On the theory of the general circulation of the atmosphere[1]

By JULIÁN ADEM, *Instituto de Geofísica, Universidad Nacional de México*[2], *México, 20, D.F.*

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ABSTRACT

Applying the conservation of energy principle to the incoming solar radiation and to the long-wave radiation, a heating mechanism of the atmosphere is introduced. A simplified emission spectrum of the atmosphere with one transparent region is used. The surface of the earth and a cloud layer are considered as black bodies.

A simplified model of the atmosphere containing the coupling between dynamical and thermodynamical processes is presented. The equations that give the temperature, the excess of radiation and the mean zonal wind for each day of the year are derived. The temperature is given by a second order differential equation that can readily be solved. The zonal wind and the excess of radiation are explicit functions of the temperature.

If in the computations we postulate that the addition of energy by processes other than radiation is a constant, the solution agrees remarkably well with the observed values, except in the months of May, June and July.

It is shown that using only meridional turbulent diffusion transport of heat by large eddies, the required transport is accomplished. The theoretical solution requires a transport with the following exchange coefficients: 5×10^{10} cm^2 per sec in February and in March, 4.5×10^{10} in October, 4×10^{10} in January, 3.5×10^{10} in November, 3×10^{10} in September and in December, 2×10^{10} in April, 0.25×10^{10} in August, and zero in May, June and July. There are two maxima of the value of the exchange coefficient: one in February and the other in October. The order of magnitude of the values of the exchange coefficient of our solution is in complete agreement with the value corresponding to the migratory cyclones and anticyclones of the middle latitudes, treated as turbulent eddies, which is equal to 5×10^{10}.

The theoretical solution correctly gives the position and strength of the jet stream, except in May, June and July. Therefore we can conclude that, except in these months, the mean daily circulation is mainly a result of the transfer of energy by radiation processes and the turbulent meridional transport of heat by the migratory cyclones and anticyclones.

1. Introduction

The energy that maintains the atmospheric circulation is solar radiation and, therefore, a fundamental problem of meteorology is to explain quantitatively how this transformation from radiant energy into mechanical energy is accomplished. The purpose of this paper is to present a simplified theory by which this coupling between thermodynamical and dynamical processes takes place. An attempt is made to give an explanation to the problem of the maintenance of the *mean state of the atmosphere*, and a quantitative theory is developed by which the mean temperature, the mean excess of radiation and the mean wind in the troposphere are explicitly given by simple formulas.

2. Temperature, pressure and density

Let us consider an atmosphere with a constant lapse rate equal to β. The temperature T^* can therefore be expressed by

$$T^*(\varphi,\psi,z,t) = -\beta[z - H(\varphi,\psi,t)] + T(\varphi,\psi,t), \quad (1)$$

where φ is the latitude, ψ the longitude, z the height measured from the surface of the earth and t is the time. For $z = H$ the temperature is equal to T. We will assume that for $z > H$ there

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exist no absorbing substances, so that the surface $z = H$ is the radiating upper boundary.

Assuming hydrostatic pressure and using the equation of a perfect gas, by integration we obtain:

$$p^* = p\left(1 + \frac{\beta(H-z)}{T}\right)^{g/R\beta}, \qquad (2)$$

$$\varrho^* = \varrho\left(1 + \frac{\beta(H-z)}{T}\right)^{g/R\beta - 1}, \qquad (3)$$

where g is the acceleration of gravity, R the gas constant, p^* the pressure, ϱ^* the density, $p = (p^*)_{z=H}$ and $\varrho = (\varrho^*)_{z=H}$. We shall assume that ϱ is constant.

The variables p, H, T and ϱ can be written as

$$\left.\begin{aligned} p &= p_0 + p', \\ H &= H_0 + H', \\ T &= T_0 + T', \\ \varrho &= \varrho_0, \end{aligned}\right\} \qquad (4)$$

where p', H' and T' are small departures from the constant values p_0, H_0 and T_0 respectively and ϱ_0 is a constant.

The mean temperature is given by

$$T_m = \frac{1}{H}\int_0^H T^* dz = T_{m_0} + T'_m, \qquad (5)$$

where $T_{m_0} = T_0 + \beta H_0/2$ and $T'_m = T' + \beta H'/2$.

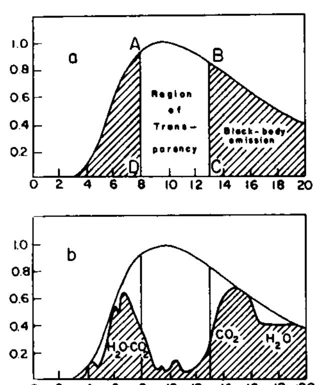

Fig. 1. (a) is the emission spectrum model used in this paper and (b) is the observed spectrum at sea level, for $T = 300$ K.

3. Atmospheric radiation

To compute the outgoing terrestrial radiation, SIMPSON (1928) considered as a working hypothesis that a layer of the atmosphere, containing 0.3 mm of precipitable water and 0.06 grams of CO_2 per sq cm, emits as a black body for wave-lengths from 5.5 μ to 7 μ and from 14 μ to the end of the spectrum, being transparent for wave-lengths from 8.5 μ to 11 μ. He assumed besides that the radiation is partially emitted for wave-lengths from 7 μ to 8.5 μ and from 11 μ to 14 μ.

We shall take a simplification of the already simplified Simpson's model and consider that the atmosphere emits as a black body from 0 to 8 μ and from 13 μ to ∞, while being transparent in the region from 8 μ to 13 μ, as shown in Fig. 1a. The observed spectrum at sea level for $T = 300$ K is shown in Fig. 1b (JOHNSON, 1954, p. 128).

In order to compute the radiation emitted by this simplified model of the atmosphere, we need to find the value of the area A B C D of Fig. 1a. To obtain it we integrate the black body spectral curve between the limits $\lambda_1 = 8$ μ and $\lambda_2 = 13$ μ:

$$\left.\begin{aligned} F(T^*, \lambda_1, \lambda_2) &= \int_{\lambda_1}^{\lambda_2} C_1 \lambda^{-5} e^{-C_2/\lambda T^*} d\lambda \\ &= \Bigg[C_1 e^{-C_2/\lambda T^*}\Bigg(\frac{\lambda^{-3}T^*}{C_2} + \\ &\quad + \frac{3\lambda^{-2}T^{*2}}{C_2^2} + \frac{6\lambda^{-1}T^{*3}}{C_2^3} + \frac{6T^{*4}}{C_2^4}\Bigg)\Bigg]_{\lambda_1}^{\lambda_2}. \end{aligned}\right\} \qquad (6)$$

Therefore, within the framework of our model

$$E(T^*) = \sigma T^{*4} - F(T^*, 8\ \mu, 13\ \mu) \qquad (7)$$

is the energy in calories per min, per cm^2, emitted by a horizontal boundary of an atmospheric layer. The temperature T^* is given in Kelvin degrees, $C_1 = (5538)\ 10^2$ cal μ^4 min^{-1} cm^{-2}, $C_2 = 14350$ μ K, and $\sigma = (8215)\ 10^{-14}$ cal cm^{-2} K^{-4} min^{-1}, σT^{*4} is the energy emitted by a black body and $F(T^*, 8\ \mu, 13\ \mu)$ is given by (6) for $\lambda_1 = 8$ μ and $\lambda_2 = 13$ μ.

We can linearize this function if we consider a basic constant temperature T^*_0 and small departures $T^{*\prime}$ from it. We take $T^* = T^*_0 + T^{*\prime}$ and expand in a Taylor's series. Therefore, for small $T^{*\prime}$ we can use the linear formula

$$E(T_0^* + T^{*\prime}) = E(T_0^*) + \left(\frac{dE}{dt}\right)_{T_0^*} T^{*\prime}. \quad (8)$$

Let us take $T^*_0 = 260\ K$ and $T^{*\prime} = 15$ K. If we use (8) instead of the complete series expansion, the error is less than 1.5 per cent. If we take $T^{*\prime} = 30$ K the error is about 4.5 per cent.

4. The solar radiation

The solar radiation received by the earth has been studied by MILANKOVITCH (1920) who gives the expression

$$I = \frac{I_0}{R_1^2} [\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos(\omega_0 + \psi)], \quad (9)$$

where I is the insolation or solar energy received in the absence of an atmosphere per unit of time on a unit area of the earth's surface; I_0 is the solar constant, which is defined as the total solar energy received per unit of time on a unit of surface and at the mean distance from the Sun to the earth in the absence of an atmosphere; φ and ψ are the latitude and longitude of the considered place; R_1 is the ratio of the distance from the earth to the sun at a given instant and the mean distance from the earth to the sun; ω_0 is the value of the sun's hour angle at a zero meridian from which we measure the longitude (from west to east), and δ is the declination of the sun. The parameters R_1, δ and ω_0 are known functions of time that can be given numerically.

The solar constant computed by the Smithsonian Institution is equal to 1.946 gram-calories per cm², per min.

By integration, we can compute from equation (9) the insolation received in different periods of time. For example, the insolation received during one day is given by

$$I_d = \frac{\tau I_0}{\pi R_1^2}(\omega_1 \sin\varphi \sin\delta + \sin\omega_1 \cos\varphi \cos\delta), \quad (10)$$

where ω_1 is obtained from equation $\cos\omega_1 = -\tan\varphi \tan\delta$ and τ is the duration of the solar day. For the case of the polar regions, where no night exists during a period longer than one day, $\omega_1 = \pi$.

5. The albedo of the earth

We can consider that the total albedo has three parts. The solar radiation returned from the top of the clouds, that returned from the gases of the atmosphere, and that returned from the surface of the earth.

Using the estimates of LONDON (1959) and HOUGHTON, JOHNSON (1954) considers that the albedo is 34 per cent for an atmosphere with 50 per cent cloud cover. The values presented in Table 1 are based on Johnson's data and on the assumption that absorption of solar radiation by the atmosphere above the cloud cover is negligible.

Let us assume that the fractional cloud cover is equal to ε. Let α_1, α_2 and α_3 be the percentages of solar radiation absorbed by the surface of the earth, the gases of the atmosphere and the cloud cover, respectively; then $\alpha_1 + \alpha_2 + \alpha_3 = 100 - \text{albedo}$. From Table 1 we can obtain α_1, α_2, α_3 and the albedo as functions of ε, as follows:

$$\left.\begin{aligned} \alpha_1 &= 48 - 2\varepsilon, \\ \alpha_2 &= 34(1-\varepsilon), \\ \alpha_3 &= 4\varepsilon, \\ \text{albedo} &= 18 + 32\varepsilon. \end{aligned}\right\} \quad (11)$$

TABLE 1.

	Albedo				Absorption of energy						
					By Surface				By atmosphere	By clouds	
Cloud cover ε	From surface	From atmosphere	From clouds	Total	Direct	Through clouds	Through atmosphere	Total α_1	α_2	α_3	Total
1.00	0	0	50	50	0	46	0	46	0	4	50
0.50	2	7	25	34	19	23	5	47	17	2	66
0	4	14	0	18	38	0	10	48	34	0	82

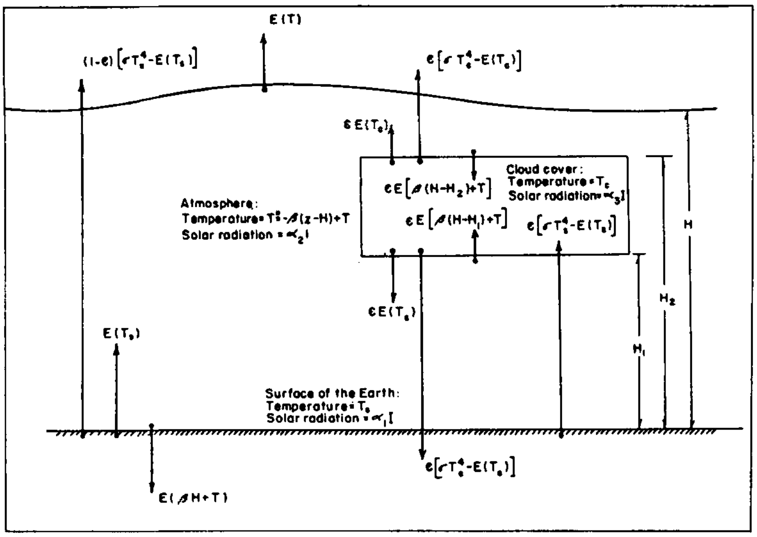

Fig. 2. Schematic representation of the balance of radiation in the troposphere, with a fractional cloud cover equal to ε.

6. The excess of radiation

Within the framework of the model discussed in Section 3, the flux emitted from an atmospheric surface is a function of only the temperature and is given by (7).

We assume that the atmosphere has a fractional cloud layer equal to ε. The lower boundary of this layer is at $z = H_1$ and the upper boundary at $z = H_2$.

Let T_c be the temperature of the cloud cover and T_s the temperature at the surface of the earth.

Fig. 2 shows the balance of radiation schematically.

Per cm^2 the atmosphere absorbs solar radiation equal to $\alpha_2 I$ and long-wave radiation from the surface of the earth and the cloud cover equal to $E(T_s) + 2\varepsilon E(T_c)$. Besides, it emits energy equal to $E(T) + E(\beta H + T) + \varepsilon E(\beta(H - H_2) + T) + \varepsilon E(\beta(H - H_1) + T)$. Therefore, the excess of radiation E_A is given by the following equation:

$$E_A = -E(T) - E(\beta H + T) + E(T_s) + 2\varepsilon E(T_c) - \varepsilon E(\beta(H - H_2) + T) - \varepsilon E(\beta(H - H_1) + T) + 0.34(1 - \varepsilon)I, \tag{12}$$

where $E(T)$, $E(\beta H + T)$, $E(T_s)$, $E(T_c)$, $E(\beta(H - H_2) + T)$ and $E(\beta(H - H_1) + T)$ are given by (7), I is given by (9) and where we have made use of the second of equations (11) to evaluate the coefficient of I.

Similarly, for the excess of radiation E_s at the surface of the earth, we can write the following equation:

$$E_s = -\sigma T_s^4 + E(\beta H + T) + \varepsilon(\sigma T_c^4 - E(T_c)) + (0.48\text{–}0.02\,\varepsilon)I, \tag{13}$$

where the coefficient of I has been evaluated using the first of equations (11).

Finally, the excess of radiation E_c at the cloud layer is given by:

$$E_c = \varepsilon\{E(\beta(H - H_2) + T) + E(\beta(H - H_1) + T) + \sigma T_s^4 - E(T_s) - 2\sigma T_c^4 + 0.04\, I\}, \tag{14}$$

where we have made use of the third of equations (11) to compute the coefficient of I.

$$\left.\begin{aligned} T_s &= T_{s0} + T_s', \\ T_c &= T_{c0} + T_c', \end{aligned}\right\} \tag{15}$$

where T_s' and T_c' are small compared with the values of T_{s0} and T_{c0}.

Substituting (4) and (15) in (12), (13) and (14) and using (5), we obtain the following equations:

$$E_A = (A_2 + \varepsilon A_2')T_m' + A_3 T_s' + \varepsilon A_4 T_c' + [A_5 + \beta A_2/2 + \varepsilon(A_5' + \beta A_2'/2)]H' + A_6 + \varepsilon A_7 + 0.34(1 - \varepsilon)I. \tag{16}$$

$$Es = B_2 T'_m + B_3 T'_s + \varepsilon B_4 T'_c + (B_5 + \beta B_2/2)H' + \varepsilon B_7 + (0.48\text{–}0.02\,\varepsilon)I. \quad (17)$$

$$Ec = \varepsilon(D_2 T'_m + D_3 T'_s + D_4 T'_c + (D_5 + \beta D_2)H' + D_6 + 0.04I). \quad (18)$$

where

$$A_2 = -\left(\frac{dE}{dT}\right)_{T=T_0} - \left(\frac{dE}{dT}\right)_{T=\beta H_0 + T_0},$$

$$A'_2 = -\left(\frac{dE}{dT}\right)_{T=\beta(H_0 - H_1)+T_0} - \left(\frac{dE}{dT}\right)_{T=\beta(H_0 - H_1)+T_0},$$

$$A_3 = \left(\frac{dE}{dT}\right)_{T=T_{s0}}, \; A_4 = 2\left(\frac{dE}{dT}\right)_{T=T_{c0}},$$

$$A_5 = -\beta\left(\frac{dE}{dT}\right)_{T=\beta H_0 + T_0},$$

$$A'_5 = \beta A'_2, \; A_6 = E(T_{s0}) - E(T_0) - E(\beta H_0 + T_0),$$

$$A_7 = 2E(T_{c0}) - E(\beta(H_0 - H_2) + T_0) - E(\beta(H_0 - H_1) + T_0),$$

$$B_4 = 4\sigma T_{c0}^3 - \left(\frac{dE}{dT}\right)_{T=T_{c0}},$$

$$B_5 = \beta B_2, \; B_6 = E(\beta H_0 + T_0) - \sigma T_{s0}^4,$$

$$B_7 = \sigma T_{c0}^4 - E(T_{c0}), \; D_2 = -A'_2,$$

$$D_3 = 4\sigma T_{s0}^3 - \left(\frac{dE}{dT}\right)_{T=T_{s0}},$$

$$D_4 = -8\sigma T_{c0}^3, \; D_5 = \beta D_2,$$

$$D_6 = -2\sigma T_{c0}^4 + E(\beta(H_0 - H_2) + T_0) + E(\beta(H_0 - H_1) + T_0) + \sigma T_{s0}^4 - E(T_{s0}).$$

7. The energy equation

The equation of conservation of energy can be written as

$$\varrho^* \frac{D}{Dt}(cT^* + W) = \nabla \cdot \varrho^* K \nabla (cT^* + W) + E_1 + E_2 + E_3 + E_4. \quad (19)$$

The first member of (19) is the rate of change of energy per unit volume, where c is the specific heat at constant volume of the atmosphere, cT^* is the thermal energy per unit mass and W the latent heat energy per unit mass. The first term of the second member is the transport by turbulent diffusion, where K is the exchange coefficient of the turbulent transport. E_1 is the rate at which thermal energy is added by radiation; E_2 is the rate at which thermal energy is added by conduction; E_3 is the rate at which latent-heat energy is added and E_4 is the molecular transformation function (MILLER, 1951).

Integrating from zero to H_0, substituting the values of T^*, ϱ^* and T'_m given by (1), (3) and (5), considering that

$$\int_0^{H_0} E_1 dz - \varepsilon \int_{H_1}^{H_2} E_1 dz = E_A,$$

and neglecting the terms of the turbulent transport that contain the product $(\partial\varrho'/\partial\varphi)(\partial T'/\partial\varphi)$ we obtain the following equation:

$$(A_0 + \varepsilon A'_0)\frac{D}{Dt}\left(T'_m + \frac{\beta H'}{2}\right) + [(A_1 - A_0) - \varepsilon(A'_1 - A'_0)]\frac{\partial}{\partial t}\left(T'_m + \frac{\beta H'}{2}\right) - (A_1 - \varepsilon A'_1)K\nabla\cdot\nabla\left(T'_m + \frac{\beta H'}{2}\right) - E_A = G, \quad (20)$$

where

$$G = -\int_0^{H_0} \varrho^* \frac{DW}{Dt} dz + \varepsilon \int_{H_1}^{H_2} \varrho^* \frac{DW}{Dt} dz + \int_0^{H_0} (\nabla\cdot\varrho^* K \nabla W + E_2 + E_3 + E_4 + W) dz - \varepsilon \int_{H_1}^{H_2} (\nabla\cdot\varrho^* K \nabla W + E_2 + E_3 + E_4 + W) dz,$$

and

$$A_0 = \frac{2c}{gH_0}\left\{\frac{R}{g + R\beta}[(p_0^* T_0^*)_{z=0} - p_0 T_0] - p_0 H_0\right\},$$

$$A'_0 = \frac{2c}{gH_0}\left\{\frac{R}{g + R\beta}[(p_0^* T_0^*)_{z=H_1} - (p_0^* T_0^*)_{z=H_2}] + H_1 (p_0^*)_{z=H_1} - H_2 (p_0^*)_{z=H_2}\right\},$$

$$A_1 = \frac{c}{g}[(p_0^*)_{z=0} - p_0],$$

$$A'_1 = \frac{c}{g}[(p_0^*)_{z=H_1} - (p_0^*)_{z=H_2}],$$

where T_0^* and p_0^* are given by (1) and (2) when $H = H_0$ and $T = T_0$. The first member of equation (20) contains the rate of change of thermal energy and the rate of change of thermal energy due to addition of energy by radiation and turbulent transport. The second member is the function G that globally represents the terms of rate of change of latent heat energy, the conduction of sensible heat and the molecular transformation function. The material derivative D/DT that appears in (20) contains the horizontal mean velocity.

8. The mean caily circulation in the troposphere

The equations that describe the mean daily conditions in the troposphere can be obtained by averaging with respect to t in the interval τ corresponding to each day and with respect to ψ in the interval from zero to 2π.

We define the mean daily temperature as

$$\overline{T}'_m = \frac{1}{2\pi\tau}\int_t^{t+\tau} dt \int_0^{2\pi} T'_m d\psi,$$

and in a similar way define the other mean daily variables.

Applying the averaging operator

$$\frac{1}{2\pi\tau}\int_t^{t+\tau} dt \int_0^{2\pi} [\quad] d\psi, \tag{21}$$

to equation (20) we obtain:

$$-\frac{A_1 - \bar{\varepsilon}A_1'}{r_0^2}\overline{K}\left(1 + \frac{\beta A}{2}\right)\frac{d^2\overline{T}'_m}{d\varphi^2} = \bar{E}_A + \overline{G}, \tag{22}$$

where we have assumed that $\overline{\varepsilon K d^2 T'_m/d\varphi^2} = \bar{\varepsilon}\overline{K} d^2\overline{T}'_m/d\varphi^2$, and that

$$\overline{\frac{D}{Dt}(T'_m + \beta H'/2)} \quad \text{and} \quad \overline{\frac{\partial}{\partial t}(T'_m + \beta H'/2)}$$

are negligible.

If we assume that $\overline{\varepsilon T'_m} = \bar{\varepsilon}\overline{T}'_m$, $\overline{\varepsilon T'_s} = \bar{\varepsilon}\overline{T}'_s$, $\overline{\varepsilon T'_c} = \bar{\varepsilon}\overline{T}'_c$ and $\overline{\varepsilon I} = \bar{\varepsilon} I_d$, the averaging leaves equations (16), (17) and (18) unchanged, except that instead of I we must put I_d ,as given by (10), and instead of ε, E_s, E_c, T'_m, T'_s and T'_c we must put $\bar{\varepsilon}$, $\bar{E}$, $\bar{E}_s$, $\bar{E}_c$, $\overline{T}'_m$, $\overline{T}'_s$ and $\overline{T}'_c$, respectively. If we give the value of $\bar{\varepsilon}$ we can solve the resulting system of equations simultaneously with respect to $\overline{T}'_m$, $\overline{T}'_s$ and $\overline{T}'_c$, obtaining a solution of the type

$$\overline{T}'_m = c_0 + c_1 I_d + c_2 \overline{H}' + c_3 \bar{E}_A + c_4 \bar{E}_s + c_5 \bar{E}_c, \tag{23}$$

$$\overline{T}'_s = b_0 + b_1 I_d + b_2 \overline{H}' + b_3 \bar{E}_A + b_4 \bar{E}_s + b_5 \bar{E}_c, \tag{24}$$

$$\overline{T}'_c = d_0 + d_1 I_d + d_2 \overline{H}' + d_3 \bar{E}_A + d_4 \bar{E}_s + d_5 \bar{E}_c. \tag{25}$$

The components of the mean horizontal wind are given by

$$\left.\begin{aligned} u &= \frac{1}{H_0}\int_0^{H_0} u^* dz, \\ v &= \frac{1}{H_0}\int_0^{H_0} v^* dz, \end{aligned}\right\} \tag{26}$$

where u^* and v^* are the components of the wind in the direction of parallels and meridians, respectively.

The wind field can be computed using the geostrophic relations combined with the equation of state:

$$\left.\begin{aligned} fu &= -\frac{R}{r_0}\frac{\partial T'_m}{\partial\varphi}, \\ fv &= \frac{R}{r_0\cos\varphi}\frac{\partial T'_m}{\partial\psi}, \end{aligned}\right\} \tag{27}$$

where we have assumed that

$$\int_0^{H_0}\frac{T^*}{\varrho^*}\frac{\partial\varrho^*}{\partial\varphi}dz = 0, \qquad \int_0^{H_0}\frac{T^*}{\varrho^*}\frac{\partial\varrho^*}{\partial\psi}dz = 0.$$

These last conditions are equivalent to assuming

$$(\partial\varrho^*/\partial\varphi)_{z=H_0/2} = (\partial\varrho^*/\partial\psi)_{z=H_0/2} = 0,$$

and are satisfied if

$$H' = AT'_m, \tag{28}$$

where $\quad A = 2H_0/(4T_0 + \beta H_0)$.

Applying the averaging operator (21) to equations (27) we obtain:

$$f\bar{u} = -\frac{R}{r_0}\frac{\partial\overline{T}'_m}{\partial\varphi}, \tag{29}$$

$$\bar{v} = 0. \tag{30}$$

Equation (29) allows us to compute the mean zonal wind and (30) implies that the mean daily meridional component of the wind is zero. This does not mean that we are dealing with a zonal wind, in fact v must be different from zero to have turbulent transport.

Let w^* be the vertical component of the wind which is assumed to be small compared with the horizontal components. If we integrate the continuity equation between zero and H_0 and neglect $\int_0^{H_0} (\partial w^*/\partial z)dz$ and $\int_0^{H_0} [(D\varrho^*/Dt)/\varrho^*]dz$, we obtain the non-divergent continuity equation $\nabla \cdot \boldsymbol{v} = 0$, where $\boldsymbol{v} = (u,v)$. Applying the averaging operator (21) to this equation and using (30) we see that the resulting equation is identically satisfied by our solution.

From (22) and (23) we obtain

$$\boxed{c_7\overline{K}\frac{d^2\overline{T}'_m}{d\varphi^2} - c_8\overline{T}'_m = -c_1 I_d - c_0 + J,} \tag{31}$$

where $c_7 = -c_3(A_1 - \varepsilon A_1')(1+\beta A/2)/r_0^2$

and $c_8 = 1 - A(c_2 + \beta/2)$.

The function J is given by

$$J = c_3\overline{G} - c_4\overline{E}_s - c_5\overline{E}_c. \tag{32}$$

Solving equation (31) we can obtain the mean daily temperature distribution. To get the solution, we need to know the function J. In these preliminary computations we shall incorporate J in a rough way assuming that it is equal to a constant such that the computed temperature is equal to the observed temperature at $\varphi = \pi/4$.

When $\overline{K} \neq 0$, since $c_8/c_7\overline{K} > 0$, the solution of (31) is the following:

$$\overline{T}'_m = \frac{c_1}{2c_6c_7\overline{K}}\left[e^{-c_6\varphi}\int I_d e^{c_6\varphi}d\varphi - e^{c_6\varphi}\int I_d e^{-c_6\varphi}d\varphi\right] + \frac{c_0 + J}{c_6^2 c_7\overline{K}} + K_1 e^{-c_6\varphi} + K_2 e^{c_6\varphi}, \tag{33}$$

where $c_6 = (c_8/c_7\overline{K})^{\frac{1}{2}}$.

To determine K_1 and K_2 in this solution we shall assume that the gradient of temperature is equal to zero at the equator and at the poles. We shall therefore take

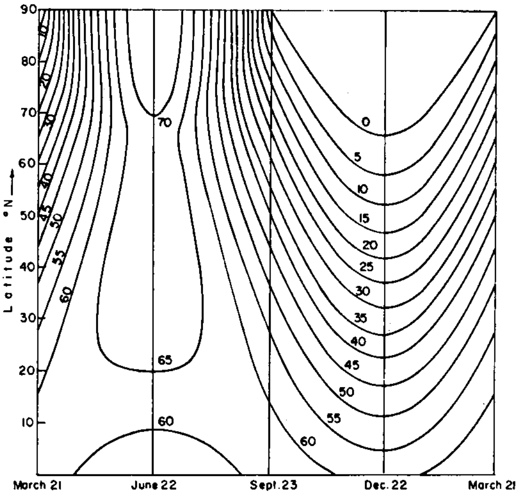

Fig. 3. Mean daily insolation in the absence of an atmosphere, in 10^{-2} cal min^{-1} cm^{-2}.

$$\left(\frac{d\overline{T}'_m}{d\varphi}\right)_{\varphi=0} = \left(\frac{d\overline{T}'_m}{d\varphi}\right)_{\varphi=\pi/2} = 0. \tag{34}$$

When $\overline{K} = 0$ the solution of (31) is

$$\overline{T}'_m = \frac{c_1}{c_8}I_d + \frac{c_0}{c_8} - \frac{J}{c_8}. \tag{33'}$$

The function I_d that enters in solution (33) is the mean daily insolation which is given by (10) and shown in Fig. 3.

We shall compute the solution corresponding to the average conditions of cloudiness and will therefore take $\bar{\varepsilon} = 0.5$. Furthermore, in the computations we shall use the following numerical values:

$r_0 = 6.36 \times 10^8$ cm
$\Omega = 7.292 \times 10^{-5}$
$R = 0.287 \times 10^7$ cm^2 seg^{-2} (°K)$^{-1}$
$\beta = 6.5 \times 10^{-5}$ (°K) cm^{-1}
$g = 980$ cm seg^{-2}
$H_0 = 11 \times 10^5$ cm
$T_0 = 216.5$°K
$T_{s0} = 288$°K
$T_{c0} = 263.5$°K
$c = 0.717 \times 10^7$ cm^2 seg^{-2} (°K)$^{-1}$
$\bar{\varepsilon} = 0.5$
$p_0 = 0.226 \times 10^6$ g cm^{-1} sec^{-1}
$H_1 = 3 \times 10^5$ cm
$H_2 = 4.5 \times 10^5$ cm
$\varrho_0 = 0.364$ g cm^{-3} 10^{-3}.

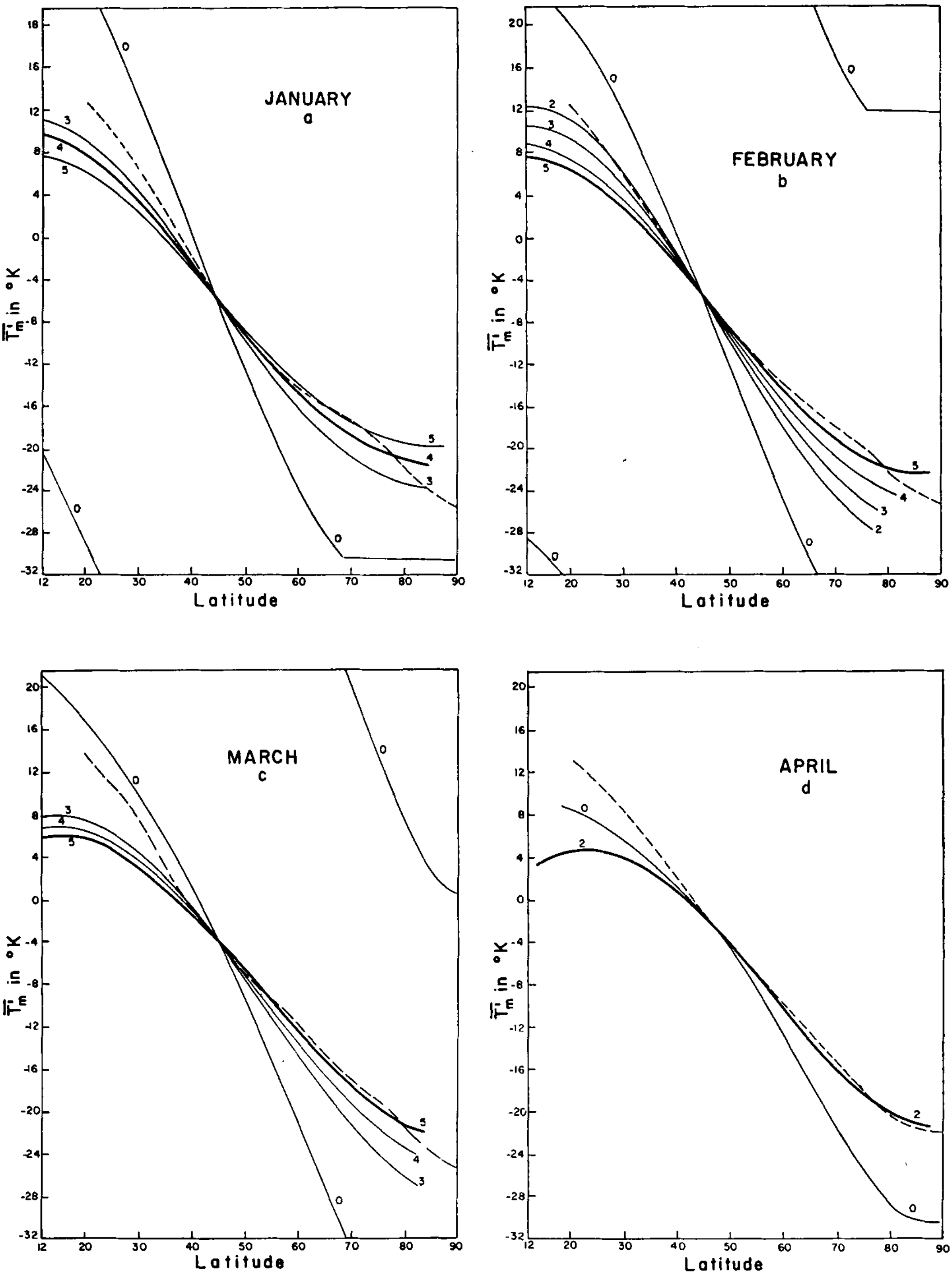

Fig. 4. Theoretical temperature for January, February, March and April corresponding to different values of the exchange coefficient K. The dashed line is the observed temperature.

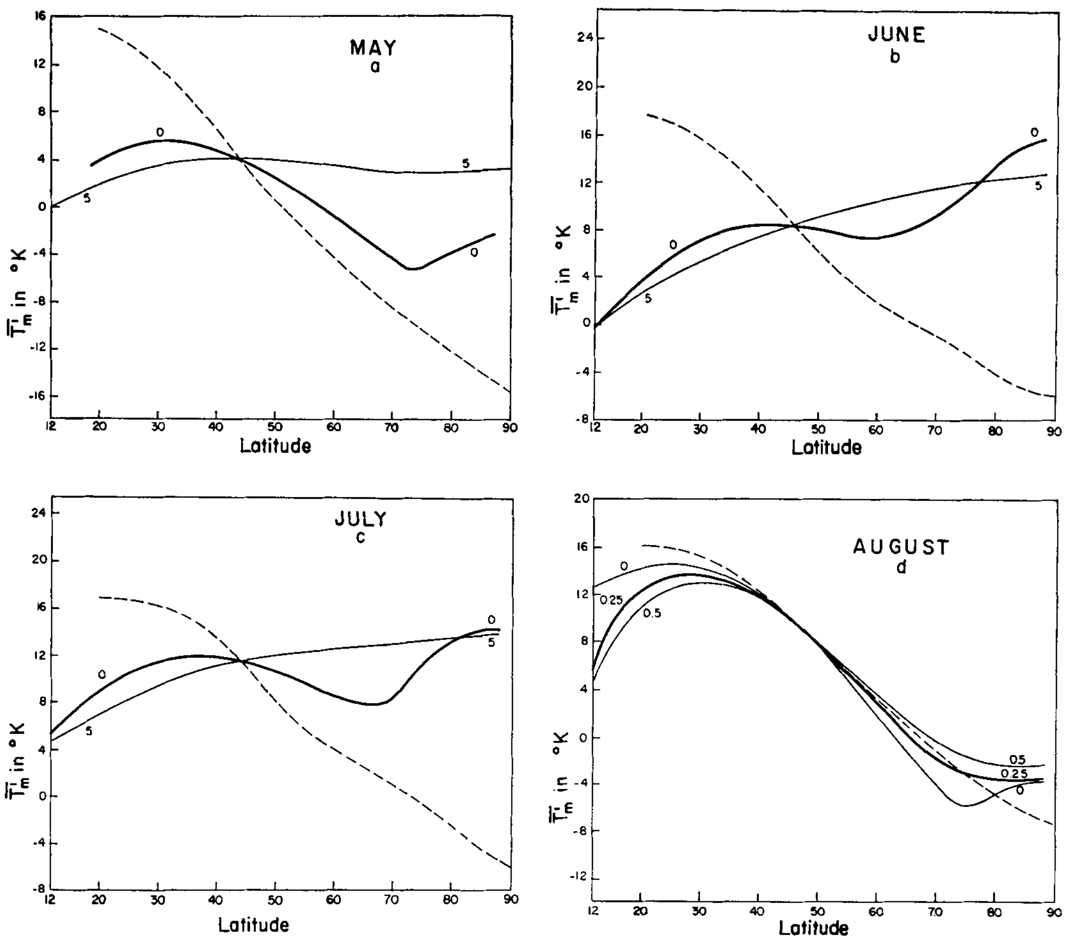

Fig. 5. Theoretical temperature for May, June, July and August corresponding to different values of the exchange coefficient $\overline{K}$. The dashed line is the observed temperature.

The computed values of the coefficients of equations (23), (24) and (25) are:

$c_0 = -59.353$, $c_1 = 125.818$, $c_2 = -30.944 \times 10^{-6}$,
$c_3 = -232.2$, $c_4 = -176.2$, $c_5 = -176.5$,
$b_0 = -59.96$, $b_1 = 153.746$, $b_2 = 24.662 \times 10^{-6}$,
$b_3 = -168.2$, $b_4 = -259.8$, $b_5 = -152.3$,
$d_0 = -54.88$, $d_1 = 119.784$, $d_2 = 27.182 \times 10^{-6}$,
$d_3 = -185.6$, $d_4 = -174.4$, $d_5 = -313.4$.

These coefficients correspond to the case in which the units of $\overline{T}'_m$, $\overline{T}'_s$ and $\overline{T}'_c$ are Kelvin degrees, those of I_d, $\bar{E}$, $\bar{E}_s$ and $\bar{E}_c$ are cal cm^{-2} seg^{-1} and those of H' are cms.

The computed value of the coefficients that enter in the other equations are the following: $A = 2.347 \times 10^3$, $A_1 = 5.755 \times 10^9$, $A'_1 = 9.0538 \times 10^8$, $C_6 = 4.61 \times 10^5$ $\overline{K}^{-\frac{1}{2}}$, $C_7 = 4.688 \times 10^{-12}$ and $C_8 = 0.996$, where we have used cms, grams, seg and °K as units.

Solution (33) has been computed for the 15th of every month and for different values of the exchange coefficient $\overline{K}$. The results for all the months are shown in Figs. 4, 5 and 6. The dashed curve is the observed mean monthly temperature taken from the 700 mb mean charts prepared by the U.S. WEATHER BUREAU (1952) and the others are the computed temperatures labeled with the corresponding exchange coefficient $\overline{K}$ (in 10^{10} cm^2 per sec). The solution that best agrees with the observed values is drawn with a thicker line.

Using (29) and the computed values of $\overline{T}'_m$ we can obtain the mean zonal wind. The results

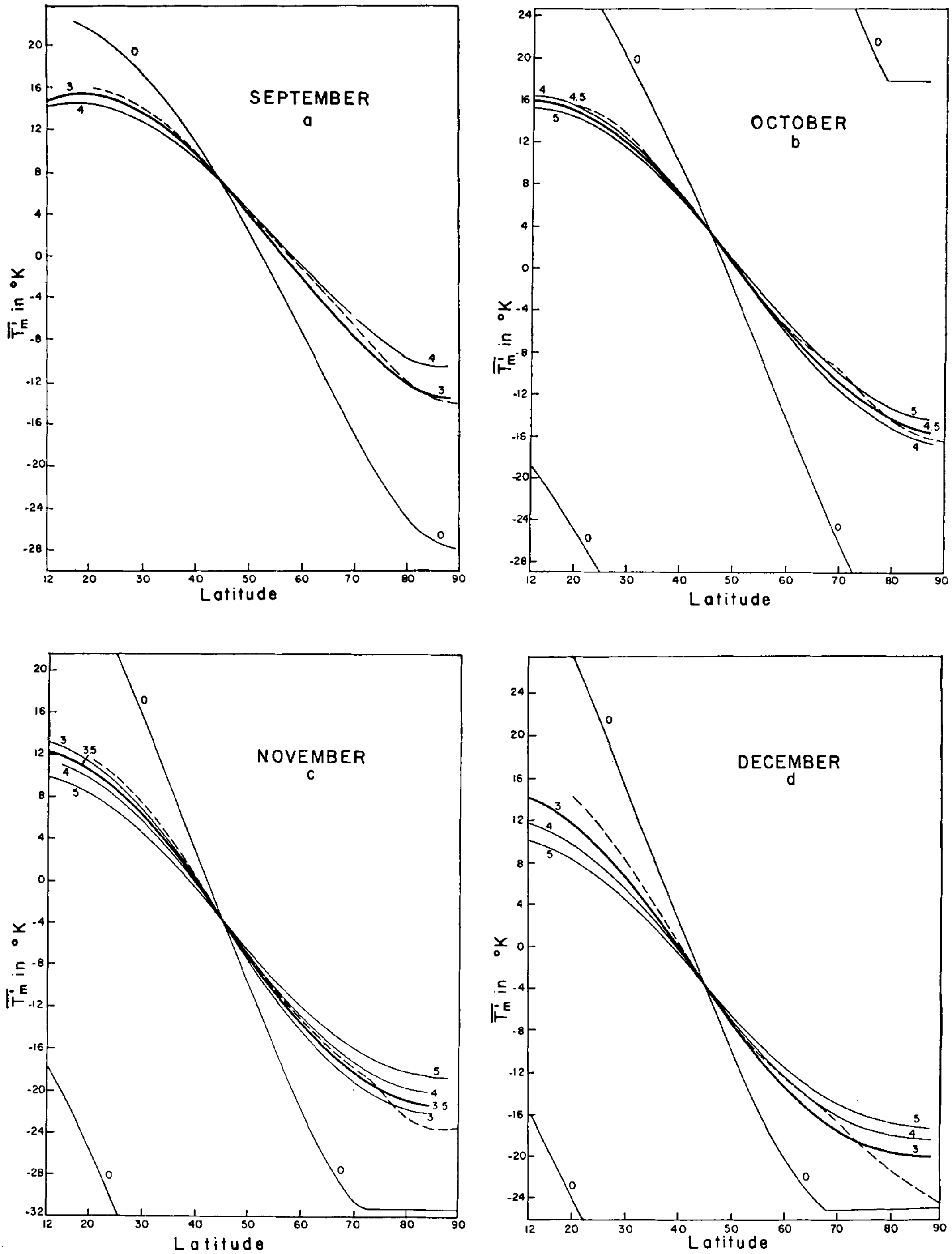

Fig. 6. Theoretical temperature for September, October, November and December, corresponding to different values of the exchange coefficient $\bar{K}$. The dashed line is the observed temperature.

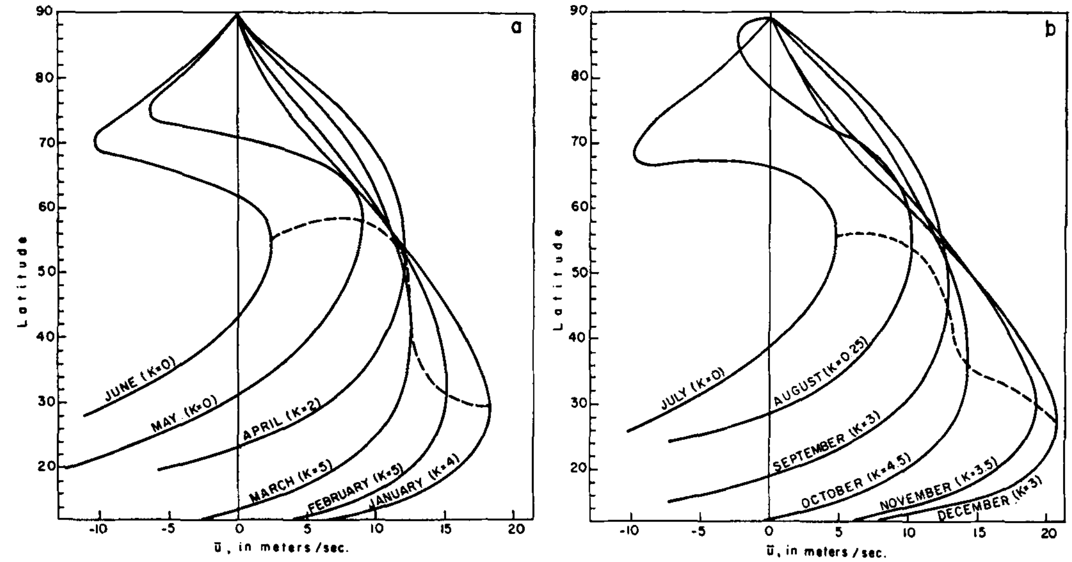

Fig. 7. Theoretical mean zonal wind at 500 milibars.

for all the months of the year are shown in Fig. 7.

From (23) and (28) we can obtain the following expression for the excess of radiation in the troposphere:

$$\bar{E}_A = -\frac{1}{c_3}(c_1 I_d + c_0) + \frac{1}{c_3}(1 - c_2 A)\overline{T}'_m - \frac{1}{c_3}(c_4 \bar{E}_s + c_5 \bar{E}_c). \quad (35)$$

The last term of the second member of (35) is a linear function of the excess of radiation in the oceans, continents and clouds and enters in the solution combined linearly with the function $\overline{G}$ as shown by (32). Since we have postulated that J is a constant, we do not need to know the values of $\bar{E}_s$, $\bar{E}_c$ and $\overline{G}$ to obtain the solution for $\overline{T}'_m$ and u.

Fig. 8*a* shows the net radiation excess in the troposphere, as given by London. Figure 8*b* shows the theoretical radiation excess computed from formula (35) when we take $c_4\bar{E}_s + c_5\bar{E}_c = 0$.

Formula (35) can be used to obtain an estimate of the term $-(1/c_3)\ (c_4\bar{E}_s + c_5\bar{E}_c)$ by substituting instead of $\bar{E}_A$ the net radiation excess given by London. The result is obtained by subtracting 8*b* from 8*a* and is shown in Fig. 8*c*. Fig. 8*d* represents the function J. From (32) we can compute the function $\overline{G}$ that represents the addition of latent heat energy and of thermal energy by conduction. The function $\overline{G}$ is represented in Fig. 8*e*. Finally, in formula (35) the term $E_0 = -(1/c_3)\ (c_1 I_d + c_0)$ represents the excess of radiation in the troposphere assuming that the mean temperature in the troposphere is a constant equal to T_{m0}, and that the temperatures at the cloud layer and at the surface of the earth are also constant and equal to T_{c0} and T_{s0}, respectively. The function E_0 is shown in Fig. 8*f*.

Discussion of the results and conclusions

With the exception of the months of May, June and July, the above computations show a remarkable agreement between the theoretical and the observed values of the temperature, the zonal wind and excess of radiation in the troposphere.

The failure of the solution in the months of May, June and July is apparently due to the fact that at that time of the year the gradient with respect to the latitude of the excess of radiation is small and of the same order of magnitude than that of the energy different from radiation (latent heat energy, vertical conduction of sensible heat, etc.). We therefore need to specify this energy, during these months, with more precision, and the error has arisen from having postulated that the function

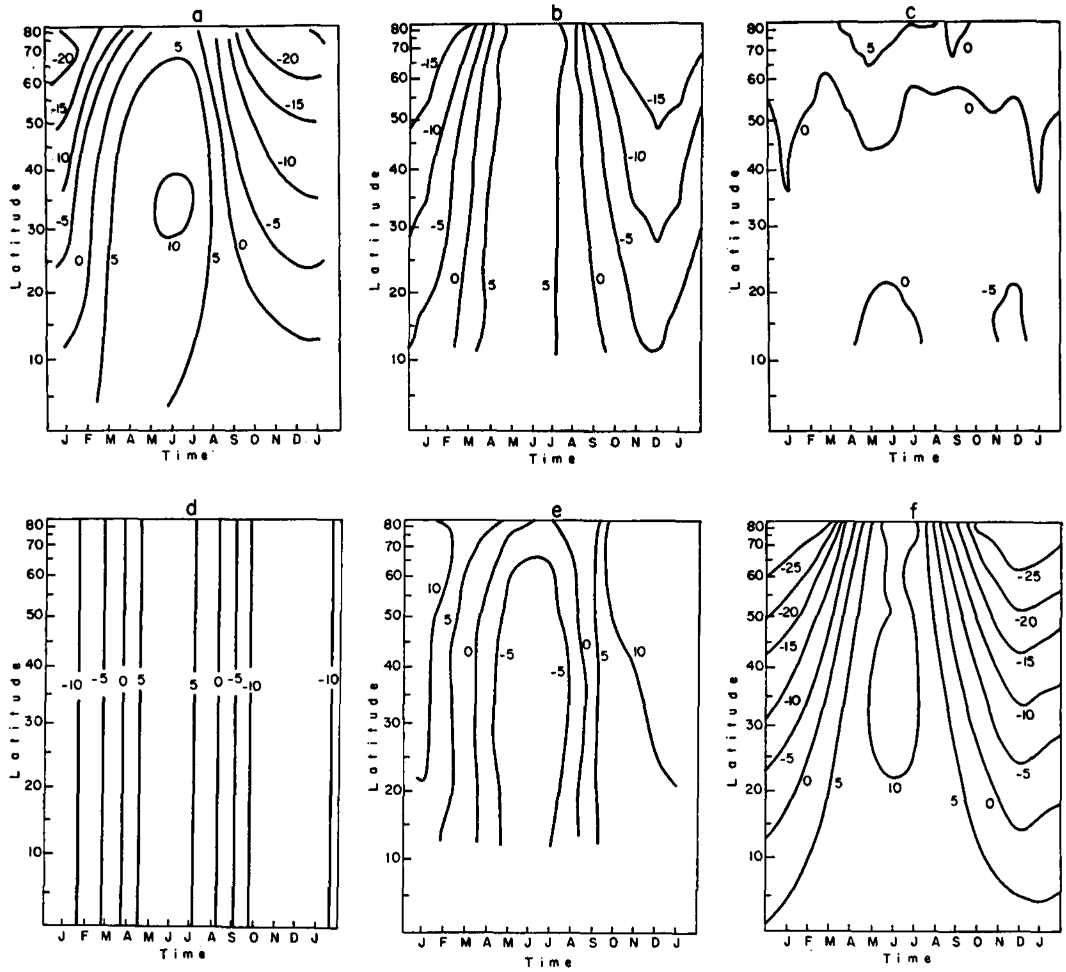

Fig. 8. The net excess of radiation in the troposphere: 8*a* is the distribution given by London and 8*b* the theoretical one; 8*c* is the difference between 8*a* and 8*b*; 8*d* is the function *J*; 8*e* is the function *G*; 8*f* is the net excess of radiation, assuming a mean temperature, constant with latitude.

J is a constant. This can be seen from a comparison of the net excess of radiation (Fig. 8*a*) and the function $\bar{G}$ that contains the addition of energy by processes different from radiation (Fig. 8*e*). We see that except in the months of May, June and July, the function $\bar{G}$, compared with the excess of radiation, is approximately a constant with latitude and $c_4\bar{E}_s + c_5\bar{E}_c = 0$. Therefore, the computations have shown that, except in these months, the radiation processes are the dominant ones, and all other processes appear globally as a constant.

The computations have also shown that the main mechanism of meridional transport is turbulent diffusion transport of heat through the larger eddies (migratory cyclones and anticyclones of middle latitudes). The exchange coefficient of our solution is equal to 5×10^{10} cm^2 per sec in February and in March, 4.5×10^{10} in October, 4×10^{10} in January, 3.5×10^{10} in November, 3×10^{10} in September and in December, 2×10^{10} in April, 0.25×10^{10} in August, and zero in May, June and July. This is in complete agreement with the order of magnitude of the $\bar{K}$ corresponding to the migratory cyclones and anticyclones of the middle latitudes, treated as turbulent eddies, which is equal to 5×10^{10} (Stewart, 1945, p. 468). This mechanism of meridional transport of heat has been used already by Defant (1921), and Lettau (1939).

We obtain two maxima in the values of $\bar{K}$, one in October and another in February; and the value of $\bar{K}$ goes to zero when we approach the summer.

The value of $\bar{K}$ is a measure of the transport required in the different months of the year. In Figs. 4, 5 and 6 the curves labeled with zero correspond to the case in which there is no transport. We see that the transport is maximum in October and in February; and it is almost zero in August. In May, June and July it is zero, but a deficiency of energy exists due to the wrong specification, as a constant, of the function J. As was pointed out by FJØRTOFT (1960), in winter the transport reduces the gradient of temperature almost to a half.

The theoretical solution predicts the position and strength of the jet stream, as shown in Fig. 7. The accuracy is very good except in May, June and July. Therefore, with the exception of these months, the mean daily circulation is mainly a result of the radiation processes and the turbulent transport of heat by the migratory cyclones and anticyclones.

Since we are using the geostrophic relations, the zonal wind becomes unrealistic as we approach to the equator.

It is worth mentioning that, since T', T_s' and T_c' are small compared with T_0, T_{0s} and T_{0c}, respectively, the solution is correct and independent of the arbitrary choice of the basic temperatures T_0, T_{0s} and T_{0c}.

The contribution from the cloud layer to the radiation balance depends on its temperature: $T_c = T_{0c} + T_c'$. Since $T_c' \ll T_{0c}$ and we have taken $T_{0c} = 263.5°\text{K}$, our cloud layer hypothesis is in substantial agreement with the one used by SIMPSON (1928) who in his computations took a cloud temperature of 261°K. A detailed study on the effect of the cloud cover on the general circulation will be given elsewhere.

9. Dynamic model of the troposphere, containing radiation processes

Instead of the geostrophic relations (27) we can use the general equations of horizontal motion, including the frictional terms, and combine them with the continuity equation and equations (16), (17), (18) and (20) to obtain a model, containing, in a simplified way, the coupling between the dynamical and the thermodynamical processes in the atmosphere. The variables are: the mean temperature (T_m'), the mean horizontal components in the velocity (u and v), the excess of radiation in the troposphere (E_A), the height of the upper radiating boundary of the troposphere (H'), the temperature of the cloud layer (T_c') and the temperature of the surface of the earth (T_s'). It is a system of seven equations and seven unknowns.

The equations contain the following parameters: insolation (I), cloudiness (ε), the Coriolis parameter (f), the skin friction coefficient, the exchange coefficient of turbulent difusion (K), the specific heat at constant volume at the surface of the earth (C_s), the density at the surface of the earth (ϱ_s), the excess of radiation at the cloud layer (E_c), the excess of radiation at the surface of the earth (E_s) and the function G.

We can add to the system two more equations: one of conservation of energy at the layer below the surface of the earth (continents and oceans) and another of conservation of energy at the cloud layer. In this case we take E_s and E_c as variables.

The function G, that contains the latent heat energy and the conduction of sensible heat, can be incorporated as a function of T_m' and other variables, instead of as a known parameter.

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