The Stability Properties and Energy Transformations of the Two-layer Model of Variable Static Stability[1]

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Abstract

The design and performance of simple atmospheric models is briefly reviewed as an introduction to the question of energetic consistency. The two-layer model of Lorenz appears to be the simplest fully-consistent formulation, and the freedom of this model from the constraints upon the Coriolis parameter and static stability characteristic of the usual quasi-geostrophic models is felt to be particularly important. As a prelude to actual numerical integration, the baroclinic stability properties and the energy transformations of this two-layer model are examined. It is found that the perturbations are here significantly more unstable than when the static stability is constrained in the usual manner. For the finite-amplitude disturbances, it is found that some of their available potential energy is transferred to the mean state through the stabilizing action of the systematic rising of the warmer air. These results suggest that the variation of the static stability both promotes the growth of new disturbances and limits the growth of the mature disturbances—effects which would appear to remedy some of the conspicuous performance errors of the conventional two-level models.

1. Introduction and review of baroclinic modeling

During the past decade a number of simple (two-level or two-parameter) models of the atmosphere were designed for the purposes of numerical weather prediction. As is well known, these models are built around the familiar quasi-geostrophic approximation, and make use of a simplified vorticity equation for frictionless, hydrostatic and adiabatic flow (ELIASSEN, 1952; CHARNEY and PHILLIPS, 1953; THOMPSON, 1953; SAWYER and BUSHBY, 1953). Several of these models have been tested under a variety of conditions, and considerable experience has been accumulated with respect to their behavior with actual atmospheric initial data (BUSHBY and HINDS, 1955; THOMPSON and GATES, 1956; STAFF, JNWP UNIT, 1957). The principle characteristic errors attributable to the physical shortcomings of these models appear to be: (1) a tendency to underestimate the growth of relatively new large-scale disturbances, (2) to overestimate the growth of the relatively mature disturbances, and thereby to produce fictitiously large gradients, and (3) to display a spurious growth of anticyclonic circulation, especially in the subtropical latitudes.

The complete diagnosis of these errors is a difficult and continuing task, although certain analyses strongly suggest the probable effects of certain of the common approximations made. The isobaric vorticity equation used in these formulations may be written

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$$\frac{\partial \zeta}{\partial t} + \mathbf{V}_H \cdot \nabla(\zeta + f) - f\frac{\partial \omega}{\partial p} = 0 \qquad (1)$$

where $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V}_H$ is the vertical component of the relative vorticity ($\mathbf{k}$ being the vertical unit vector and ∇ the isobaric gradient operator), f the Coriolis parameter, $\omega = dp/dt$ the vertical motion relative to an isobaric surface, and $\mathbf{V}_H$ the horizontal wind. We note at once that the neglected terms representing the vertical advection of vorticity ($\omega \partial \zeta / \partial p$), the so-called twisting term ($\mathbf{k} \cdot \nabla \omega \times \partial \mathbf{V}_H / \partial p$), and the divergence term ($-\zeta \partial \omega / \partial p$) may account for the systematic errors noted above. The contribution of the twisting term in particular has been estimated by REED and SANDERS (1953) and by ELIASSEN (1955), who find that it evidently plays a significant role in the maintenance of the intense gradients associated with a frontal zone. For the larger-scale disturbances as viewed with the usual prediction grids, this term and the vertical vorticity advection appear to be less important than the other terms, although their effects often appear on a large scale ('ARNASON and CARSTENSEN, 1959). The replacement of the divergence term $-(\zeta + f) \partial \omega / \partial p$ by $-f \partial \omega / \partial p$ in (1) has also been noted as a probable cause of the models' tendency to overdevelop the anticyclonic circulation and to distort its characteristic asymmetry relative to the cyclonic circulations (STAFF, JNWP UNIT, 1957).

In addition to these simplifications of the vorticity equation, the horizontal velocity field in these models is usually represented by the quasi-geostrophic relations

$$\left.\begin{aligned} \mathbf{V}_H &= f^{-1} \mathbf{k} \times \nabla \phi \\ \zeta &= f^{-1} \nabla^2 \phi \end{aligned}\right\} \qquad (2)$$

where $\phi = gz$ is the geopotential. This approximation leads to a systematic overestimate of the vorticity in both cyclonic and anticyclonic circulations, especially in the lower latitudes; it is evidently responsible for much of the "spurious anticyclogenesis" observed with the geostrophic barotropic model (GATES, 1957; SHUMAN, 1957), and contributes to similar errors in the simple baroclinic models (CHARNEY and PHILLIPS, 1953; THOMPSON and GATES, 1956; SHUMAN, 1956; PHILLIPS, 1958). The approximation (2) and its consequent neglect of the vorticity advection by the divergent wind has also been cited as responsible for errors of type (2) noted above (THOMPSON, 1956; PHILLIPS, 1958).

The adiabatic equation accompanying (1) and (2) in the baroclinic models may be written in isobaric coordinates as

$$\frac{\partial T}{\partial t} + \mathbf{V}_H \cdot \nabla T + \omega \Gamma = 0, \qquad (3)$$

where $\Gamma = \partial T / \partial p - \alpha / c_p$ is a measure of the static stability, α is the specific volume, and c_p is the isobaric specific heat. Here the temperature T is in turn related to the geopotential through the hydrostatic equation

$$\frac{\partial \phi}{\partial p} + \alpha = 0. \qquad (4)$$

In the conventional two-level model the stability Γ is assigned a constant value in both space and time. In view of the generally more unstable stratification in the vicinity of developing cyclones as compared to anticyclones, this approximation may systematically overestimate the effects of vertical motion on the temperature distribution, and hence underestimate the cyclonic development. As in the vorticity equation, use of the quasi-geostrophic approximation in (3) entails the neglect of the advection of temperature by the divergent portion of the horizontal wind, and this may lead to further systematic error.

2. Consistency and baroclinic modeling

The efforts to reduce the characteristic errors of the conventional two-level models have taken a number of directions, all more or less focused on the removal of constraints inherent in the system (1)—(3). As a first step the geostrophic approximation (2) may be replaced by the more general balance equation,

$$\nabla^2 \phi - \nabla \cdot (f \nabla \psi) - 2\left[\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right] = 0. \qquad (5)$$

In the barotropic model this step results in the virtual elimination of the anticyclogenesis errors of type (3) noted in § 1 (SHUMAN, 1957). A similar although less spectacular improvement is also made by use of the balance equation (or at least a non-divergent wind) in the simple baroclinic models (BUSHBY and HUCKLE, 1956; PHILLIPS, 1958).

Further improvement in the models' performance might also be expected by the inclusion of the neglected vorticity and heat transfer mechanisms. In particular, the inclusion of the divergent-wind vorticity advection and the vertical vorticity advection in (1) has been proposed by THOMPSON (1956), and the inclusion of the twisting term has been proposed on a number of occasions. Similarly, one might consider temperature advection by the divergent wind in (2), and allow the static stability to vary.

Viewed as independent "corrections" to the system (1)—(3), these modifications lead to a large number of possible models. Due both to the relative difficulty of incorporating these effects and to the inconclusive results of the few such tests that have been made (CHARNEY and PHILLIPS, 1953; CHARNEY, GILCHRIST, and SHUMAN, 1956), an extensive program of term-by-term generalization of the simple system (1)—(3) has apparently not been undertaken. This is perhaps fortunate in view of the consistency requirements to be placed upon the models, as first discussed by HOLLMANN (1956) and in a more comprehensive fashion by LORENZ (1957).

These consistency requirements may be most easily seen if we consider the complete vorticity equation for frictionless flow,

$$\frac{\partial\zeta}{\partial t} + \mathbf{V}_H \cdot \nabla(\zeta + f) + \omega\frac{\partial\zeta}{\partial p} - (f+\zeta)\frac{\partial\omega}{\partial p} + \\ + \mathbf{k}\cdot\nabla\omega\times\frac{\partial\mathbf{V}_H}{\partial p} = 0. \qquad (6)$$

By rearrangement and use of standard identities, this equation may be written in the form

$$\frac{\partial\zeta}{\partial t} + \left[J(\psi, \zeta+f) + \nabla\cdot((\zeta+f)\nabla\chi) + \right. \\ \left. + (\zeta+f)\frac{\partial\omega}{\partial p}\right] + \left[\frac{\partial}{\partial p}(\omega\zeta) - \zeta\frac{\partial\omega}{\partial p}\right] - \\ - \left[(\zeta+f)\frac{\partial\omega}{\partial p}\right] + \left[\frac{\partial}{\partial p}(\mathbf{k}\cdot\nabla\omega\times(\mathbf{k}\times\nabla\psi)) + \right. \\ \left. + \nabla\cdot\mathbf{k}\times\left(\frac{\partial\omega}{\partial p}\mathbf{k}\times\nabla\psi\right) + \zeta\frac{\partial\omega}{\partial p}\right] = 0. \qquad (7)$$

Here use has been made of the isobaric continuity equation

$$\nabla^2\chi + \frac{\partial\omega}{\partial p} = 0, \qquad (8)$$

where χ is the potential function of the divergent portion of the horizontal wind, and ψ is the stream function of the rotational portion of the horizontal wind. The four groups of bracketed terms in (7) represent the corresponding four terms in (6), with the last two members of the first bracketed group representing the vorticity advection by the divergent wind.

If (7) is now integrated over the atmosphere's total mass, assuming $\omega = 0$ at both the upper and lower limits of pressure, we find

$$\frac{\partial}{\partial t}\iint \zeta\, dp\, ds = 0, \qquad (9)$$

where ds is a fractional element of area on an isobaric surface. Thus the total vorticity of the atmosphere is conserved in the absence of friction (and with $\omega = 0$ at $p = p_0$), and (9) is consequently an integral requirement to be placed upon simplified versions of the vorticity equation if spurious sources and sinks of vorticity are to be avoided in the mean. It is evident from (7) that this consistency condition can be met in several ways. For example, if either vorticity advection by the divergent wind or the divergence term is neglected, then the other should be neglected also, or if only the former is neglected, the latter should be replaced by $-f\partial\omega/\partial p$ with f taken as a constant. (Another possible selection is considered in § 3.) Further, the vertical vorticity advection and the twisting term must also either both be included or both neglected. Viewed in this manner the simplified vorticity equation (1) is consistent in the sense of (9) only when $\mathbf{V}_H$ is considered non-divergent and when $f = f_0$ in $-f\partial\omega/\partial p$. Use of the geostrophic approximation (2) in this connection is therefore a consistent procedure. In some numerical integrations f has not been considered constant in the above manner (GATES, POCINKI, and JENKINS, 1956). This question of consistency in simple baroclinic models has also recently been considered by 'ARNASON and CARSTENSEN (1959) and by WIIN-NIELSEN (1959).

We may apply similar considerations to the adiabatic equation (3), and for this purpose write it in the form

$$\frac{\partial T}{\partial t} + \left[J(\psi, T) + \nabla\cdot(T\nabla\chi) + \frac{\partial}{\partial p}(\omega T) - \right. \\ \left. - \omega\frac{\partial T}{\partial p}\right] + \left[\omega\frac{\partial T}{\partial p} - \frac{\alpha\omega}{c_p}\right] = 0. \qquad (10)$$

Here the two groups of bracketed terms represent the corresponding two terms of (3), with the last three members of the first bracketed group representing the temperature advection by the divergent wind. Integration of (10) over the atmosphere's mass as before yields

$$\frac{\partial}{\partial t}\iint c_p\, T\, dp\, ds = \iint \omega\alpha\, dp\, ds. \tag{11}$$

The left-hand side here represents the variation of the total potential and internal energy of the atmosphere in hydrostatic equilibrium, and since total energy is conserved in adiabatic frictionless flow, we may regard the right-hand side of (11) as the transformation of total potential and internal energy into kinetic energy, accomplished by the familiar systematic rising of warm air (WHITE and SALTZMAN, 1956).

We note that both temperature advection by the divergent wind and a variable static stability are present in (10), and hence are implicit in the integral relation (11). If the former is neglected, say through the geostrophic approximation, and the static stability $\Gamma = \partial T/\partial p - \alpha c_p$ is assigned a constant value in the manner of the conventional two-level model, then we obtain instead of (11) the result

$$\frac{\partial}{\partial t}\iint c_p\, T\, dp\, ds = 0, \tag{12}$$

which implies that the mean temperature of an isobaric surface cannot change. In this case, therefore, no net conversions between kinetic and potential energy (as here defined) are possible. The quasi-geostrophic models with constant static stability are thus artificially constrained in the sense that during the development stage ($\omega\alpha < 0$) the local conversion of potential energy into kinetic energy may be systematically underestimated, while during the occlusion and decay stage ($\omega\alpha > 0$) the potential energy conversion may be overestimated. In broad terms, one might interpret this constraint as contributing to the characteristic errors of types (1) and (2) discussed in § 1.

It is, however, still possible to preserve the consistency condition (11) with a different set of simplifications. If one neglects the temperature advection with the divergent wind, say by the use of (2), and in addition assigns the lapse rate $\partial T/\partial p$ a constant value rather than the static stability Γ,[1] then (11) results directly. Such a formulation might therefore serve as a consistent baroclinic model, although it still neglects possibly important processes.

3. The Lorenz two-layer model

A simple yet consistent model with a minimum of constraints has been designed by LORENZ (1957, 1960). This model is free of many of the limitations of the usual quasi-geostrophic formulations and may provide a significant step short of the primitive equations themselves. After a review of the two-layer Lorenz formulation, we shall consider its baroclinic stability properties and energy transformations as a prelude to actual numerical integrations.

From (7) we may note that a consistent simplification is to write

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2\psi + f) + \nabla\cdot(f\nabla\chi) = 0\,, \tag{13}$$

where $\nabla^2\psi = \zeta$ and where the variation of the Coriolis parameter has not been restricted. LORENZ (1957, 1960) has shown that consistent with this form of the vorticity equation are the adiabatic and "balance" equations in the forms

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) + \nabla\cdot(\Theta\nabla\chi) + \frac{\partial}{\partial p}(\omega\,\Theta) = \\ = 0, \tag{14}$$

$$\nabla^2\phi - \nabla\cdot(f\nabla\psi) = 0, \tag{15}$$

where Θ is the potential temperature. Here (14) is equivalent to (10) without approximation, and (15) represents the "linear" portion of the balance equation (5), again without restriction of the Coriolis parameter; to this extent the system (13)—(15) may be termed a "non-geostrophic" formulation. By applying (13) and (14), together with the continuity equation (8), to a two-layer atmosphere as illustrated in fig. 1, we may obtain the system

$$\frac{\partial \nabla^2\psi}{\partial t} + J(\psi, \nabla^2\psi + f) + J(\tau, \nabla^2\tau) = 0 \tag{16}$$

$$\frac{\partial \nabla^2\tau}{\partial t} + J(\tau, \nabla^2\psi + f) + J(\psi, \nabla^2\tau) - \\ - \nabla\cdot(f\nabla\chi) = 0 \tag{17}$$

[1] In the mean winter atmosphere over North America, the measure $\partial T/\partial p$ is nearly as constant in the troposphere as is the measure Γ (GATES, 1961).

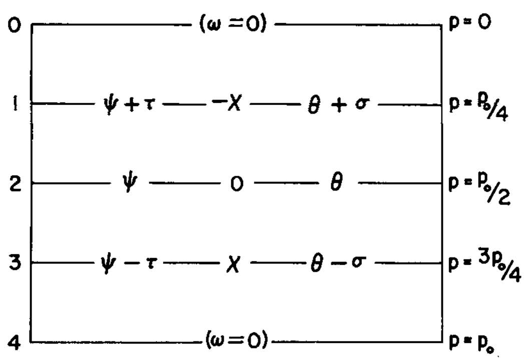

Figure 1. The information levels and primary dependent variables of the Lorenz two-layer atmospheric model of variable static stability.

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) + J(\tau, \sigma) - \nabla \cdot (\sigma \nabla \chi) = 0 \quad (18)$$

$$\frac{\partial \sigma}{\partial t} + J(\psi, \sigma) + J(\tau, \Theta) - \nabla \Theta \cdot \nabla \chi = 0 \quad (19)$$

$$2^{-\varkappa-1} R \nabla^2 \Theta - \nabla \cdot (f \nabla \tau) = 0. \quad (20)$$

Here $\psi = (\psi_1 + \psi_3)/2$, $\Theta = (\Theta_1 + \Theta_3)/2$, $\tau = (\psi_1 - \psi_3)/2$, $\sigma = (\Theta_1 - \Theta_3)/2$, and $\chi = (\chi_3 - \chi_1)/2$ where the subscripts refer to the levels shown in fig. 1; R is the specific gas constant for dry air, and $\varkappa = R/c_p$. To obtain (16)—(19) we have applied (13), (14), and (8) to both levels 1 and 3, and evaluated vertical finite-differences over $p_0/2$ (p_0 = 1,000 mb assumed constant), with the boundary conditions $\omega = 0$ at both $p = 0$ and $p = p_0$. We have also used the approximation $\Theta_2 = (\Theta_1 + \Theta_3)/2$. Equation (20) is obtained by differentiating (15) with respect to pressure, using the hydrostatic relation (4), and evaluating $\partial\psi/\partial p$ by finite differences over $p_0/2$ about the level 2. This is the system given by LORENZ (1957, 1960), except for a slight difference in the coefficient of $\nabla^2\Theta$ in (20). The continuity equation accompanying (16)—(20) is

$$\frac{2\omega}{P_0} - \nabla^2 \chi = 0, \quad (21)$$

where ω is the vertical motion at level 2.

Equations (16) and (17) represent the "mean" and "thermal" vorticity equations, respectively, while (18) and (19) provide for the variation of the mean temperature and static stability. Equation (20) represents a form of the thermal wind equation. Upon integration over the entire atmosphere (or other suitable closed region), we obtain the results $\frac{\partial}{\partial t}\int \nabla^2 \psi \, ds = 0$,

$$\frac{\partial}{\partial t}\int \nabla^2 \tau \, ds = 0, \frac{\partial}{\partial t}\int \Theta \, ds = 0, \frac{\partial}{\partial t}\int \sigma ds = - \int \Theta \nabla^2 \chi \, ds, \text{ and } \frac{\partial}{\partial t}\int \nabla^2 \Theta \, ds = 0;$$

this system is therefore consistent in the sense of (9) and (11), and as suggested by Lorenz, probably constitutes the simplest such formulation. It is interesting to note that it is the variation of the static stability σ in (19) which actually provides for the transformation of potential into kinetic energy in the mean.

4. Stability properties of the linearized system

In order to examine the stability of small amplitude perturbations in the baroclinic atmosphere described by this model, we may partition the dependent variables into a basic state (denoted by the subscript zero) and a superposed perturbation (denoted by a prime) in the usual manner. For convenience, we shall consider the basic or undisturbed state to be a uniform non-divergent zonal flow; hence $\chi_0 = 0$. Moreover, we shall regard ψ_0, τ_0, Θ_0 and σ_0 as linear functions of y (distance northward) alone, and regard the perturbations as functions of x (distance eastward) and t (time) only. Although these assumptions somewhat oversimplify the problem, they probably do not conceal the basic stability properties of the present model as compared with those of the usual two-level model determined in the same manner.

Assuming perturbations of the form exp $[i k (x - ct)]$, where $k = 2\pi/L$ is the wave number, L the wavelength, and c the wave speed, we find that the existence of non-trivial perturbations in the linearized system requires

$$\begin{vmatrix} k^2(c-U)+\beta & -k^2U^\star & 0 & 0 & 0 \\ -k^2U^\star & k^2(c-U)+\beta & 0 & 0 & -ikf \\ \partial\Theta_0/\partial y & \partial\sigma_0/\partial y & -(c-U) & U^\star & -ik\sigma_0 \\ \partial\sigma_0/\partial y & \partial\Theta_0/\partial y & U^\star & -(c-U) & 0 \\ 0 & -f & R2^{-\varkappa-1} & 0 & 0 \end{vmatrix} = 0. \quad (22)$$

Here $U = -\partial\psi_0/\partial y$ is the mean zonal wind, $U^\star = -\partial\tau_0/\partial y$ is the vertical wind shear of the basic state, and $\beta = \partial f/\partial y$. Upon requiring the zonal flow to satisfy the thermal wind equation (20), and $\partial^2 U/\partial p^2 = 0$, we have the relations

$$\frac{\partial\Theta_0}{\partial y} = -2^{\varkappa+1} f U^\star R^{-1}, \qquad (23)$$

and

$$\frac{\partial\sigma_0}{\partial y} = 2^{\varkappa}(1-\varkappa) f U^\star R^{-1}. \qquad (24)$$

Expansion of (22) with the aid of these relations leads after some manipulation to the cubic frequency equation

$$(c-U)^3 + m_1(c-U)^2 + m_2(c-U) + m^3 = 0, \qquad (25)$$

where

$$m_1 = \frac{\beta}{k^2}\left(1 + \frac{r_1 - r_2}{r_1 + 1}\right), \qquad (26)$$

$$m_2 = \left(\frac{\beta}{k^2}\right)^2 \left(\frac{r_1 - r_2}{r_1 + 1}\right) - (U^\star)^2 \left(\frac{r_1 - 1}{r_1 + 1}\right), \qquad (27)$$

$$m_3 = \frac{\beta}{k^2}(U^\star)^2 \left(\frac{-r_2}{r_1 + 1}\right). \qquad (28)$$

Here we have introduced the dimensionless ratios

$$r_1 = \frac{Rk^2\sigma_0}{2^{\varkappa+1} f^2}, \qquad (29)$$

$$r_2 = \frac{(1-\varkappa)k^2 U^\star}{2\beta}, \qquad (30)$$

which depend critically upon the stability and shear of the basic state. The frequency equation (25) reduces to that of the usual two-level model when the underlined terms in (22)—those dependent upon a variable static stability—are omitted. The familiar frequency equations for the barotropic and advective models may also be obtained by introduction of the appropriate simplifications.

The frequency equation (25) is seen to depend upon the wave number of the disturbance and the parameters of the basic state in an involved manner; for the present purposes, however, only the critical conditions separating unstable and stable (or neutral) perturbations need to be determined, and we may proceed directly to this result by noting that the unstable waves correspond to the existence of complex conjugate roots of (25). This occurs only when the discriminant

$$D \equiv 27 m_3^2 - m_1^2 m_2^2 + 4 m_1^3 m_3 - 18 m_1 m_2 m_3 + 4 m_2^3 > 0, \qquad (31)$$

and thus the desired stability criterion is given by $D = 0$. This condition, however, is still an awkward functional relation, and has therefore been determined numerically as follows. For each 500 km wavelength to 8,000 km and for each 2 m sec^{-1} shear to 20 m sec^{-1}, the value of D was computed and the curve $D = 0$ positioned graphically in the stability diagram. Figures 2 and 3 show the results of these calculations for selected σ_0, and extend the preliminary results given earlier (GATES, 1961). These figures also show the growth rate of the unstable perturbations in the form of the time required for a doubling of the amplitude, calculated as above for the unstable waves from

$$t_D = (2\pi c_i)^{-1} L \ln 2$$

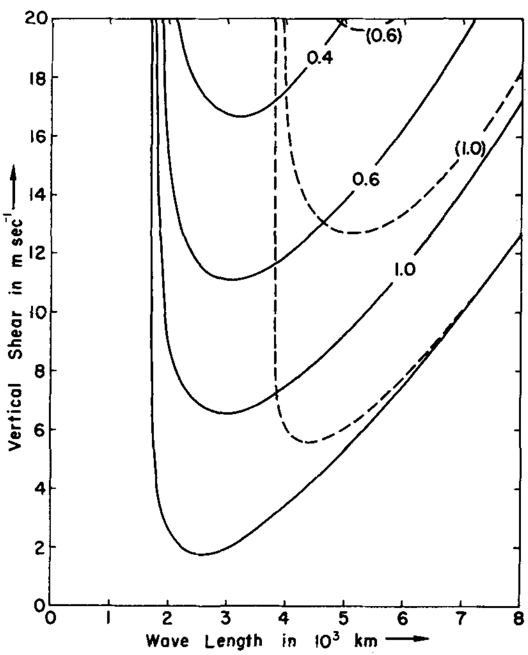

Figure 2. The baroclinic stability properties of the two-layer model of variable static stability. The solid curves give the doubling time (in-days) of unstable disturbances, with the lower curve representing the critical stability conditions. Here the vertical shear is measured over $p_0/4$, the basic state's static stability $\sigma_0 = 15$ deg (measured over $p_0/4$) at 45 N, and the basic zonal current $U = 20$ m sec^{-1}.
The dashed curves are for the conventional two-level model of constant static stability.

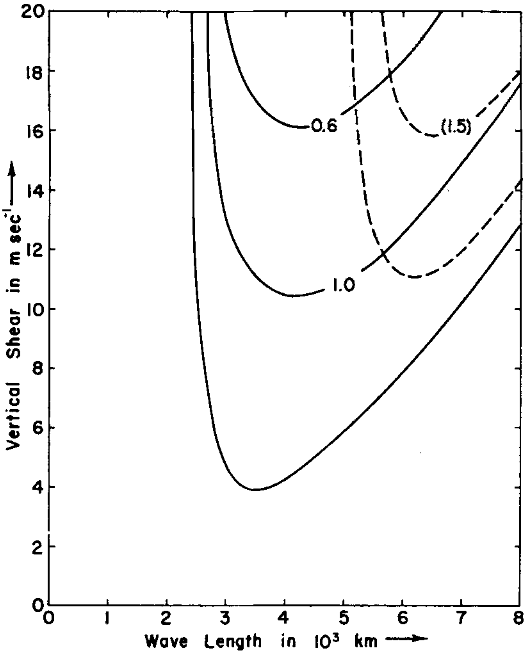

Figure 3. Same as fig. 2, but with $\sigma_0 = 30$ deg.

where L is the wavelength and c_i is the positive imaginary part of the phase speed.[2]

The case $\sigma_0 = 15$ deg (fig. 2) is representative of the stability in winter for mid-latitudes, and shows the presence of a minimum unstable shear and a shortest unstable wave, both of which, however, are considerably less than those given by the usual two-level model (PHILLIPS, 1954). Disturbances as short as 2,000 km wavelength may be unstable in the present model with $\sigma_0 = 15$ deg (compared with 3,600 km in the conventional model), and it is just for these shorter synoptic waves that the effects of the self-determination of static stability could be expected to be the most important. For the longer waves, the critical stability conditions approach those of the usual two-level model. The growth rates of the unstable waves are here greater for all values of the shear than are those of the conventional model; evidently the variation of the static stability has a destabilizing effect upon the growing baroclinic disturbance, and is most pronounced for the shorter waves. For the wave lengths greater than 3,000 km the critical curve of fig. 2 is very close to that of the advective model (SUTCLIFFE, 1951).

For the larger static stability $\sigma_0 = 30$ deg, representative of winter-time conditions at higher latitudes, fig. 3 shows an even greater departure from the stability criteria of the usual models. Here waves as short as 2,500 km may be unstable (compared with 5,100 km in the conventional case), and the minimum unstable shear is somewhat less than that of the usual two-level model. Comparison with fig. 2 shows that the unstable waves' growth rates are here everywhere less than those of the case $\sigma_0 = 15$ deg, as would be expected. Finally, we may note that the growth rates of the longer waves are relatively insensitive to the static stability of the basic state, and the critical stability curves in the cases $\sigma = 15$ and 30 deg are practically coincident for wave lengths greater than 6,000 km.

In comparing the present model's stability characteristics with those of the usual 2-level model, one should also examine the reduction of (22) with $\partial\sigma_0/\partial y = 0$. In this case the left-hand portion of the critical curve of figs. 2 and 3 is shifted toward longer wavelengths by about 1000 km, with the right-hand portion remaining essentially intact. These new critical stability curves are the same as those for the usual 2-level model with *halved* static stability.

These results suggest that the variation of static stability in the present model plays an important role in at least the initial growth of baroclinic disturbances, and the increased growth rates found for the shorter waves may help to reduce some of the systematic error of the conventional models noted in § 1.

5. Energy transformation functions

Having shown that the stability characteristics of relatively young disturbances in the present model are significantly different than those of the conventional two-level model, it now becomes of interest to examine the role of variable static stability in the energy trans-

[2] From the algebraic solution of the cubic equation (25) we find

$$c_i = 2^{-1}\sqrt{3}\,(a-b), \text{ where}$$

$$a, b = \frac{1}{3\sqrt[3]{2}}\{(9m_1 m_2 - 2m_1^3 - 27m_3) \pm [(9m_1 m_2 - 2m_1^3 - 27m_3)^2 + 4\,(3m_2 - m_1^2)^3]^{1/2}\}^{1/3}$$

formations of the model. To this end let us first consider the kinetic energy per unit area,

$$K = \frac{p_0}{2g} (\nabla \psi \cdot \nabla \psi + \nabla \tau \cdot \nabla \tau), \quad (32)$$

and seek from (16) and (17) the behavior of the spatially-integrated kinetic energy for both the zonal average motion and for the disturbances.

Writing $() = \overline{()} + ()'$, where $\overline{()}$ denotes the zonal average in the usual manner, we may average (16) and (17) and form equations in $\partial\bar{\psi}/\partial t$, $\partial\bar{\tau}/\partial t$, $\partial\psi'/\partial t$, and $\partial\tau'/\partial t$.

Multiplying these equations by $\bar{\psi}$, $\bar{\tau}$, ψ', and τ', respectively, and integrating over the earth's surface we find the results

$$\frac{\partial}{\partial t}\int \bar{K} ds = -\frac{p_0}{g}\int \bar{\psi}_y \frac{\partial}{\partial y}(\overline{\psi_x' \psi_y'} + \overline{\tau_x' \tau_y'})\, ds - \frac{p_0}{g}\int \bar{\tau}_y \frac{\partial}{\partial y}(\overline{\tau_x' \psi_y'} + \overline{\psi_x' \tau_y'})\, ds - \frac{R}{2^\varkappa g}\int \bar{\Theta}\,\bar{\omega}\, ds, \quad (33)$$

and

$$\frac{\partial}{\partial t}\int K' ds = \frac{p_0}{g}\int \bar{\psi}_y \frac{\partial}{\partial y}(\overline{\psi_x' \psi_y'} + \overline{\tau_x' \tau_y'})\, ds + \frac{p_0}{g}\int \bar{\tau}_y \frac{\partial}{\partial y}(\overline{\tau_x' \psi_y'} + \overline{\psi_x' \tau_y'})\, ds - \frac{R}{2^\varkappa g}\int \overline{\Theta' \omega'}\, ds, \quad (34)$$

where $\int \bar{K} ds = \frac{p_0}{g}\frac{1}{2}\int (\nabla\bar{\psi} . \nabla\bar{\psi} + \nabla\bar{\tau} . \nabla\bar{\tau}) ds$ is the kinetic energy of the mean zonal (non-divergent) motion, and $\int K' ds = \frac{p_0}{g}\frac{1}{2}\int (\nabla\psi' \cdot \nabla\psi' + \nabla\tau' \cdot \nabla\tau')\, ds$ is the kinetic energy of the disturbances. Here use has been made of the zonal and disturbance forms of (8), (20), and (21). Equations (33) and (34) are the same as those found for the usual two-level quasi-geostrophic model by PHILLIPS (1956), and are not directly dependent upon the static stability or its possible variations.

Let us now seek the corresponding equations for the available potential energy. As discussed by LORENZ (1957), the total available potential energy per unit area in the present model is given by

$$\int A ds = \frac{R p_0}{2^{\varkappa+1} g}\left[\frac{\int \Theta^2 ds - (\int \Theta ds)^2 + \int \sigma^2 ds - (\int \sigma ds)^2}{\int \sigma ds + (\int \sigma ds)_m}\right], \quad (35)$$

where $(\int \sigma ds)_m$ is the maximum (stable) stability which could be obtained by adiabatic re-stratification, and may be considered a constant. Here $\int A ds$ is essentially a measure of the total variance of the potential temperature in each of the two layers of the model. By now writing $\Theta = \bar{\Theta} + \Theta'$ and $\sigma = \bar{\sigma} + \sigma'$ in the usual manner, we have

$$\int \bar{A} ds = \frac{R p_0}{2^{\varkappa+1} g}\left[\frac{\int \bar{\Theta} ds - (\int \bar{\Theta} ds)^2 + \int \bar{\sigma}^2 ds - (\int \bar{\sigma} ds)^2}{\int \sigma ds + (\int \sigma ds)_m}\right] \quad (36)$$

and

$$\int A' ds = \frac{R p_0}{2^{\varkappa+1} g}\left[\frac{\int (\Theta')^2 ds + \int (\sigma')^2 ds}{\int \sigma ds + (\int \sigma ds)_m}\right] \quad (37)$$

for the total available potential energy of the zonal flow and of the disturbances, respectively.

Considering first the zonal case, we may differentiate (36) and obtain

$$\frac{\partial}{\partial t}\int \bar{A} ds = \frac{R p_0}{2^{\varkappa+1} g}\left[\frac{\partial}{\partial t}\int \bar{\Theta}^2 ds + \frac{\partial}{\partial t}\int \bar{\sigma}^2 ds - \frac{\partial}{\partial t}(\int \bar{\sigma} ds)^2\right]\left[\int \sigma ds + (\int \sigma ds)_m\right]^{-1}$$
$$- \frac{R p_0}{2^{\varkappa+1} g}\left[\int \bar{\Theta}^2 ds - (\int \bar{\Theta} ds)^2 + \int \bar{\sigma}^2 ds - (\int \bar{\sigma} ds)^2\right]\left[\int \sigma ds + (\int \sigma ds)_m\right]^{-2} \frac{\partial}{\partial t}\int \bar{\sigma} ds, \quad (38)$$

noting that $\frac{\partial}{\partial t}\left(\int \Theta ds\right)^2 = 0$ for adiabatic motion with $\omega = 0$ at both $p = 0$ and $p = p_0$. Evaluating each time derivative in (38) from (18) and (19), we readily find

$$\frac{\partial}{\partial t}\int \bar{\Theta}^2 ds = 2\int \bar{\Theta}_y(\overline{\Theta' \psi_x'} + \overline{\sigma' \tau_x'} - \overline{\sigma' \chi_y'} - \bar{\sigma}\,\bar{\chi}_y)\, ds,$$

$$\frac{\partial}{\partial t}\int \bar{\sigma}^2 ds = 2\int \bar{\sigma}_y(\overline{\sigma' \psi_x'} + \overline{\Theta' \tau_x'} - \overline{\Theta' \chi_y'} - \bar{\sigma}\,\bar{\chi}_y)\, ds - 2\int \bar{\sigma}\,(\overline{\Theta' \nabla^2 \chi'})\, ds.$$

Noting further that

$$\frac{\partial}{\partial t}\int \bar{\sigma} ds = \int \left(\nabla \bar{\Theta} \cdot \nabla \bar{\chi} + \overline{\nabla \Theta' \cdot \nabla \chi'}\right) ds,$$

writing (21) for both the zonal flow and disturbances, and using the relation

$$\left(\int \sigma ds\right)_m^2 = -\left(\int \Theta ds\right)^2 + \int (\Theta)^2 ds + \int (\sigma)^2 ds, \quad (39)$$

we find after some manipulation the result

$$\frac{\partial}{\partial t}\int \bar{A} ds = \frac{R p_0}{2^\varkappa g\left[\int \sigma ds + \left(\int \sigma ds\right)_m\right]}\int \left[\bar{\Theta}_y\left(\overline{\Theta' \psi'_x} + \overline{\sigma' \tau'_x} - \overline{\sigma' \chi'_y}\right) + \bar{\sigma}_y\left(\overline{\sigma' \psi'_x} + \overline{\Theta' \tau'_x} - \overline{\Theta' \chi'_y}\right)\right] ds +$$

$$+ \frac{R}{2^\varkappa g}\int \bar{\Theta}\bar{\omega} ds + \frac{R}{2^\varkappa g}\left[\frac{\int \sigma ds - \left(\int \sigma ds\right)_m}{\int \sigma ds + \left(\int \sigma ds\right)_m}\right]\int \bar{\Theta}\bar{\omega} ds + \frac{2R}{2^\varkappa g}\int \left[\frac{\int \sigma ds - \bar{\sigma}}{\int \sigma ds + \left(\int \sigma ds\right)_m}\right]\overline{\Theta' \omega'} ds. \quad (40)$$

We may note that if we were to regard σ as an absolute constant $\sigma \equiv \sigma_0$, then the last two terms of (40) would drop out, and we would have simply

$$\frac{\partial}{\partial t}\int \bar{A} ds = $$

$$= \frac{R p_0}{2^{\varkappa+1} g \sigma_0}\int \bar{\Theta}_y\left(\overline{\Theta' \psi'_x}\right) ds + \frac{R}{2^\varkappa g}\int \bar{\Theta}\bar{\omega} ds. \quad (41)$$

This reduction corresponds to the energy transformation obtained with the familiar two-level models (PHILLIPS, 1954 and 1956), when A is suitably defined.

Turning now to the eddy available potential energy, we differentiate (37), giving

$$\frac{\partial}{\partial t}\int A' ds = \frac{R p_0\left[\frac{\partial}{\partial t}\int (\Theta')^2 ds + \frac{\partial}{\partial t}\int (\sigma')^2 ds\right]}{2^{\varkappa+1} g\left[\int \sigma ds + \left(\int \sigma ds\right)_m\right]}$$

$$- \frac{R p_0\left[\int (\Theta')^2 ds + \int (\sigma')^2 ds\right]}{2^{\varkappa+1} g\left[\int \sigma ds + \left(\int \sigma ds\right)_m\right]^2}\frac{\partial}{\partial t}\int \sigma ds. \quad (42)$$

From (18) and (19) we then find

$$\frac{\partial}{\partial t}\int (\Theta')^2 ds + \frac{\partial}{\partial t}\int (\sigma')^2 ds =$$

$$- 2\int \bar{\Theta}_y\left(\overline{\Theta' \psi'_x} + \overline{\sigma' \tau'_x} - \overline{\sigma' \chi'_y}\right) ds$$

$$- 2\int \bar{\sigma}_y\left(\overline{\sigma' \psi'_x} + \overline{\Theta' \tau'_x} - \overline{\Theta' \chi'_y}\right) ds + \frac{4}{p_0}\int \bar{\sigma}\,\overline{\Theta' \omega'} ds,$$

with which we may then rewrite (42) after some rearrangement as

$$\frac{\partial}{\partial t}\int A' ds = \frac{-R p_0}{2^\varkappa g\left[\int \sigma ds + \left(\int \sigma ds\right)_m\right]}\int \left[\bar{\Theta}_y\left(\overline{\Theta' \psi'_x} + \overline{\sigma' \tau'_x} - \overline{\sigma' \chi'_y}\right) + \bar{\sigma}_y\left(\overline{\sigma' \psi'_x} + \overline{\Theta' \tau'_x} - \overline{\Theta' \chi'_y}\right)\right] ds +$$

$$+ \frac{R}{2^\varkappa g}\int \overline{\Theta' \omega'} ds - \frac{R}{2^\varkappa g}\left[\frac{\int \sigma ds - \left(\int \sigma ds\right)_m}{\int \sigma ds + \left(\int \sigma ds\right)_m}\right]\int \bar{\Theta}\,\bar{\omega} ds + \frac{2R}{2^\varkappa g}\int \left[\frac{\int \sigma ds - \bar{\sigma}}{\int \sigma ds + \left(\int \sigma ds\right)_m}\right]\overline{\Theta' \omega'} ds. \quad (43)$$

It is now convenient to collect the above results in the compact form

$$\frac{\partial}{\partial t}\int \bar{K} ds = \left\{K' \cdot \bar{K}\right\} + \left\{\bar{A} \cdot \bar{K}\right\}, \quad (44)$$

$$\frac{\partial}{\partial t}\int K' ds = -\left\{K' \cdot \bar{K}\right\} + \left\{A' \cdot K'\right\}, \quad (45)$$

$$\frac{\partial}{\partial t}\int \bar{A} ds = -\left\{\bar{A} \cdot \bar{K}\right\} - \left\{\bar{A} \cdot A'\right\} + \left\{A' \cdot \bar{A}\right\}_1 + $$

$$+ \left\{A' \cdot \bar{A}\right\}_2, \quad (46)$$

$$\frac{\partial}{\partial t}\int A' ds = \left\{\bar{A} \cdot A'\right\} - \left\{A' \cdot K'\right\} - \left\{A' \cdot \bar{A}\right\}_1 -$$

$$- \left\{A' \cdot \bar{A}\right\}_2, \quad (47)$$

where the energy transformation functions are given by

$$\{K'\cdot\bar{K}\} = \frac{-p_0}{g}\int\Big[\bar{\psi}_y\frac{\partial}{\partial y}\left(\overline{\psi'_x\psi'_y}+\overline{\tau'_x\tau'_y}\right)+ \bar{\tau}_y\frac{\partial}{\partial y}\left(\overline{\tau'_x\psi'_y}+\overline{\psi'_x\tau'_y}\right)\Big]\,ds, \tag{48}$$

$$\{\bar{A}\cdot\bar{K}\} = -\frac{R}{2^{\varkappa}g}\int\bar{\Theta}\,\bar{\omega}\,ds, \tag{49}$$

$$\{A'\cdot K'\} = -\frac{R}{2^{\varkappa}g}\int\overline{\Theta'\,\omega'}\,ds, \tag{50}$$

$$\{\bar{A}\cdot A'\} = \frac{-Rp_0}{2^{\varkappa}g\left[\int\sigma\,ds+(\int\sigma ds)_m\right]}\int\Big[\bar{\Theta}_y\left(\overline{\Theta'\psi'_x}+\overline{\sigma'\tau'_x}-\overline{\sigma'\chi_y}\right)+\bar{\sigma}_y\left(\overline{\sigma'\psi'_x}+\overline{\Theta'\tau_x}-\overline{\Theta'\chi_y}\right)\Big]\,ds, \tag{51}$$

$$\{A'\cdot\bar{A}\}_1 = \frac{R}{2^{\varkappa}g}\left[\frac{\int\sigma ds-(\int\sigma ds)_m}{\int\sigma ds+(\int\sigma ds)_m}\right]\int\bar{\Theta}\,\bar{\omega}\,ds, \tag{52}$$

$$\{A'\cdot\bar{A}\}_2 = \frac{2R}{2^{\varkappa}g}\int\left[\frac{\int\sigma ds-\bar{\sigma}}{\int\sigma ds+(\int\sigma ds)_m}\right]\overline{\Theta'\omega'}\,ds. \tag{53}$$

The transformations (48)—(51) correspond to those found in the usual two-level model, although (51) is here modified by the stability variations. The transformations (52) and (53), however, do not occur in the conventional model, and represent effects directly due to the variations of static stability. We may note that these transformations modify the exchange of available potential energy between the zonal flow and the eddies by the same processes which are responsible for the exchange between potential and kinetic energy, i.e., the $\overline{\Theta\omega}$ and $\overline{\Theta'\omega'}$ correlations.

In the middle and high latitudes of significant $\overline{\Theta'\omega'}$, the condition $\sigma > \int\sigma ds$ will generally prevail, since the predominantly lower static stability of the region below, say, 30 deg N will depress the areal mean stability $\int\sigma ds$ relative to σ in the remaining half hemisphere. Hence the transformation $\{A'\cdot\bar{A}\}_2$ will serve in favor of that of the zonal mean, and in this sense acts as a brake upon the disturbances' growth. Further, since $\int\sigma ds < (\int\sigma ds)_m$ generally, the transformation $\{A'\cdot\bar{A}\}_1$ will also act to increase the zonal available potential energy at the expense of the eddy available potential energy when $\int\bar{\Theta}\bar{\omega}\,ds < 0$, although this effect is probably somewhat less than that of $\{A'\cdot\bar{A}\}_2$ and may act in the opposite direction.

From (44)—(47) we may construct an energy diagram in the manner of PHILLIPS (1956), shown in fig. 4. Here the indicated direction of energy transfer is that characteristic of the growing mid-latitude disturbance. As in the conventional model, the growth of the disturbance kinetic energy depends upon the systematic rising of the warmer air relative to the cooler air. As noted above, however, this same process, through its effective increase of the static stability, will simultaneously reduce the supply of eddy available potential energy and hence inhibit excessive disturbance growth. The variation of the static stability thus serves both to encourage and then to control the development, and may ultimately suppress the development altogether at the end of the disturbance's life cycle and thereby set the stage for new development.

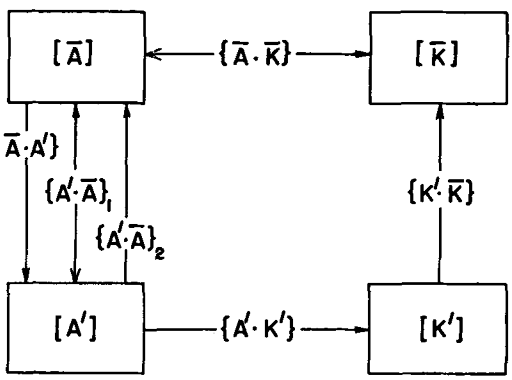

Figure 4. A schematic energy transformation diagram for the two-level model of variable static stability. The indicated directions of transfer are those characteristic of growing disturbances in middle latitudes, with the transformations $\{\bar{A}\cdot\bar{K}\}$ and $\{A'\cdot\bar{A}\}_1$ possibly proceeding in either direction. Here A denotes available potential energy, K denotes kinetic energy, and the bar and prime denote the zonal average and the perturbations, respectively. The symbol [ ] denotes spatial integration.

6. Summary and conclusions

The two-layer model designed by LORENZ (1957, 1960) appears to be the simplest baroclinic model which possesses consistent relations between the total kinetic and available potential energy, and between the available potential energy and the static stability. The model's variation of static stability is perhaps its most significant generalization over conventional two-level models. The Coriolis parameter is allowed full variability, and this feature may help to reduce the spurious anticyclogenesis characteristic of the quasi-geostrophic formulations.

The analysis of the baroclinic stability properties of the linearized equations shows that the synoptic-scale perturbations are unstable at approximately half the vertical wind shear required for instability in the usual two-level model; this feature may help the present model to remedy the systematic underprediction of relatively young disturbances noted with the conventional model. The analysis of the energy transformations reveals a mechanism whereby the supply of available potential energy to the disturbances is regulated through the variation of the static stability; this feature may help the present model to remedy the systematic overprediction of relatively mature disturbances noted with the conventional model. In order to test adequately these conjectures, a program of comparative numerical integration is now in progress, using northern hemisphere data from the International Geophysical Year.

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