The Stability of Thermoclinic Jets

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(Manuscript received February 6, revised version August 28, 1961)

Abstract

A laminar jet, in a "shallow layer" of water of density $\varrho - \Delta\varrho$ is in geostrophic equilibrium above a much deeper and resting layer of water of density ϱ. We compare the quasi-hydrostatic energy releasing process in this thermoclinic model with other baroclinic and barotropic models. It is first shown that a small amplitude disturbance can grow only if there is a transfer of *kinetic energy* from the horizontal variations in the mean jet and that conversions of mean potential energy can only occur concomitantly with, or as a finite amplitude result of this process. We also examine the perturbation equation for values of the stream Rossby number which are much less than unity and show that the finite depth of the thermocline inhibits inflectional instability of the jet. The generalization of Lord Rayleigh's theorem shows that the jet is stable if the gradient of *potential* vorticity does not vanish. In general, there is a critical value of the ratio of the radius of deformation $\left(f^{-1}\sqrt{gH_0\Delta\varrho/\varrho}\right)$ to the half width of the jet above which quasi-geostrophic disturbances will start to grow. It is proposed that meanders of the synoptic Gulf Stream will develop when the jet becomes "excessively" deep and that the effect of such instabilities is to limit the depth of the main thermocline in the interior of the ocean.

1. Statement of the Problem

We shall call the geostrophic laminar current, which is shown schematically in Fig. 1, a thermoclinic jet since it is a useful abstraction (ROSSBY, 1936); STOMMEL, 1953) of the thermal structure of the Gulf Stream. It consists of a homogeneous layer of light water of mean depth H_0 lying above a much deeper layer of water which is denser by an amount $\Delta\varrho \ll \varrho$. The latter is resting in a coordinate system which is rotating at $\omega = f/2$ rads./sec. The velocity in the upper layer is $V^*(x)$ directed along the y axis (or into the page) and this is balanced by the hydrostatic pressure gradient due to the elevation of the free surface. The characteristic width of the jet is denoted by $2L$ and V is the maximum speed.

The mutual adjustment of the free surface and the interface, which is necessary to bring about this inviscid equilibrium, allows us to express the hydrostatic pressure gradient in the upper layer in terms of its thickness $[h^*(x)]$ as: $g'dh^*/dx$ where $g' = g\Delta\varrho/\varrho$ is a "reduced" value of gravity. Under the conditions stated below[1] a similar simplification may be made for a perturbation $[h'(x, y, t)]$ which is introduced upon this mean field, so that if $h_T = h^* + h'$ denotes the total thickness of the upper layer then $g'\nabla h_T$ is the horizontal pressure gradient

[1] The hydrostatic (or shallow water) approximation may be defended when the mean depth of the top layer (H_0) and the bottom layer (H_b) are both small compared to the horizontal scale of the perturbation (L). On the other hand it may be shown that the ratio of the pressure gradients in the two layers is small to order H_0/H_b provided the stream Rossby number (R) and the rotational Froude number (F) are of magnitude unity or less (see eq. 10 for definitions of R, F). Thus the validity of this approximation and the analogy of the two layer model in Fig. 1 with the one layer model in Fig. 2, depend on the inequalities $H_0/L \ll H_b/L \ll 1$ when $R \leq 1$ and $F \leq 1$.

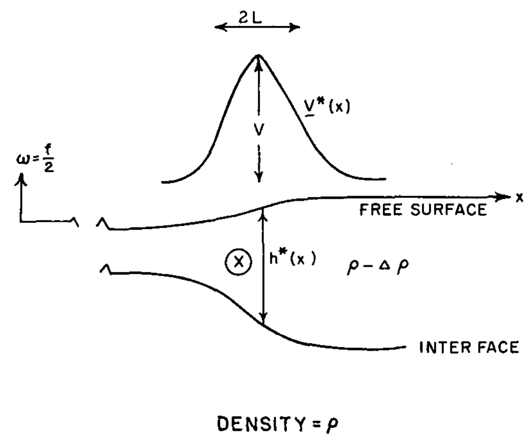

Fig. 1. A schematic diagram of a two layer thermoclinic jet. The lower layer of density ϱ is infinitely deep and resting. The geostrophically balanced velocity field $V^*(x)$ is directed along the y axis (into the page) and its maximum value is V. The section shown is supposed to occur at very large distances from the axis of the rotating system (f-plane).

therein. When these conditions are fulfilled then the dynamical model in Fig. 1 is analogous to the single fluid model shown in Fig. 2, in which gravity is reduced to g'. Although short-wave Helmholtz instabilities, associated with the interfacial shear in Fig. 1, have been filtered by the hydrostatic approximation it will be seen that either model has available potential energy as well as kinetic energy of horizontal shear. Both of these energy sources can be tapped by a superimposed perturbation. In Section 2 we shall examine certain integrals of the motion and show that the kinetic energy releasing process is the more fundamental one in the thermoclinic model. If an initially small disturbance releases potential energy from the mean flow it can only do so with a concomitant release of kinetic energy by the action of the Reynolds stress. This model then is fundamentally different from, and a logical extreme to, models having a uniformly distributed baroclinic zone (CHARNEY, 1947; EADY, 1949) and which release only potential energy in the limit of small Rossby number. KOTCHIN'S (1932) model of a Margules front separating two layers of the *same* thickness exhibits a similar potential energy conversion for long wave-length disturbances. SOLBERG (1930) has also considered the polar front instability in which both fluids are of infinite vertical extent. My understanding is that the instability mechanism here is due to vertical shear across the interface.

The classical Rayleigh-Tollmien theory (LIN, 1955) of two-dimensional inviscid shearing flow has established that a symmetric jet is unstable. HAURWITZ and PANOFSKY (1950) have suggested this as a mechanism for meanders in the Gulf Stream (see FUGLISTER and WORTHINGTON, 1951, for the observational data which supports this picture). For zonal barotropic currents on a rotating sphere KUO (1948) has generalized Lord Rayleigh's criterion to show that an extreme value of absolute vorticity is necessary for instability of the profile. The thermoclinic model allows a modification of this energy releasing process by virtue of the vertical velocities which work against the gravitational field. Nevertheless, a simple generalization of the foregoing criteria is possible which shows that "large-scale" instabilities will occur only when the profile has a maximum (or minimum) of *potential* vorticity. We point out the stabilizing influence of the finite depth of the thermocline and suggest the relevance of this quantity to the problem of the stability of the Gulf Stream.

2. The Energetics of Thermoclinic Jets

If $\mathbf{V}(x, y, t)$ denotes the total horizontal component of velocity in Figure 2 (or in the upper layer of Fig. 1) and $h_T(x, y, t)$ the thickness of the layer then the momentum and continuity equations are—

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + 2\boldsymbol{\omega} \times \mathbf{V} = -g' \nabla h_T \quad (1)$$

$$\frac{\partial h_T}{\partial t} + \nabla \cdot \mathbf{V} h_T = 0 \quad (2)$$

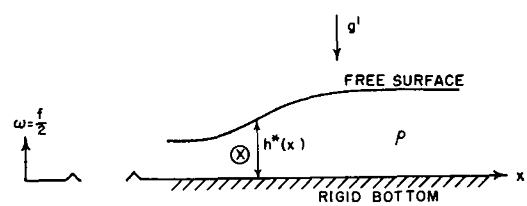

Fig. 2. A single layer system which is equivalent to Fig. 1 with respect to the dynamics of quasi-hydrostatic disturbances. $g' = g\Delta\varrho/\varrho$ is the "reduced" value of gravity.

Let us resolve the velocity into a mean component $\mathbf{V}^*(x)$ directed along the y axis and a perturbation $\mathbf{V}'(x, y, t)$ which has magnitudes (u', v') along (x, y). With a similar decomposition of $h_T = h^* + h'$ the linearized perturbation equations are—

$$\frac{\partial \mathbf{V}'}{\partial t} + 2\boldsymbol{\omega} \times \mathbf{V}' + \mathbf{V}^* \cdot \nabla \mathbf{V}' + \mathbf{V}' \cdot \nabla \mathbf{V}^* = -g'\nabla h' \tag{3}$$

$$\frac{\partial h'}{\partial t} + \nabla \cdot (\mathbf{V}^* h' + \mathbf{V}' h^*) = 0 \tag{4}$$

An interesting energy equation may now be formed by multiplying (3) by $h^*\mathbf{V}'$ and (4) $g'h'$. Denoting an average value over the entire horizontal plane by a bar we then get

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{h^*\mathbf{V}'^2} + \frac{1}{2}\overline{h^*\mathbf{V}^* \cdot \nabla \mathbf{V}'^2} + + \overline{h^*\mathbf{V}' \cdot \mathbf{V}' \cdot \nabla \mathbf{V}^*} = -g'\overline{h^*\mathbf{V}' \cdot \nabla h'} \tag{5}$$

$$\frac{\partial}{\partial t}\frac{1}{2}g'\overline{h'^2} + g'\overline{h'\nabla \cdot \mathbf{V}^* h'} + g'\overline{h'\nabla \cdot \mathbf{V}' h^*} = 0 \tag{6}$$

In proceeding further we make use of the boundary condition that the normal component of $\mathbf{V}'$ vanishes at the x-boundaries and the downstream variation is well behaved (e.g. periodic in y). Note that the second term in (5) vanishes because the undisturbed velocity field has no horizontal divergence and also $\nabla \cdot h^*\mathbf{V}^* = 0$. Likewise, the second term in (6) vanishes and these equations become—

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{h^*\mathbf{V}'^2} = -\overline{h^*\mathbf{V}' \cdot \mathbf{V}' \cdot \nabla \mathbf{V}^*} + g'\overline{h'\nabla \cdot \mathbf{V}' h^*} \tag{7}$$

$$\frac{\partial}{\partial t}\frac{1}{2}g'\overline{h'^2} = -g'\overline{h'\nabla \cdot \mathbf{V}' h^*} \tag{8}$$

Equation (7) may be interpreted as a statement of the rate of increase of the perturbation kinetic energy in the undisturbed depth h^*. The first term on the right represents a conversion of kinetic energy as the result of the Reynolds stress acting on the horizontal shear of the mean field. The second term represents work done against the gravitational field and will have a counterpart in the increase or decrease of the potential energy of the mean field. Note that in (8) there is an equal and opposite transfer to perturbation potential energy so that the sum of (7) and (8) is—

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\overline{h^*\mathbf{V}'^2} + \frac{1}{2}g'\overline{h'^2}\right\} = -\overline{h^*\mathbf{V}' \cdot \mathbf{V}' \cdot \nabla \mathbf{V}^*} = -\overline{h^* u' v' \frac{\partial V^*_{(x)}}{\partial x}} \tag{9}$$

Since both integrands on the left side of (9) are positive definite it follows that spontaneous instabilities can only occur if $-\overline{h^*\mathbf{V}' \cdot \mathbf{V}' \cdot \nabla \mathbf{V}^*}$ is positive, i.e., the mean field must do work on the perturbation through the Reynolds stress. If this term were zero or negative neither $\mathbf{V}'$ nor h' could increase with time. This energy equation implies that the simple meander theory of STOMMEL (1953) is untenable, for in his model $h^*(x)$ is linear and $\partial V^*/\partial x = 0$. In that case $^1/_2\overline{h^*V'^2} + {}^1/_2 g'\overline{h'^2}$ is conserved and a small disturbance could not grow unless the boundaries were squeezed. We also conclude that in the thermoclinic model, which we are about to consider in more detail, the shearing mechanism is fundamental to instability despite the availability of potential energy as represented by the departures of the surfaces from equi-potentials. This statement does not imply that there is no conversion of mean potential energy to the perturbation, but rather that if such occurs it can only be as a concomitant to the shearing mechanism. It should be borne in mind that this conclusion is based on the initial hypothesis that the bottom layer is very deep. Accordingly, this approximation may have the effect of eliminating certain modes of instability which are of interest.

3. The Stability of Quasi-Geostrophic Disturbances

The stability of the dynamical system (eq. (3) and (4)) depends on two "external" nondimensional parameters:

$$R \equiv V/fL \qquad F \equiv f^2L^2/g'H_0 \tag{10}$$

The stream Rossby number (R) measures the ratio of inertial to coriolis force, for perturbations which have characteristic horizontal dimensions of order L (the half width of the jet) or greater. The rotational Froude number (F) is a measure of the vertical velocity of the perturbation, since the ratio of total horizontal convergence to either component ($\partial u'/\partial x$) is of order RF. One would like to discuss the stability characteristics for all values of R and F, using suitable profiles which describe the form of the mean flow. However, the analytical difficulties are such that we must satisfy ourselves with partial solutions, each of which is valid over certain ranges of the parameters. For example, when $F \ll 1$ the barotropic non-divergent theory is applicable and the necessary condition for instability requires that the jet have an inflection point. The case which is considered below is equivalent to an asymptotic[2] expansion of (3) and (4) in the Rossby number and is valid for $R \ll 1$ and F of order unity or less (as compared with the conclusion in Sec. 2 where the Rossby number may be as large as unity).

Equations (1) and (2) imply the conservation of the vertical component of potential vorticity. If $\zeta^*(x)$ denotes the undisturbed value of relative vorticity and $\zeta'(x, y, t)$ the perturbation vorticity then we have:

$$\left(\frac{\partial}{\partial t} + V^*(x)\frac{\partial}{\partial y} + v'\frac{\partial}{\partial y} + u'\frac{\partial}{\partial x}\right)\frac{f+\zeta^*+\zeta'}{h^*+h'} = 0 \tag{11}$$

When linearized eq. (11) becomes

$$\frac{1}{h^*(x)}\left(\frac{\partial}{\partial t} + V^*(x)\frac{\partial}{\partial y}\right)\zeta' - \frac{f+\zeta^*}{h^*}\cdot$$
$$\cdot\left(\frac{\partial}{\partial t} + V^*\frac{\partial}{\partial y}\right)h' + u'\frac{\partial}{\partial x}\frac{f+\zeta^*}{h^*} = 0 \tag{12}$$

At this stage we introduce the geostrophic approximations:

$$v' = g'/f\,\frac{\partial h'}{\partial x} \qquad u' = -g'/f\,\frac{\partial h'}{\partial y}$$

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = g'/f\,\nabla^2 h'$$

and therefore "correct to order R" eq. (12) becomes

$$\left(\frac{\partial}{\partial t} + V^*\frac{\partial}{\partial y}\right)\nabla^2 h' - \frac{f}{g'}\left(\frac{f+\zeta^*}{h^*}\right)\cdot$$
$$\left(\frac{\partial}{\partial t} + V^*\frac{\partial}{\partial y}\right)h' - h^*\left(\frac{\partial}{\partial x}\frac{f+\zeta^*}{h^*}\right)\frac{\partial h'}{\partial y} = 0 \tag{13}$$

Let

$$h' = \phi(x)e^{i\alpha(y-ct)},\quad c \equiv a + ib \tag{14}$$

and denote $P(x) = (f+\zeta^*)/h^*$ as the undisturbed potential vorticity. We then obtain the eigenvalue equation

$$\phi''(x) - [\alpha^2 + fP(x)/g']\phi - \frac{h^*P'(x)\phi}{V^*(x)-c} = 0$$
$$\phi(\pm\infty) = 0 \tag{15}$$

where a prime denotes differentiation with respect to x. We now parallel Lord Rayleigh's technique (LIN, 1955) by multiplying (15) by the conjugate eigen-function ($\hat{\phi}$) and average the result over all values of x. When the imaginary part is set equal to zero one obtains

$$b\int_{-\infty}^{+\infty}\frac{h^*P'(x)\phi\hat{\phi}}{(V^*-a)^2+b^2}\,dx = 0 \tag{16}$$

It then follows that if there are unstable eigen functions (i.e. $b > 0$) then $P'(x)$ must vanish at some point which is at a finite distance from the centre of the jet. If $P'(x)$ were always of the same sign then (16) could not be satisfied. It is therefore necessary that the potential vorticity

$$P(x) = \frac{f+\zeta^*(x)}{h^*(x)} \tag{17}$$

have a maximum (or minimum) interior to the endpoints of the jet, in order that the latter be unstable. The derivation of this criterion is such that it can only be relied upon in the limit of small Rossby number ($\zeta^* \ll f$). Utilizing this approximation we may alternatively conclude that the flow is stable if

$$h^*\frac{d}{dx}\frac{f+\zeta^*}{h^*} = V^{*\prime\prime}(x) - \frac{f+\zeta^*}{g'h^*}fV^*$$
$$\cong V^*(V^{*\prime\prime}/V^* - f^2/g'h^*)$$

is never zero. Let us consider the class of velocity profiles for which $V^{*\prime\prime}/V^*$ is always finite and let max. $V^{*\prime\prime}/V^*$ denote its greatest value

[2] The y-wavelength is restricted to be of order L or greater and the time scale (period) of order L/V or greater. These assertions lead to a self consistent expansion of (3) and (4) whose zero order term is identical to that (15a) which is obtained by the simple derivation given below.

(for example: $V^*(x) \neq 0$ except at $x = \pm \infty$ where V^* goes to zero as some positive power of $1/x$). From the above equation it then follows that the mean field is stable if

$$\frac{f^2}{g' H_{+\infty}} > \max. \frac{V^{*\prime\prime}(x)}{V^*(x)}$$

$$H_{+\infty} \equiv h^*(\infty) \tag{18}$$

Therefore a given $V^*(x)$ profile (corresponding to sufficiently small Rossby number) can be stabilized with respect to quasi-geostrophic modes by a uniform reduction of the thickness of the upper layer (other things being equal) to the point where (18) is satisfied.

On the other hand, it is easy to establish *instability* at sufficiently large h^* by noting that in this limit $(F \ll 1)$ $P(x) \to 0$ while $h^* P'(x) \to d\zeta^*/dx$. In this case (15) reduces to

$$\phi''(x) - \alpha^2\phi - \frac{V^{*\prime\prime}(x)}{V^* - c}\phi = 0$$

$$\phi(\pm\infty) = 0 \tag{19}$$

which is mathematically identical to the classical equation for two-dimensional inviscid instability. Tollmien's theorem (LIN, 1955) tells us that (for symmetrical V^*) there are marginally unstable solutions in the vicinity of certain discrete wave-numbers (α). Therefore, with a given profile, there must exist a maximum critical depth $[(H_{+\infty})]_{cr.}$ above which the mean field is unstable and this number is bounded by:

$$0 < \frac{f^2}{g'(H_{+\infty})_{cr.}} < \max. \frac{V^{*\prime\prime}}{V^*(x)} \tag{20}$$

The phase speed of the neutral disturbance is equal to the mean speed of the jet at the potential vorticity extremum.

More precise specifications of these quantities require the solution of the eigenvalue equation (15), which is equivalent (correct to order R) to the following non-dimensional equation:

$$\phi''(\xi) - [\alpha_0^2 + F]\phi - \frac{\overline{W}''(\xi) - F\overline{W}(\xi)}{\overline{W} - c_0}\phi = 0$$

$$\phi(\pm\infty) = 0 \tag{15a}$$

where

$$\xi = x/L, \qquad V^*(x) = V\overline{W}(\xi)$$

$$c_0 = c/V \qquad \alpha_0 = \alpha L.$$

Although (15a) is difficult to solve in general, it is possible to obtain particular solutions for the case of the "Bickley Jet": $W = \text{sech}^2 x\, \xi$ (SAVIC, 1941). By direct substitution one finds that $\phi = \text{sech}^2\, \xi$ satisfies (15a) provided $^1/_6\, \alpha_0^2 = c_0 = {}^1/_6\,(4 - F)$. Therefore long quasi-stationary waves ($\alpha_0 \sim 0$, $c_0 \sim 0$) are neutral at $F \sim 4$. Using Tollmien's perturbation technique (LIN, loc. cit.) one can readily show that unstable solutions exist in the neighborhood of the neutral ones and that $F < 4$ is a sufficient condition for the instability of a Bickley Jet.

4. Discussion

Let us use the Bickley Jet profile to make some order of magnitude estimates of the depth of the upper layer which is necessary for instability. Consider water with a vertical temperature difference of 10° C (or air with a 1° C increase in potential temperature), $f = 10^{-4}$ sec^{-1}, and a jet half width of $L = 50$ km. Then $F = 4$ when $H_{+\infty} = 250$ meters. When this critical depth is exceeded the jet will be unstable to long waves provided the Rossby number is sufficiently small. To fulfill this, and validate the asymptotic theory, the amplitude of the jet must be less than $V \ll fL = 5$ meters/sec.

As regards the Gulf Stream, it should be pointed out that observations of the mean shear indicate that the characteristic Rossby number is not small but of order unity. Furthermore the restriction on the depth of the bottom layer in this model is such as to preclude other types of modes which are associated with the baroclinic structure of the jet. Nevertheless it is believed that the criterion for instability which is expressed by an extremum in the potential vorticity has a range of applicability which is greater than that which is indicated by the formal limits of validity of this theory. The relevance of this criterion to the stability of the Gulf Stream is supported by the observation of STOMMEL (1958 Fig. 65, based on a section at Lat. 38° N) that the layer between 17° C and 19° C is approximately one of constant potential vorticity. The disruption of the Gulf Stream north of Cape Hatteras may act as a control on the maximum depth of the thermocline in the interior of the ocean as well as along the western boundary.

Acknowledgement

The simple derivation of the instability criterion in Sec. 3 and the connection of the model with the criteria of Rayleigh and Kuo is the result of a suggestion made by Prof. Jule Charney of M.I.T. who read an earlier manuscript. This work was supported by funds from the U.S. Office of Naval Research and is W.H.O.I. contribution No. (1166).

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