The baroclinic instability of a "2 $\frac{1}{2}$ dimensional model atmosphere" in which the upper level is non-divergent

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ABSTRACT

The baroclinic instability of a "2 $\frac{1}{2}$ dimensional model atmosphere" is examined by the method of small wave perturbations. The model is very similar to that of ESTOQUE (1956) but somewhat less restrictive. It is shown that the assumption of an upper isobaric level of non-divergence leads to the following unusual results: (1) Instability is impossible if the Rossby beta-parameter is zero. (2) The vertical shears required for instability increase linearly with the beta-parameter. (3) Only positive vertical shears lead to instability. (4) The instability is of the linear type as opposed to the more usual exponential type.

1. Introduction

PETTERSSEN (1955) used the idea of a level (not necessarily isobaric) of (horizontal) non-divergence to study the factors influencing development at sea level. He found that if the level of non-divergence is known, then the sea level divergence and hence the vorticity tendency can be computed from the physical state of the atmosphere below the level of non-divergence. Furthermore, during cyclogenesis the level of non-divergence is usually somewhere around 500–600 mb.

Petterssen's study stimulated ESTOQUE (1956) to present a model for cyclone development which can be integrated graphically. The model is essentially a "2 $\frac{1}{2}$ dimensional model" resembling the model of CHARNEY and PHILLIPS (1953). The principal assumption of Estoque was that the upper information level is a level of non-divergence.

Estoque's model integrated graphically has been extensively used in the United States to predict the 1000 mb level flow. Moreover, since late in 1958, the Joint Numerical Weather Prediction Group, Suitland, Maryland, has been using a model nearly identical to Estoque's model.

To the writer's knowledge Estoque's model has never been tested theoretically for baroclinic instability. It is the aim of this paper to test for baroclinic instability a model very similar to Estoque's model but somewhat less restrictive. It will be shown that the assumption of a level of non-divergence leads to the following results:

(1) Instability is impossible if the Coriolis parameter is constant.
(2) However, the vertical shears required for instability increase linearly with the Rossby beta-parameter.
(3) Only positive vertical shears lead to instability.
(4) The instability is of the linear type as opposed to the more usual exponential type.

2. Mathematical formulation of the model

CHARNEY and PHILLIPS (1953) proposed a "2 $\frac{1}{2}$ dimensional model" of the atmosphere. In their model the vorticity equation is used (see Fig. 1) at levels 1 and 3 and the thermodynamic equation is used at level 2. The model is quasi-geostrophic. Turning of the vortex tubes and the vertical advection of vorticity are neglected. The term $(f+\zeta)(\partial\omega/\partial p)$ is approximated by $f(\partial\omega/\partial p)$, where f is the Coriolis parameter, ζ

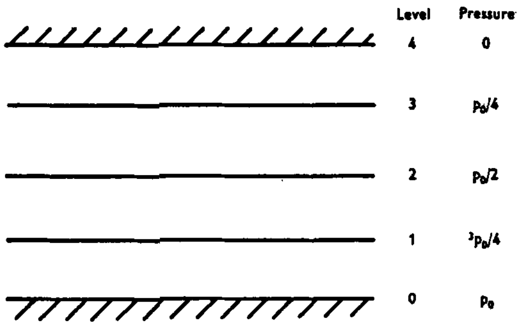

Fig. 1. Schematic diagram of the pressure levels used by CHARNEY and PHILLIPS (1953).

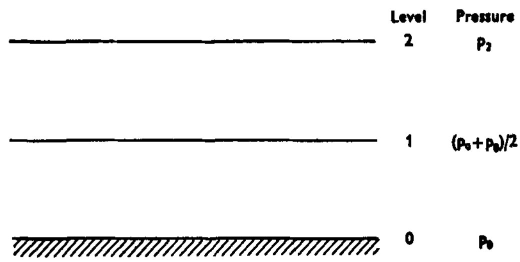

Fig. 2. Schematic diagram of the pressure levels used in the model of ESTOQUE (1956). Level 2 is assumed to be a level of (horizontal) non-divergence.

is the vertical component of vorticity, p is pressure, $\omega \equiv dp/dt$, t is time and d/dt is the substantial time derivative. ω is set to zero at levels 0 and 4. Finally, friction and non-adiabatic heating are neglected. The unknowns are φ_3, φ_1 and ω_2 where φ is the geopotential of an isobaric surface.

PHILLIPS (1954) has examined the baroclinic instability of the above model. The qualitative results of his theoretical study will be compared with those derived from the model to be next described.

ESTOQUE's (1956) model is similar to that of Charney and Phillips above. The vorticity equation is used (see Fig. 2) at levels 0 and 2 and the thermodynamic equation is used at level 1. In general p_2 will not be $p_0/2$, where p_0 is the surface pressure. ω is set to zero at level 0. Estoque used all the simplifications in the equations that Charney and Phillips used plus some additional ones to be described.

In writing down the governing equations we use an x–y–p Cartesian coordinate system and the β-plane approximation. Here x increases to the east and y to the north.

The vorticity equations at levels 2 and 0 are respectively

$$\frac{\partial}{\partial t}\nabla^2\varphi_2 + \frac{1}{f}J(\varphi_2, \nabla^2\varphi_2) + \beta\frac{\partial \varphi_2}{\partial x} = f^2\left.\frac{\partial \omega}{\partial p}\right)_2, \quad (2.1)$$

$$\frac{\partial}{\partial t}\nabla^2\varphi_0 + \frac{1}{f}J(\varphi_0, \nabla^2\varphi_0) + \beta\frac{\partial \varphi_0}{\partial x} = f^2\left.\frac{\partial \omega}{\partial p}\right)_0, \quad (2.2)$$

where $\beta \equiv df/dy$, an assumed constant and J is the Jacobian determinant.

The thermodynamic equation as in PHILLIPS (1954) is approximated by

$$\lambda_1^2\left[\frac{\partial}{\partial t}(\varphi_2-\varphi_0) + \frac{1}{f}J(\varphi_0, \varphi_2)\right] = f^2(\Delta p)^{-1}\omega_1, \quad (2\cdot3)$$

where $\Delta p \equiv p_0 - p_2$ and λ_1^2 is a static stability parameter assumed constant:

$$\lambda_1^2 \equiv (\Phi_2 - \Phi_0)^{-1} f^2\, \Theta_1/(\Theta_2 - \Theta_0).$$

Φ_i is a standard geopotential of the ith level and Θ is the potential temperature. (2.3) is based upon conservation of potential temperature.

At the present stage we have as unknowns

$$\varphi_0, \varphi_2, \omega_1, \left.\frac{\partial \omega}{\partial p}\right)_0 \quad \text{and} \quad \left.\frac{\partial \omega}{\partial p}\right)_2$$

but we have only three equations at our disposal. To complete the system Estoque made assumptions equivalent to the following:

$$\left.\begin{aligned} &\left.\frac{\partial \omega}{\partial p}\right)_2 = 0, \\ &\omega_1 = b\,\omega_2(x, y, t), \\ &\left.\frac{\partial \omega}{\partial p}\right)_0 = -c(\Delta p)^{-1}\omega_2(x, y, t), \end{aligned}\right\} \quad (2.4)$$

where b and c are given non-dimensional constants.

In order to render the system (2.1) (2.2) and (2.3) suitable for graphical integration Estoque further assumed that

$$\frac{1}{f}J(\varphi_0, \zeta_0 + f) \cong 0,$$

$$\frac{1}{f}J(\varphi_2, \zeta_0 + f) \cong 0.$$

We shall not make these assumptions (which are rather questionable) so that the model to be analyzed is somewhat less restrictive than that of Estoque.

Using (2.4) we obtain the completed system to be analyzed:

$$\frac{\partial}{\partial t}\nabla^2\varphi_2+\frac{1}{f}J(\varphi_2\nabla^2\varphi_2)+\beta\frac{\partial\varphi_2}{\partial x}=0, \quad (2.5)$$

$$\frac{\partial}{\partial t}\nabla^2\varphi_0+\frac{1}{f}J(\varphi_0,\nabla^2\varphi_0)+\beta\frac{\partial\varphi_0}{\partial x}=-c\omega^*, \quad (2.6)$$

$$\lambda_1^2\left[\frac{\partial}{\partial t}(\varphi_2-\varphi_0)+\frac{1}{f}J(\varphi_0,\varphi_2)\right]=b\omega^*, \quad (2.7)$$

where $\omega^*\equiv f^2(\Delta p)^{-1}\omega_2$.

Eliminating ω^* between (2.6) and (2.7) we obtain

$$\frac{\partial}{\partial t}\nabla^2\varphi_0+\frac{1}{f}J(\varphi_0,\nabla^2\varphi_0)+\beta\frac{\partial\varphi_0}{\partial x}+a\lambda_1^2\left[\frac{\partial}{\partial t}(\varphi_2-\varphi_0)+\frac{1}{f}J(\varphi_0,\varphi_2)\right]=0, \quad (2.8)$$

where $a\equiv c/b$. It can be seen from (2.8) that Estoque's hypotheses (2.4) can be reduced to one concerning a; namely the ratio

$$a\equiv-(\Delta p)\left.\frac{\partial\omega}{\partial p}\right)_0\omega_1^{-1}.$$

The complete system is now (2.5) and (2.8). It should be noted that (2.5) contains no references to φ_0 and hence the solution for φ_2 is from (2.5) alone. This solution must be entered into (2.8) to obtain φ_0. Thus φ_2 acts to some extent like a forcing agent on φ_0. Consequently, if the system is to exhibit baroclinic instability it must be of the resonance type. This qualitative comment will be validated analytically in Art. 3.

In order to distinguish our model from that of Charney and Phillips we henceforth refer to our model as Model I and that of Charney and Phillips as Model II.

3. The baroclinic instability of model I

We postulate basic uniform equilibrium currents at levels 0 and 2 plus small perturbations such that

$$\varphi_0=\bar\varphi_0+\varphi_0',$$

$$\varphi_2=\bar\varphi_2=\varphi_2',$$

and

$$U_0\equiv-\frac{1}{f}\frac{\partial\bar\varphi_0}{\partial y},$$

$$U_2\equiv-\frac{1}{f}\frac{\partial\bar\varphi_2}{\partial y},$$

$$\frac{\partial\bar\varphi_0}{\partial x}=\frac{\partial\bar\varphi_2}{\partial x}=0.$$

Linearizing (2.5) and (2.8) and dropping the primes, the equations for the perturbed motion are

$$\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right)\nabla^2\varphi_2+\beta\frac{\partial\varphi_2}{\partial x}=0, \quad (3.1)$$

$$\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\nabla^2\varphi_0+\beta\frac{\partial\varphi_0}{\partial x}+a\lambda_1^2\left[\frac{\partial}{\partial t}(\varphi_2-\varphi_0)+U_0\frac{\partial\varphi_2}{\partial x}-U_2\frac{\partial\varphi_0}{\partial x}\right]=0. \quad (3.2)$$

To obtain the solution for φ_2 write

$$\varphi_2=A_2\cos\mu y e^{i(lx-nt)},$$

where A_2 is assumed real and $i\equiv\sqrt{-1}$. (3.1) then gives

$$n=l(U_2-\beta/k^2),$$

where $k^2\equiv\mu^2+l^2$.

The solution for φ_0 is obtained by entering the solution för φ_2 into (3.2) and writing for φ_0

$$\varphi_0=\hat\varphi_0(t)\cos\mu y)e^{ilx}.$$

The result is the following equation:

$$A\frac{d\hat\varphi_0}{dt}+iB\hat\varphi_0=-iCe^{-int}, \quad (3.3)$$

where
$$A\equiv a\lambda_1^2+k^2,$$
$$B\equiv lk^2(U_0-\beta/k^2+a\lambda_1^2U_2/k^2),$$
$$C\equiv a\lambda_1^2(n-U_0l)A_2.$$

A, B, and C are real constants. The right-hand member of (3.3) represents the "forcing" of φ_0 by φ_2.

Neglecting the term in C for the moment, we obtain the "free" solution for $\hat{\varphi}_0$ by writing

$$\hat{\varphi}_0 = De^{-i\nu t},$$

where D is an arbitrary constant and ν is to be determined. From (3.3) with the right side zero we find

$$\nu = B/A. \tag{3.4}$$

The contribution of the term in C is obtained by writing

$$\hat{\varphi}_0 = Ee^{-int},$$

where E is to be determined from (3.3). The result is

$$E = \frac{C/A}{n-\nu}.$$

The general solution of (3.3) is thus

$$\hat{\varphi}_0(t) = De^{-i\nu t} + \frac{C/A}{n-\nu}e^{-int},$$

where as yet D is arbitrary. To determine D we impose the following condition at $t=0$:

$$\hat{\varphi}_0(0) = A_0^{(0)}e^{i\varepsilon}.$$

Thus, $A_0^{(0)}$ is the amplitude (real) of φ_0 at $t=0$ and ε is the phase shift between levels 0 and 2 at $t=0$. The complete solution is

$$\hat{\varphi}_0(t) = \frac{C/A}{n-\nu}(e^{-int} - e^{-i\nu t}) + A_0^{(0)}e^{i(\varepsilon-\nu t)} \tag{3.5}$$

If we multiply (3.5) by $\cos \mu y e^{ilx}$ and take the real part we obtain the real part of φ_0. Whence,

$$\text{Real } \varphi_0 = \frac{C/A}{n-\nu}\cos \mu y\,[\cos(lx-nt) - \\ -\cos(lx-\nu t)] + A_0^{(0)}\cos\mu y\cos(lx-\nu t+\varepsilon). \tag{3.6}$$

After some algebraic manipulation we find

$$C/A = A_2(n-\nu)\,a/\alpha + \frac{A_2 al\beta(a-\alpha)}{k^2\alpha(a+\alpha)},$$

and
$$n-\nu = \frac{la}{a+\alpha}\left[\frac{\alpha}{a}(U_2-U_0) - \beta/k^2\right],$$

where
$$\alpha \equiv k^2\lambda_1^{-2}.$$

The condition $\nu = n$ implies

$$\frac{\alpha}{a}(U_2-U_0) = \beta/k^2. \tag{3.7}$$

Moreover as ν tends to n, we have asymptotically

$$C/A \to \frac{A_2 al\beta}{k^2\alpha}\left(\frac{a-\alpha}{a+\alpha}\right),$$

and

$$\frac{\cos(lx-nt) - \cos(lx-\nu t)}{n-\nu} \to t\sin(lx-nt).$$

The last result can be proven using L'Hospital's rule.

Combining the above results we have when $\nu = n$

$$\text{Real } \varphi_0 = \frac{al\beta(a-\alpha)}{k^2\alpha(a+\alpha)}A_2 t\cos\mu y\sin(lx-nt) \\ + A_0^{(0)}\cos\mu y\cos(lx-nt+\varepsilon), \tag{3.8}$$

while the solution for φ_2 gives

$$\text{Real } \varphi_2 = A_2\cos\mu y\cos(lx-nt). \tag{3.9}$$

It should be noted that if (3.7) is not satisfied, the maximum value of Real φ_0 is

$$\text{Real } \varphi_0]_{\max} = \frac{2C/A}{n-\nu} + A_0^{(0)},$$

which is bounded and can be made as small as we please by making A_2 and $A_0^{(0)}$ sufficiently small. Thus, if (3.7) is not satisfied the flow must be reckoned stable. When (3.7) is satisfied the amplitude of Real φ_0 increases linearly with time. Thus, (3.7) is the criterion for instability of Model I.

4. Qualitative discussion of the results

(a) It is seen from (3.6) that if $\beta = 0$, no instability is possible. However, if $\beta \neq 0$ then the shear required for instability increases linearly with β. These are unusual aspects of Model I.

(b) From (3.8) we see that if $\alpha < a$, the developing disturbance (represented by the first term) at level 0 is $\frac{1}{4}$ wavelength ahead of the disturbance at level 2, while if $\alpha > a$ the developing disturbance at level 0 is $\frac{1}{4}$ wavelength behind the disturbance at level 2. In a numerical example to be presented in Art. 5 we shall show that the case $\alpha > a$ leads to very weak amplification so that $\alpha = a$ determines the effective lower limit of wavelength for baroclinic instability.

In Model II, the lower limit of wavelength is calculated from $\alpha = 2$ (PHILLIPS, 1954).

(c) Since α, β, k^2 and presumably a are all positive, (3.7) shows that only positive vertical shears lead to instability. In Model II positive and negative shears are equally unstable.

(d) Let $L = 2\pi/l$ and consider a diagram with axes L and $U_2 - U_0$. Model I exhibits true instability only along a line while Model II is unstable in a semi-infinite area on that diagram. Of course, the solution for φ_0 is a continuous function of $U_2 - U_0$ so that near the line of true instability of Model I the solution will be nearly the same as on that line.

(e) The instability is of the linear type[1] while Modell II exhibits exponential type instability.

(f) For $\alpha < a$ the amplifying disturbance is always $\frac{1}{4}$ L ahead of the disturbance at level 2. In Model II, the phase shift depends upon the parameters of the problem but the amplifying disturbance is always ahead of the upper level disturbance.

5. Numerical example

We first consider the problem of determining a. Treating ω as continuous we may write

$$\omega = Q(p)\,\omega_2(x, y, t),$$

where

$$Q(p_2) = 1,$$

$$Q(p_0) = 0,$$

$$\left.\frac{dQ}{dp}\right)_2 = 0.$$

In the annexed table we consider three tentative forms of $Q(p)$ and their associated a.

[1] Professor Norman A. Phillips has informed me that NEMCHINOV (1958) (1959) has found linear baroclinic instability in a model related to Model I.

$Q(p)$	b	c	$a \equiv c/b$
$\sin\left[\frac{\pi}{2}(\Delta p)^{-1}(p_0 - p)\right]$	$2^{-\frac{1}{2}}$	$\pi/2$	$2^{-\frac{1}{2}}\pi \doteqdot 2.22$
$1 - \frac{(p-p_2)^2}{(\Delta p)^2}$	$\frac{3}{4}$	2	$\frac{8}{3} \doteqdot 2.67$
$\frac{p_0 - p}{\Delta p}$, $p \geqslant p_2$	$\frac{1}{2}$	1	2

These examples give a in the range 2 to 3. Obviously, the last form is the one that should be used for comparison with Model II since it is there assumed

$$\left.\frac{\partial\omega}{\partial p}\right)_3 = \frac{\omega_2}{p_0/2},$$

$$\left.\frac{\partial\omega}{\partial p}\right)_1 = -\frac{\omega_2}{p_0/2}.$$

Moreover, if the last form were used the lower limit of instability in Model I would be calculated from $\alpha = 2$, exactly as in Model II. There is, however, very little observatical evidence to suggest which of the three forms is most appropriate to the atmosphere. Here, we shall use $a = 2.22$ following ESTOQUE (1956), who used the trigonometric form.

Following ESTOQUE (1956) we take as level 2 the 500 mb level and as level 0 the 1000 mb level. After Phillips we take $\mu = \pi/2W$ so that at $y = \pm W$ there are rigid walls. With $W = \frac{1}{6}\pi Re$, where Re is the mean radius of the earth, $2W$ is equivalent to 60° of latitude. Let H_i be the standard height of the ith pressure level and g be the acceleration of gravity, f and β are evaluated at 45°N. The assumed numerical values are

$$H_2 - H_0 = 5.5 \text{ km},$$
$$\Theta_2 - \Theta_0 = 30^\circ \text{ K},$$
$$\Theta_1 = 306^\circ \text{ K},$$
$$f = 1.03 \times 10^{-4} \text{ sec}^{-1},$$
$$\beta = 1.62 \times 10^{-11} \text{ m}^{-1} \text{ sec}^{-1},$$
$$g = 9.8 \text{ m sec}^{-2},$$
$$Re = 6.63 \times 10^6 \text{ m},$$

whence,

$$\mu^2 = 2.23 \times 10^{-13} \text{ m}^{-2},$$
$$\lambda_1^2 = 1.64 \times 10^{-12} \text{ m}^{-2}.$$

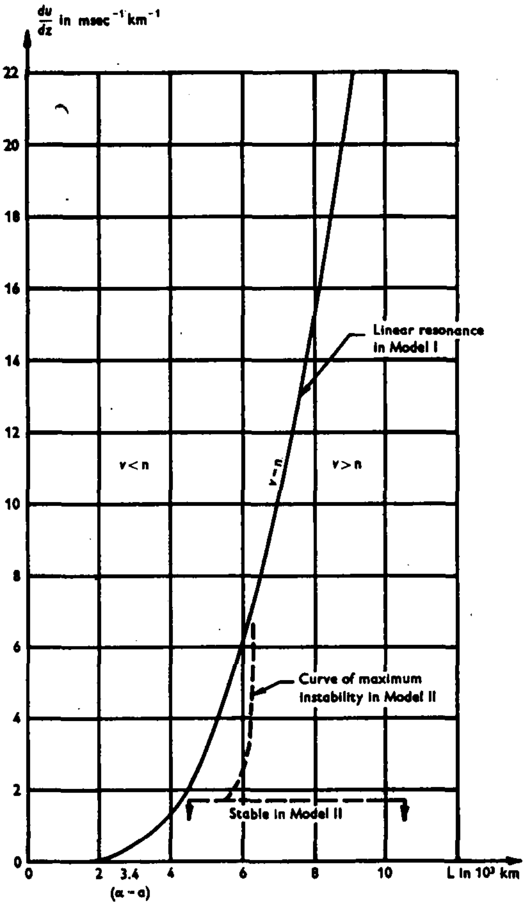

Fig. 3. Shears required *vs.* wavelength for linear instability in Model I. Dashed curve is from Fig. 1, PHILLIPS (1954) and is the curve of maximum instability in Model II.

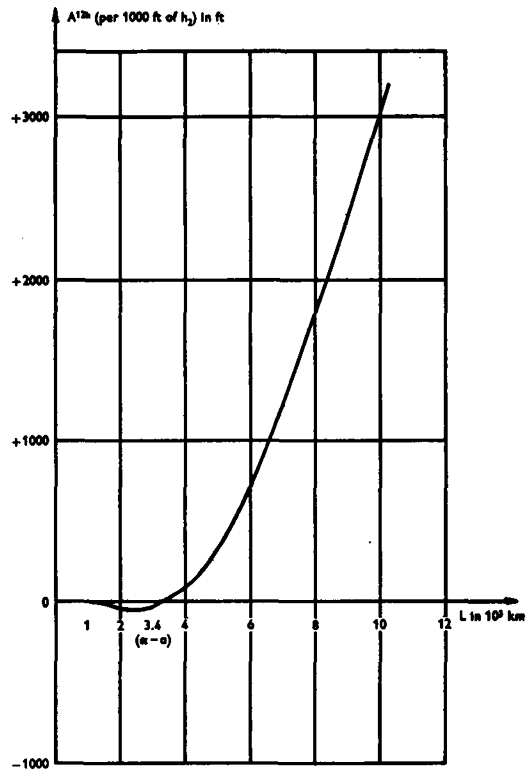

Fig. 4. Amplitude of the baroclinic development after 12^h at 1000 mb expressed in ft per 1000 ft of h_2 in Model I.

The instability criterion (3.7) has been used to calculate dU/dZ as a function of L. The results are displayed in Fig. 3. The dashed curve in Fig. 3 is from Fig. 1 in PHILLIPS (1954) and is the curve of maximum instability in Model II. The two curves are in rough qualitative agreement. Apparently, Model I requires somewhat greater shears than Model II for maximum instability.

We next examine the growth rate in Model I. Let h_2 be the amplitude in feet of the perturbation at level 2 and A^τ be the amplitude in feet of the amplifying perturbation at level 0 after time τ. We have

$$A^\tau = \frac{a\, l\, \beta\, (a-\alpha)}{k^2\, \alpha\, (a+\alpha)}\, h_2\, \tau. \tag{5.1}$$

Fig. 4 is a plot of A^τ vs. L for $h_2 = 1000$ ft and $\tau = 12^h$. It can be seen that at least moderate development can be achieved in 12^h. Thus, for $dU/dZ = 7\,\mathrm{m\ sec^{-1}\ km^{-1}}$, $L = 6000$ km, we find a development of about 750 ft at level 0 after 12^h.

6. Conclusions

It has been shown that Model I, which is essentially the ESTOQUE (1956) model, exhibits baroclinic instability of an unusual type. Art. 4 summarizes the unusual features of the model. It has been demonstrated that these unusual features do not depend upon the pressure at level 2. Thus, changing p_2 from 500 to 600 mb will not change the general theoretical results but will alter only the numerical values of U_2, A_2 and λ_1^2.

It might be noted from (5.1) that A^{τ} is proportional to h_2. Thus, intense baroclinic development at level 0 requires an intense pre-existing trough at level 2. In many observed cases development at 500 and 1000 mb takes place simultaneously. Model I applied to those cases will probably fail to adequately predict the cyclogenesis. However, the success of the model claimed by ESTOQUE (1956) would suggest that a large percentage of observed cyclogeneses could be predicted with Model I.

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