On the diurnal variation of surface temperature

By OLOV LÖNNQVIST, *The Swedish Meteorological and Hydrological Institute, Stockholm.*

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ABSTRACT

The global short-wave radiation is the main external factor which determines the diurnal temperature variation at the surface of the earth. Therefore, there should be a close connection between the diurnal variation of the global radiation, which is normally quasi-symmetrical relative to noon, and that of the surface temperature, which on the contrary has a very characteristic non-symmetrical appearance. It is assumed that the surface temperature can be represented with sufficient approximation by a small number of terms of a Fourier series, and that the energy loss by various effects, such as evaporation, is proportional to the surface temperature. In doing so it can easily be shown that each harmonic term of the temperature variation is determined by the respective term of the diurnal variation of the global radiation; the changes in amplitude and phase angle of each term being functions of the heat capacity and thermometric conductivity of the soil and the coefficient for the rate of loss of energy by non-conduction. The formulae arrived at are successfully tested against observed data published by Franssila. In doing so the first four terms of the Fourier series are used.

Characteristic features of the diurnal variation of insolation and surface temperature

Disregarding in this study the effects of advection of energy, insolation is the external factor determining the temperature at the surface of the earth. The insolation consists of the short-wave global radiation, i.e. radiation from sun and sky reduced by the albedo of the surface, and the long-wave back radiation from the atmosphere. Since the latter is quasi-constant throughout the 24 hours in case of a clear sky, it will be disregarded at this stage. Furthermore the back radiation from the atmosphere is usually combined with the black-body radiation of the surface into long-wave net radiation and can thus be accounted for when introducing surface cooling effects in the theory, as done below.

Fig. 1*a* shows a typical diurnal variation of short-wave global radiation. This case was observed by FRANSSILA (1936) and represents the mean conditions at Pälkäne, Finland, during the cloudless period August 6–8, 1934. Since FRANSSILA did not observe short-wave radiation separately the present author has computed it from the net short-wave plus long-wave radiation observed by Franssila, by assuming a constant relative long-wave ne radiation throughout the period. As shown in the figure, the radiation is on the whole symmetric relative to local noon although there is in fact a slight skewness in the curve, hardly visible from the figure. The duration of daylight is 16.6 hours as shown by the symbols for sunrise and sunset.

Fig. 1*b* shows the simultaneous curve for the temperature at the surface of the earth. This curve has all the typical features of the diurnal surface-temperature variation, viz. the sharp-edged minimum about one hour after sunrise, the very steep rise in temperature early in the forenoon, the quasi-parabolic maximum about one and a half hours after noon, the inflexion point about two hours before sunset and the less and less steep drop in temperature during the night.

Our problem is now to find the connection between the two curves, i.e. to show the reason for the typical non-symmetrical features of the temperature curve, and—if possible—to give the means of finding the temperature curve corresponding to any given radiation curve.

SUTTON (1953) has discussed various heat transfer problems though none formulated exactly in this way. To the author's knowledge only part of our problem has been dealt with

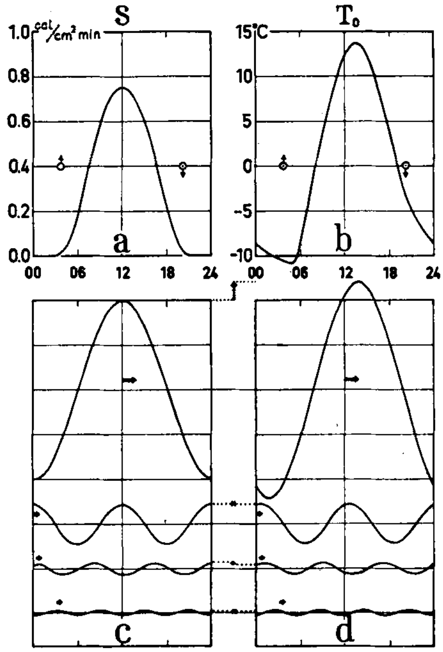

Fig. 1. Harmonic analysis of the diurnal variation of insolation (a and c) and surface temperature (b and d) according to Franssila's observations at Pälkäne, August 6–8, 1934. Sunrise and sunset are indicated by symbols. Horizontal arrows indicate computed displacements of the insolation harmonics in order to obtain the corresponding temperature harmonics. Vertical arrows and the dotted lines indicate the computed changes in the respective amplitudes.

so far, viz. the fall of temperature on a clear night. This particular question was first studied by Brunt (1932) and has subsequently been dealt with by Philipps (1940), Jaeger (1945), Groen (1947) and Reuter (1951). Frost prediction methods based on Brunt's parabolic formula for nocturnal surface temperatures have been given by Dufour (1938) and Lönnqvist (1945). In spite of the partial success in applying Brunt's formula in practical frost prediction, the formula itself can be seriously criticized, because of many simplifying assumptions made in formulating the problem mathematically. The most serious one of these assumptions seems to be the introduction of a sudden step in the radiative flux at sunset which gives a vertical tangent to the temperature curve at that time. As mentioned in discussing Fig. 1b above, the steepest slope in the curve is usually found one to three hours before sunset.

Theoretical Approach

We assume that the unknown diurnal variation of the surface temperature, T_0, can be written by sufficient approximation in the following way,

$$T_0 = T_m + \sum_{j=1}^{n} (p_j \cos j\omega t + q_j \sin j\omega t), \qquad (1)$$

where $\omega = 0.727 \times 10^{-4}$, provided time is given in seconds. Thus, p_j and q_j are now the unknowns of our problem.

From this assumption we know, see for instance Sutton (1953), that the temperature in the soil and at the surface, $z \leqslant 0$, can be written

$$T(z, t) = T_m(z) + \sum_{j=1}^{n} e^{\alpha_j z}[p_j \cos (j\omega t + \alpha_j z) + q_j \sin (j\omega t + \alpha_j z)]. \qquad (2)$$

Here $$\alpha_j = \sqrt{\frac{j\omega}{2\varkappa}}, \qquad (3)$$

where $\varkappa$ is the thermometric conductivity of the soil.

The solution (3) can be found by determining $\partial T/\partial t$ and $\partial^2 T/\partial z^2$ from (2) by partial differentiation and inserting these expressions in the equation of conduction of heat in a solid, viz.

$$\frac{\partial T}{\partial t} = \varkappa \frac{\partial^2 T}{\partial z^2}. \qquad (4)$$

Now, we know that the non-conductional flow of energy from above into the surface is proportional to the vertical temperature gradient in the soil at the surface ($z \leqslant 0$). Hence

$$\varkappa \varrho c \left(\frac{\partial T}{\partial z}\right)_{z=0} = S - h(T - T_h), \qquad (5)$$

where ϱc is the heat capacity of the soil, S is the global radiation corrected for albedo, h is a "cooling coefficient", and T_h is the specific temperature at which there is neither cooling nor heating by the sum of all non-conductional effects such as evaporation (or condensation) and long-wave net radiation. The validity of the last-mentioned assumption will be discussed later on. At this stage it should be mentioned only that the last term in eq. (5) could be looked upon as the effect of Newton's law of cooling.

To be able to utilize (5) we have to differentiate (2) and we obtain

$$\left(\frac{\partial T}{\partial z}\right)_{z=0} = \sum_{j=1}^{n} \alpha_j[(q_j + p_j)\cos j\omega t + (q_j - p_j)\sin j\omega t] - \frac{C}{\varkappa \varrho c}. \tag{6}$$

Here, the constant C comes from the term $T_m(z)$ and accounts for the contribution from the seasonal variation of conduction in the soil. C is positive in winter and negative in summer. Also, we need an expression for the insolation S. We choose to express S with a Fourier series with an equal number of terms as used for the temperature assumption. Thus

$$S = S_m + \sum_{j=1}^{n}[b_j \cos j\omega t + c_j \sin j\omega t], \tag{7}$$

where the constants b_j and c_j can easily be obtained for any given set of radiation data.

Now, by equalizing similar terms in eq. (5), according to (1), (6) and (7), we obtain

$$S_m + C = h(T_m - T_h), \tag{8}$$

and for each j

$$\begin{cases} \varkappa \varrho c \alpha_j (q_j + p_j) = b_j - hp_j, & (9) \\ \varkappa \varrho c \alpha_j (q_j - p_j) = c_j - hq_j. & (10) \end{cases}$$

Introducing

$$\beta_j = \varkappa \varrho c \alpha_j, \tag{11}$$

we find from (9) and (10) for each j

$$\begin{cases} p_j = \dfrac{(\beta_j + h)b_j - \beta_j c_j}{(\beta_j + h)^2 + \beta_j^2}, & (12) \\ q_j = \dfrac{\beta_j b_j + (\beta_j + h)c_j}{(\beta_j + h)^2 + \beta_j^2}. & (13) \end{cases}$$

We are interested in the change ϑ_j, in the phase angle. Obviously

$$\operatorname{tg}\vartheta_j = \frac{\dfrac{b_j}{c_j} - \dfrac{p_j}{q_j}}{1 + \dfrac{b_j}{c_j}\cdot\dfrac{p_j}{q_j}} = \frac{\beta_j}{\beta_j + h}. \tag{14}$$

We are also interested in the displacement D_j, i.e. the change of the time of the maxima for each harmonic term.

$$D_j = \frac{\vartheta_j}{j} \text{ radians} = \frac{\vartheta_j^0}{15j} \text{ hours.} \tag{15}$$

Finally we want an expression for the relation between the amplitude A_{S_j} of each harmonic term of the diurnal variation of the insolation and the amplitude A_{T_j} of the respective terms of the diurnal variation of surface temperature. Obviously

$$\frac{A_{T_j}}{A_{S_j}} = \sqrt{\frac{p_j^2 + q_j^2}{b_j^2 + c_j^2}} = \frac{1}{\sqrt{(\beta_j + h)^2 + \beta_j^2}} = \frac{1}{\beta_j \sqrt{1 + \left(1 + \dfrac{h}{\beta_j}\right)^2}}. \tag{16}$$

According to (3) and (11) we can write

$$\beta_j = \beta\sqrt{j}, \tag{17}$$

where

$$\beta = \varrho c \sqrt{\varkappa}\sqrt{\frac{\omega}{2}}. \tag{18}$$

Estimation of the constants h, T_h and β

In eq. (5) we have summarized the non-conductional cooling of the surface in the term $h(T - T_h)$. Speaking first of the cooling by long-wave net radiation to a clear sky, it can be written

$$F = 0.002\ T + 0.12 = 0.002\,(T + 60)\ \text{cal/cm}^2\text{min}, \tag{19}$$

provided temperature is given in degrees Celsius and assuming a relative net radiation equal to 27 per cent. Eq. (19) is valid with good approximation for $0 < T < 30°$C. It means that Newton's law of cooling holds as if the radiative temperature of the cloudless sky were equal to $-60°$C.

Secondly, let us look at the cooling by evaporation (and warming by condensation). In Fig. 2 the data are taken from combined observations and computations by Franssila (*op. cit.*) for two different weather situations. We are interested here especially in the conditions during the period August 4–5 and 6–8, 1934,

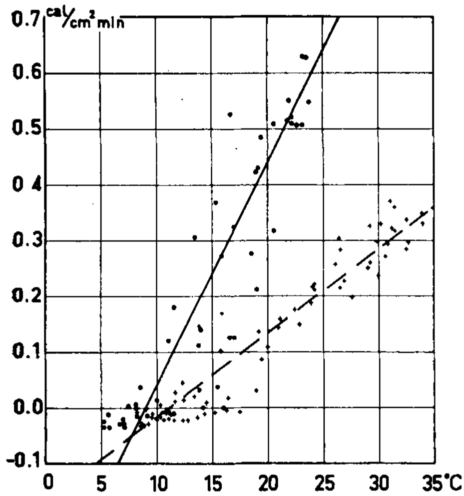

Fig. 2. The relation between surface temperature and the cooling by evaporation according to Franssila's observations. Crosses and broken line refer to the period August 4–5 and 6–8, 1934, while dots and continuous line refer to June 25–27, 1934.

given by crosses in the figure. Obviously, the cooling, W, can be represented with some approximation by the linear function

$$W = 0.015\,(T - 11)\ \text{cal/cm}^2\text{min}. \qquad (20)$$

In this case it is reasonable to interpret the temperature value 11°C in (20) as being the dew-point temperature at a certain level, knowing that the diurnal variation of the dew-point temperature in the air is small as compared with that of the temperature itself. According to Franssila's observations the average dew-point temperature one metre above the surface was equal to 11.3°C during the period in question, thus in good agreement with our interpretation.

Turbulence could give rise to a third non-conductional cooling term. Then T_h could be interpreted as the potential temperature of the free atmosphere whereas h should be a function of the wind velocity. However, since we are interested primarily in light-wind conditions, turbulence will be disregarded in this paper.

Adding up (19) and (20), we obtain

$$h(T - T_h) = F + W$$
$$= 0.017\,(T - 2.6)\ \text{cal/cm}^2\text{min}. \qquad (21)$$

Changing now the unit to cal/cm²sec, we have

$$h = 2.8 \times 10^{-4}\ \text{cal/cm}^2\text{sec deg}, \qquad (22)$$

and

$$T_h = 2.6°\text{C}. \qquad (23)$$

In Fig. 2 the dots correspond to another weather situation, viz. June 25–27, 1934. Still using (19) for the long-wave net radiation, we obtain in this case quite different values for h and T_h, viz. 7.0×10^{-4} cal/cm²sec deg and 5.7°C, respectively. This shows that the coefficients h and T_h easily vary quite a lot from one weather situation to another.

Finally, we have to estimate the value for β. Strictly speaking, β should represent conduction both in the soil and in the air, i.e. it should be a sum of two terms. Since however molecular conduction in the air is negligible as compared with soil conduction, nothing but soil conduction will be discussed below. The following table indicates values for β for some different types of soil, computed from data given by GEIGER (1961). It should be emphasized that we have used $\omega = 0.727 \times 10^{-4}$ in (18) when computing these values which means that they are valid for the diurnal wave only.

Furthermore, from the representative values for soil given by SUTTON (1953) we find $\beta = 2.5 \times 10^{-4}$, while Franssila's own investigations of the characteristics of the soil at the actual site give $\beta = 2.1 \times 10^{-4}$ cal/cm²sec deg.

TABLE 1. *Typical values for the diurnal soil-conduction coefficient β.*

Type of soil	$10^4 \times \beta$ (cal/cm² sec deg)
Snow, freshly fallen	0.3
Peaty soil, dry	0.5
Clay, dry	0.6
Sand, dry	1.0
Marshy soil, wet	1.5
Sand, wet	2.2
Clay, wet	2.3
Snow, old	2.4
Ice	3.1
Rock	3.7

Verification of the formulae

Fig. 1c shows the first four harmonics of the variation of the radiation presented in Fig. 1a. Indicated by horizontal arrows are the displace-

TABLE 2. *Comparison between observed values for b_j and c_j in cal/cm² min and observed and computed values for p_j and q_j in degrees Celsius.*

	First harmonic	Second harmonic	Third harmonic	Fourth harmonic
b_j observed	−0.398	0.088	0.027	−0.005
c_j observed	−0.012	0.002	0.010	−0.002
p_j computed	−10.8	1.9	0.4	−0.1
q_j computed	−5.4	1.0	0.4	0.0
p_j observed	−10.8	2.1	0.4	−0.2
q_j observed	−5.4	0.7	0.4	0.1

TABLE 3. *Comparison between observed and computed values for the displacement D_j in hours and the ratio of amplitudes A_{T_j}/A_{S_j} in deg per cal/cm²min.*

	First harmonic	Second harmonic	Third harmonic	Fourth harmonic
D_j computed	1.64	0.86	0.68	0.54
D_j observed	1.67	0.63	0.57	−0.8
Ratio computed	30.4	24.6	21.6	19.4
Ratio observed	30.4	24.3	19.5	39.4

TABLE 4. *Comparison between observed and computed values for the surface temperature at Pälkäne, August 6–8, 1934.*

Local time	Computed $T-T_m$ (in °C)	Observed $T-T_m$ (in °C)
00	−8.6	−8.6
02	−9.7	−9.9
04	−10.5	−10.6
06	−7.8	−8.3
08	−0.7	−0.1
10	7.3	7.2
12	12.2	12.3
14	13.7	13.5
16	10.5	9.9
18	3.8	3.8
20	−2.9	−2.8
22	−6.9	−6.5

ments D_j corresponding to eqs. (14) and (15), provided

$$h = 2.7 \times 10^{-4} \text{ cal/cm}^2\text{sec deg}, \quad (24)$$

and $$\beta = 2.3 \times 10^{-4} \text{ cal/cm}^2\text{sec deg}. \quad (25)$$

The values were chosen to give the best possible fit with observed temperatures. They

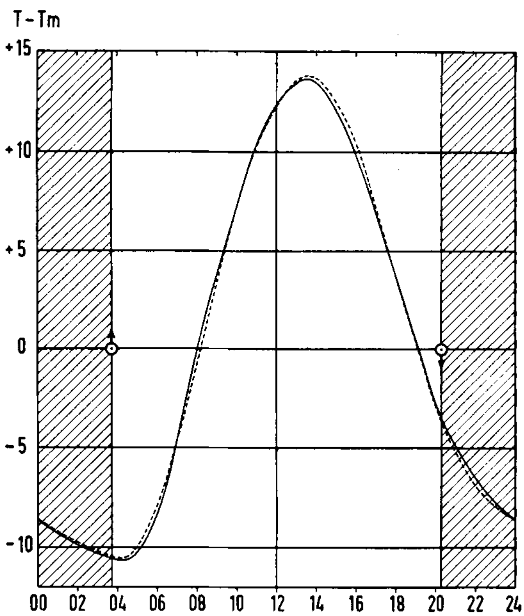

Fig. 3. Diurnal variation of the temperature at the surface of the earth. The continuous line corresponds to Franssila's measurements, while the broken line shows the variation computed from the observed insolation, assuming $h = 2.7 \times 10^{-4}$ and $\beta = 2.3 \times 10^{-4}$ cal/cm² sec deg. Sunrise and sunset are indicated by symbols. Night hours are shaded.

agree exceedingly well with the values for h and β, mentioned above. The vertical arrows and dotted lines in the figure indicate the changes in amplitude as given by eq. (16),

Fig. 1*d* shows the first four harmonics of the observed variation of surface temperature presented in Fig. 1*b*. The amplitudes and the displacements of the first, second and third harmonic term agree very well with the computed ones shown in the figure. The less good agreement as far as the fourth harmonic is concerned can be explained by the fact that this harmonic is more strongly affected, counted as a percentage, by observational errors in temperature and radiation than are the other ones.

Tables 2, 3 and 4 demonstrate in different ways the good agreement between observed and computed data in this particular case.

The values given in Table 4 are presented in Fig. 3. The full line shows the observed variation, the broken line shows the computed one. It is evident that the computed curve represents all the typical features of the observed temperature variation.

REFERENCES

BRUNT, D., 1932, Notes on radiation in the atmosphere. *Quart. J. Roy. Met. Soc.*, 58, pp. 389–418.

DUFOUR, L., 1938, Notes sur le problème de la gelée nocturne. *Inst. Roy. Mét. de Belgique. Mem. Vol. IX*, 24 pp.

FRANSSILA, M., 1936, Mikroklimatische Untersuchungen des Wärmehaushalts. *Mitt. der Met. Zentralanstalt Helsinki*, 20, 103 pp.

GEIGER, R., 1961, *Das Klima der bodennahen Luftschicht.* Fr. Vieweg & Sohn, Brunswick. 646 pp.

GROEN, P., 1947, On radiational cooling of earth's surface during the night especially with regard to the prediction of ground frost. *K. Nederl. Met. Inst. Mededeel. en Verh. Serie B.* 1, No. 9.

JAEGER, J. C., 1945, Note on the effect of wind on nocturnal cooling. *Quart. J. Roy. Met. Soc.*, 71, pp. 388–390.

LÖNNQVIST, O., 1945, A frost prediction diagram. *Arkiv f. Matem., Astr. och Fysik* 32 *A*, No. 13, 7 pp.

PHILIPPS, H., 1940, Zur Theorie der Wärmestrahlung in Bodennähe. *Gerlands Beitr. z. Geophys.* 56, pp. 229–319.

REUTER, H., 1951, Forecasting Minimum Temperatures. *Tellus*, 3, pp. 141–147.

SUTTON, O. G., 1953, *Micrometeorology.* McGraw-Hill, New York, Toronto, London, 333 pp.