary condition for instability is therefore that the perturbations must not be angularly similar at high and low wave number. The perturbation must be oriented in a non-similar way to drain energy from the original spectrum. By now it must be clear that because of the asymmetry in the bias function, the angular dependance of the spectrum is very important in determining the spectrum dynamics as can be seen from our earlier general theorems, the tendency computation and finally the spectrum stability problem. We shall now give one more example.

An equlibrium spectrum

If an isotropic spectrum is unstable to angularly dissimilar disturbances it is possible that a new system for which at least part of the energy spectrum is steady may result. It is necessary to examine the possibility that such spectra, that are non isotropic, may be steady over some range in wave space. If we again consider that we may neglect viscosity and the β effect (i.e. an inertial equilibrium problem) the relevant equation for $\varepsilon_{\mathbf{k}}$ is (14).

Consider Fig. (5).

Physically we picture the following situation. That the net energy flowing by non-linear effects into a circular band a distance k from the origin in wave space be zero. For an isotropic spectrum this is trivial but for a non-isotropic spectrum we must balance the energy coming from (or going to) high wave numbers ε_H against energy going to (or coming from) low wave numbers ε_L over the entire ring.

From (14) (and assuming dense wave distribution in $\mathbf{k}$ space) the relevant balance equaation is for a ring a distance k from the origin:

$$\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_{k'} \cdot \int_0^k F(\mathbf{k}',\mathbf{k})\,\psi_{\mathbf{k}'}\psi_{\mathbf{k}-\mathbf{k}'}\psi_{-\mathbf{k}}$$

$$+F(\mathbf{k}',-\mathbf{k})\,\psi_{\mathbf{k}'}\psi_{-\mathbf{k}-\mathbf{k}'}\psi_{\mathbf{k}}\,k'dk'$$

$$=-\int_0^{2\pi} d\theta_{k'} \int_0^{2\pi} d\theta_k \int_k^\infty k'dk'\,F(\mathbf{k}',\mathbf{k})\psi_{\mathbf{k}'}\psi_{\mathbf{k}-\mathbf{k}'}\psi_{-\mathbf{k}}$$

$$+F(\mathbf{k}',-\mathbf{k})\,\psi_{\mathbf{k}'}\psi_{-\mathbf{k}-\mathbf{k}'}\psi_{\mathbf{k}} \tag{28}$$

We make approximations similar to those used used deriving equations (15) and (16).

For $k'>k$

(a) $F(\mathbf{k}',\mathbf{k})\simeq -k'^2 \sin 2\theta_{k'k}$

(b) $\psi(\mathbf{k}-\mathbf{k}')\simeq\psi(-\mathbf{k}')$

For $k'<k$

(a) $F(\mathbf{k}',\mathbf{k})\simeq k'k \sin \theta_{k'k}$.

In order to observe the first non-vanishing contribution to k from lower wave numbers we must write:

(b) $\psi_{\mathbf{k}-\mathbf{k}'}=\psi(\mathbf{k})-(\nabla_{\alpha}\psi_{\alpha})_{\alpha-\mathbf{k}}\cdot\mathbf{k}' \ldots$

Equation (28) then becomes

$$\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_{k'} \int_0^k k'dk'kk' \sin \theta_{k'k}$$

$$\times\left[\frac{(\nabla_{\alpha}\psi_{\alpha})_{\alpha=-\mathbf{k}}\cdot\mathbf{k}'}{\psi(\mathbf{k})}-\frac{(\nabla_{\alpha}\psi_{\alpha})_{\alpha=\mathbf{k}}\cdot\mathbf{k}'}{\psi(-\mathbf{k})}\right]$$

$$\times\psi(\mathbf{k})\,\psi(-\mathbf{k})\,\psi(+\mathbf{k}')$$

$$=\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_k' \int_k^\infty k'^3 \sin 2\theta_{k'k}$$

$$\times\psi(\mathbf{k}')\,\psi(-\mathbf{k}')\,[\psi(\mathbf{k})+\psi(-\mathbf{k})]\,dk'. \tag{29}$$

We assume that we may find a solution to (29) that is angularly similar, i.e. we try

$$\psi(\mathbf{k})=k^a f(\theta_k). \tag{30}$$

Since $\quad \psi_{\mathbf{k}}^{*}=\psi_{-\mathbf{k}}; \quad f(\theta_k+\pi)=f^{*}(\theta_k).$

After working out the necessary gradients needed in equation (29) we obtain:

$$\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_{k'} \int_0^k dk'k'^{3+a}k^{2a} \sin \theta_{k'k}$$

$$\times\left\{-2 \cos \theta_{k'k}\,af(\theta)f^{*}(\theta_k)-\sin \theta_{k'k}\right.$$

$$\left.\times\frac{d}{d\theta_k}f(\theta_k)f^{*}(\theta_k)\right\}f(\theta_{k'})=\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_k'$$

$$\times\int_k^\infty k'^{2a+3} \sin 2\theta_{k'k}f(\theta'_k)f^{*}(\theta'_k)k^a$$

$$\times\{f(\theta_k)+f^{*}(\theta_k)\}d\,k'. \tag{31}$$

We can easily do the integrations over k' first. We obtain

$$\frac{1}{a+4}\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta'_k f(\theta'_k)$$

$$\times\left\{\sin 2\theta_{k'k}\,a\,f(\theta_k)f^{*}(\theta_k)+\sin^2 \theta_{k'k}\right.$$

$$\left.\frac{d}{d\theta_k}f(\theta_k)f^{*}(\theta_k)\right\}$$

$$=\frac{1}{2a+4}\int_0^{2\pi} d\theta_k \int_0^{2\pi} d\theta_{k'}f(\theta'_k)f^{*}(\theta'_k)\{f(\theta_k)$$

$$+f^{*}(\theta_k)\}. \tag{32}$$

With the condition obtained for convergence of the integrals at $k=0$ and $k=\infty$ that

$$-2>a>-4 \tag{33}$$

To solve (32) let us try

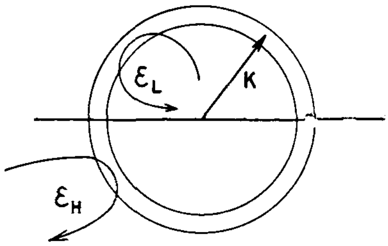

FIG. 6. Schematic picture of energy interchanges on either side of an equilibrium region acting as an energy barrier.

$$f(\theta) = A + B \sin 2\theta + C \cos 2\theta + i(D \cos \theta + E \sin \theta),$$

where all of A, B, C, D, E, are real.

After doing the required quadratures in (32) over θ_k and $\theta_{k'}$ and using the relation $\theta_{k'k} = \theta_{k'} - \theta_k$ we obtain our final relation:

$$\left(\frac{a+1}{a+4}\right)[B(E^2 - D^2) + 2DCE]$$

$$= \frac{1}{a+2}[B(D^2 - E^2) + 2DCE] \quad (34)$$

which may be re-written as

$$\left[\frac{a+1}{a+4} + \frac{1}{a+2}\right][B(E^2 - D^2) + 2DCE] = 0. \quad (35)$$

The polynomial in a that must vanish if the first bracket is to vanish is:

$$a^2 + 4a + 6 = 0.$$

This has no real roots. For a solution of (35) it is then required that:

$$B(D^2 - E^2) = 2CDE. \quad (36)$$

This condition requires that the total energy input into the ring vanish in a manner such that as much energy is returned from the ring to low wave numbers as is received. The same holds true for high wave numbers. The situation is pictured in Fig. 6. The energy balance is not a trivial one. Energy is returned at different angular positions and an energy balance is in general not maintained except for the ring as a whole. However, if $A = B = C = 0$ detailed examination of the computations indicated in (32) show that for this odd parity spectrum a trivial balance is achieved. Otherwise the more general balance indicated schematically in Fig. 6 is appropriate in which case the equilibrium middle region acts as an energy barrier for low and high wave numbers returning the energy in a re-distributed state to both ends of the spectrum[1].

We note that if $B = C = D = E = 0$ we have the trivial isotropic steady case.

In the light of equations (15) and (16) we may say that if the spectrum is isotropic at high **k** or odd parity at low **k** the spectrum will remain in complete isolation. Otherwise it is possible that the spectrum may achieve a more general equilibrium where a barrier in **k** space may prevent energy leakage but will allow each end of the spectrum to act to redistribute the *existing* energy in the other end of the spectrum. Finally, the equilibrium region itself may redistribute its energy within the ring also.

[1] It is interesting to note that if k is not very large, viscous effects, if considered, can be neglected in (28) and the results are unchanged, The previously mentioned integral constraints (BATCHELOR, 1956) do not, however, still apply in the presence of viscosity.

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