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Convection in Conditionally Unstable Atmosphere

By H. L. KUO, Massachusetts Institute of Technology[1]

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Abstract

The analysis has shown that if the stratification is unstable for ascending motion over an infinite area and stable for descending motion, and small random perturbations of various scales are introduced, the final disturbance evolved will have the dimension of a cumulus cloud. The stable stratification in the descending region has the effect of increasing the critical lapse-rate and narrowing the ascending region and making the descending motion widespread, so that the centers of the ascending regions are further apart. The distance estimated from the theoretical analysis agrees favorably with those obtained from satellite observations.

The distributions of the various perturbation quantities pertaining to a circular symmetrical disturbance have been obtained. They show many resemblances to the corresponding distributions in tropical storms. Since the main source of energy of the mature tropical storms is the latent heat of condensation, it is really tempting to attribute the formation of these large-scale disturbances directly to convection in conditionally unstable air. However, the analysis shows that some organizing mechanisms are needed in order to channel the release of the energy of unstable stratification into a large scale circulation.

1. Introduction

Free convections occur very often in nature, but are usually governed by different physical processes on different occasions. In earth's free atmosphere such free convections are maintained not by the mean temperature stratification, which is usually stable, but by the water vapour distribution. If the temperature and the water vapour distribution in the atmosphere is such that when an element is lifted, enough latent heat may be liberated by condensation so as to raise the temperature of the element above that of the surroundings, then the stratification is unstable for ascent. This combined influence of temperature and water vapour stratification is most conveniently represented by the vertical distribution of the mean equivalent potential temperature Θ_E. Instability requires saturation and an upward decrease of Θ_E. The corresponding temperature lapse-rate is the moist adiabatic γ_m.

Since most of the condensed liquid water or snow fall out as precipitation, a change of state in reverse direction occurs only in a much smaller extent in the descending branch of the circulation. Therefore the mean stratification is usually stable for descending motion. Such a system which is unstable for saturated ascension and stable when unsaturated and for descension is said to be conditionally unstable. In such a system, we must treat the ascending motion and the descending motion separately.

The problem of convection in a conditionally unstable atmosphere has been analyzed by BJERKNES (1938) and PETTERSSEN (1939) by the slice method, which takes into consideration the influence of the stable stratification in the descending region but neglects the pressure perturbation and friction. The results are that for a saturated atmosphere, whenever the lapse rate γ is greater than the moist adiabatic lapse rate γ_m, the stratification is unstable, but only those perturbations with a very small ascending region can grow.

More recently, HAQUE (1952), LILLY (1960) and KUO (1960) applied the perturbation technique to this problem, and also take into consideration the Coriolis force because of their interest in the large scale motions. The first two authors treated the motion as inviscid and adiabatic, therefore were also unable to

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determine a true critical value for $\partial\Theta_E/\partial z$ or a preferred scale of the perturbation. In Haque's discussion, only the largest perturbation with zero growth rate is considered, and he applied his result directly to the initiation of tropical storms. A similar interpretation has also been given by Lilly. However, such interpretations are highly questionable because it disregards all the other growing disturbances.

The aim of this paper is to determine the critical $\partial\Theta_E/\partial z$ for the initiation and the maintenance of convective motion in a conditionally unstable atmosphere, and to obtain the preferred scale and the nature of these convective motion. For these problems it is necessary to take into consideration the viscous effect and that of heat conduction, or some equivalent effects. It will be shown that in an atmosphere which is at rest and is either absolutely or conditionally unstable, only cumulus cloud type convection can be expected to evolve if random infinitesimal perturbations are introduced. In case of stable stratification in the descending region, the distance between the ascending centers will be relatively large, and may be as far as 20—40 kilometers apart. Such convective systems also tend to arrange into long bands when a shearing mean wind is present, as have been discussed by Kuettner (1959), Sobhag (1931), Fritz and Wexler (1960) and by Winston (1960).

The second aim of this paper is to explore the possible mechanisms which will produce an organizing effect on the convective motion for the initiation and maintenance of a large-scale circulation, and help to control the small scale convection. A number of such factors, which seem to be important for the initiation and development of the large scale tropical cyclones, will be discussed in the last section.

In addition, section two of this paper will be devoted to the derivation of a set of dynamic and thermodynamic equations which, even though simplified already, still govern almost all of the important atmospheric motions. Further simplifications can be obtained by introducing additional restrictions regarding the nature of motion.

2. General equations

In this section the equations appropriate for the class of motions in which the distributions of absolute pressure p and of density ϱ are approximately as that of a normal atmosphere at rest shall be derived.

We define the normal atmosphere as an atmosphere which has the same overall stratification, the same total mass and the same total internal energy as the actual atmosphere, and which is at hydrostatic equilibrium. Denoting by p_0, ϱ_0, T_0 the pressure, density, and absolute temperature of this normal atmosphere, then these quantities are functions of the vertical coordinate z alone, and p_0 and ϱ_0 are related by the hydrostatic equation

$$\frac{\partial p_0}{\partial z} = -g\varrho_0 \tag{2.1}$$

where g is the gravitational acceleration. For simplicity, we assume that the atmosphere satisfies the equation of state of a perfect gas.

Since T_0 is the equilibrium temperature when the fluid is at rest, its vertical distribution is determined by nonadiabatic processes such as heat conduction, radiation and addition of latent heat of water vapor by condensation and evaporation. Heat conduction alone will give a linear dependence of T_0 on z, while the other nonadiabatic processes may result in more complicated vertical variations. Here we merely consider T_0 as a known function of z.

In dealing with a gas medium it is more convenient to use the quantity $s = \log\Theta$ as a dependent variable. Here $\Theta = T(P/p)^{1-cv/cp}$ is the potential temperature, therefore s is proportional to the entropy.

Denoting the departures of the various quantities from their normal values by small letters with a prime, and assuming that

$$p'/p_0 \ll 1, \ \varrho'/\varrho_0 \ll 1 \tag{2.2}$$

we then obtain the following relations from the perfect gas equation

$$\varrho'/\varrho_0 = p'/p_0 - T'/T_0 = \frac{p'}{\gamma P_0} - s' \tag{2.3}$$

where $\gamma = c_p/c_v$, c_p and c_v being the specific heat at constant pressure and at constant volume respectively. Thus p'/p_0, ϱ'/ϱ_0, T'/T_0 and $s'(=\Theta'/\Theta_0)$ are of the same order of magnitude.

We mention that these primed quantities may have non-zero horizontal and vertical averages.

Hereafter the following symbols shall be used:

$\vec{V}$ = velocity vector with components u, v, w.

$\vec{i}, \vec{j}, \vec{k}$ = unit vectors in the directions x, y, z, with z increasing upward.

∇_1 $= \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$ denotes the horizontal delta operator.

∇ $= \nabla_1 + \vec{k}\frac{\partial}{\partial z}$ denotes the three-dimensional delta operator.

f is the Coriolis parameter

$\vec{\omega}$ $= \nabla \times \vec{V}$ is the relative vorticity vector with components ξ, η, ζ, given by $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

π $= p'/\varrho_0$

∇_1^2 $= \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$ denote the two-dimensional and the three-dimensional Laplacian operators.

ν = kinematic viscosity coefficient or its equivalence.

k = thermometric counductivity or its equivalence. We shall approximate part of the radiation process by an equivalent conductive process.

s_z $= \frac{1}{\Theta_0}\frac{\partial \Theta_0}{\partial z} = \left(\frac{g}{c_p} + \frac{\partial T_0}{\partial z}\right)/T_0$ is the stability factor of the normal dry atmosphere.

σ $= -\frac{1}{\varrho_0}\frac{\partial \varrho_0}{\partial z} = \left(\frac{g}{R} + \frac{\partial T_0}{\partial z}\right)/T_0 = \frac{g}{\gamma R T_0} + s_z$ is the density stratification factor.

Neglecting higher powers of p' and ϱ', the equations of motion can be written in the following vectorial form

$$\frac{\partial \vec{V}}{\partial t} + (f\vec{k} + \vec{\omega}) \times \vec{V} = -\nabla\left(\pi + \frac{1}{2}V^2\right) + \vec{k}(s_z \pi + g s') + \nu \nabla^2 \vec{V} \qquad (2.4)$$

For earth's atmosphere, the value of s_z is of the order of 10^{-5} m^{-1} whereas that of σ and $g/\gamma R T_0$ are about ten times larger. Since for most problems the scale height of the motion is usually smaller than $H = 1/\sigma$ ($\simeq$ 10 km), we may neglect s_z against either σ, $g/\gamma R T_0$ or $\partial/\partial z$. Using this approximation, eq. (2.4) then reduces to the following

$$\frac{\partial \vec{V}}{\partial t} + (f\vec{k} + \vec{\omega}) \times \vec{V} = -\nabla\left(\pi + \frac{1}{2}V^2\right) + g s' \vec{k} + \nu \nabla^2 \vec{V} \qquad (2.4\text{ a})$$

The assumption that ϱ'/ϱ_0 is always small implies that ϱ/ϱ_0 is approximately constant for a moving particle of fluid, i.e.,

$$\frac{1}{\varrho_0}\frac{d\varrho}{dt} = \frac{1}{\varrho_0}\left(\frac{\partial \varrho'}{\partial t} + w\frac{\partial \varrho_0}{\partial z}\right) \simeq -\sigma w$$

Consequently the approximate form of the continuity equation is given by

$$\nabla \cdot \vec{V} = \sigma w \qquad (2.5)$$

This approximation implies a limitation on the time scale of the variations.

Through the use of this approximate continuity equation, sound waves and high frequency gravitational waves whose period is less than 8 minutes are excluded from the present consideration. We mention that when the vertical scale of the motion is much smaller than the scale height $H(=1/\sigma)$, the equation of continuity reduces to that of incompressible flow.

The first law of thermodynamics can be expressed more simply in terms of s:

$$\frac{\partial s'}{\partial t} + \vec{V} \cdot \nabla s' + s_z w - k \nabla^2 s' = \frac{Q}{c_p T_0} \qquad (2.6)$$

where Q is the accession of heat to unit mass per unit time by non-adiabatic processes other than by conduction.

It may be mentioned that the equations (2.4), (2.5) and (2.6) are very similar to the equations in the (x, y, p, t) coordinate system, except that now the vertical equation of motion need not be hydrostatic.

The variable π can be eliminated from eq. (2.4) by an application of the curl operator $\nabla \times$, which yields the vorticity equation. Combining it with the continuity equation (2.5) it then takes the following form

$$\frac{d}{dt}\left(\frac{f\vec{k}+\vec{\omega}}{\varrho_0}\right)=\frac{\nu}{\varrho_0}\nabla^2\vec{\omega}-\frac{g\,\vec{k}\times\nabla s'}{\varrho_0}+ \\ +\frac{(f\vec{k}+\vec{\omega})\cdot\nabla\cdot\vec{V}}{\varrho_0} \qquad (2.7)$$

Equation (2.6) together with the three equations (2.7) and the appropriate boundary conditions suffice to determine the dependent variables u, v, w and s'.

To determine π, it is convenient to obtain another equation by applying the delta operator ∇. to eq. (2.4 a) and making use of eq. (2.5), which result in the following equation

$$\nabla^2\pi-\sigma\frac{\partial\pi}{\partial z}=g\left(\frac{\partial s'}{\partial z}-\sigma s'\right)+ \\ +f\zeta+\omega^2-\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x}\right)^2 \qquad (2.8)$$

It may be mentioned that if the horizontal scale of the motion is much larger than the vertical scale, the hydrostatic relation will be valid. Equation (2.8) then splits into two equations, one is the hydrostatic relation, the other is equivalent to the so called balance equation, and relates $\nabla_1^2\pi$ to the wind field.

For the determinations of the critical lapse rate and the preferred scales of motion for the relatively complicated cases of convection when the influence of the stable descending surroundings is taken into consideration, we shall find the energy integrals are extremely useful. Therefore we shall derive the energy equations at this point.

Multiplying eq. (2.4 a) by $\varrho_0\vec{V}$ scalarly[1] and integrating over the entire volume, making use of eq. (2.5) and assuming that the normal velocity vanishes on the boundary we obtain

$$\frac{\partial K}{\partial t}=g\int ws'\,dm-D \qquad (2.9)$$

where $K=\frac{1}{2}\int V^2\,dm$ is the total kinetic energy, $D=-\int v_i\frac{\partial\tau_{ik}}{\partial x_k}\,dm$ is the total rate of dissipation, τ_{ik} being the viscous stress, and is given by $\tau_{ik}=\nu_{(k)}\frac{\partial v_i}{\partial x_k}$, where $\nu_{(k)}$ is the viscosity coefficient in the direction of x_k, including that due to smaller eddy motions. The summation convention over repeated indices i and k has been used in τ_{ik} and in D.

Similarly, multiplication of eq. (2.6) by $g\varrho_0 s'/s_z$ and integration lead to the following thermal energy integral

$$\frac{\partial E}{\partial t}+g\int ws'\,dm=\frac{c_p}{g}\int\frac{s'Q}{T_0 s_z}\,dm-D_e \qquad (2.10)$$

where

$$E=\frac{g}{2\,s_z}\int s'^2\,dm \text{ and } D_e=-g\int\frac{k}{s_z}s'\nabla^2 s'\,dm.$$

These two equations show that for inviscid and adiabatic motions, the total energy $(K+E)$ is conserved.

For the special case of adiabatic motion in an isentropic atmosphere, eq. (2.9) alone suffices to express the energy conservation relation because we then have $\int ws\,dm=\frac{\partial}{\partial t}\int zs\,dm$, which indicates again that the increase of kinetic energy must be accompanied by an upward transport of s. A necessary but obvious modification of eq. (2.10) for this case is to remove the factor $1/s_z$ from this equation.

3. Differential equations for small perturbations

Since the main concern of this paper is to determine the critical lapse rate and to investigate the nature of the convective motion near the critical point under the influences of earth's rotation and the release of latent heat of condensation, the governing equations (2.4 a) and (2.6) can be linearized. It is then possible, and also advantageous, to obtain a partial differential equation for a single dependent variable.

Since some kind of current is usually present in the atmosphere, we shall assume the existence of a constant mean velocity u_0. The influences of a variable u_0 and of horizontal baroclinicity will be investigated later.

Elimination of s' between the third equation of (2.4 a) and eq. (2.6) yields

$$(gs_z+LL')\,w=-L'\frac{\partial\pi}{\partial z}+\frac{gQ}{c_pT_0} \qquad (3.1)$$

[1] If it is desired to use eq. (2.4), we may multiply this equation by $\varrho_0\vec{V}\overline{\Theta}/\Theta$; integration over the total mass then yields a modified kinetic energy integral in terms of $K_1=\frac{1}{2}\int\frac{V^2\overline{\Theta}}{\Theta}\,dm$, where $\overline{\Theta}$ is the mean value of Θ_0.

where $L = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} - \nu \nabla^2$ and $L' = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} - k \nabla^2$.

Note that U_0 occurs only in the operators L and L' in the combination $U_0 \frac{\partial}{\partial x}$. It can be removed by taking a coordinate system which moves with the constant speed U_0. Therefore for the disturbances which are moving with U_0, it has no effect.[1] This term disappears also when the disturbances are independent of x.

Applying the horizontal delta operator ∇_1. to eq. (2.4 a) and making use of eq. (2.5) we obtain

$$L\left(\frac{1}{\varrho_0}\frac{\partial \varrho_0 w}{\partial z}\right) = -f\zeta + \nabla_1^2 \pi \tag{3.2}$$

Elimination of ζ between (3.2) and the third of (2.7) yields

$$(f^2 + L^2)\frac{1}{\varrho_0}\frac{\partial \varrho_0 w}{\partial z} = L\nabla_1^2 \pi \tag{3.3}$$

Finally, eliminating π between (3.1) and (3.3) and introducing the new dependent variable $\psi = \varrho_0 w e^{\sigma z/2}$ we obtain the following equation in ψ:

$$L'(f^2 + L^2)\left(\frac{\partial^2 \psi}{\partial z^2} - \frac{\sigma^2}{4}\psi\right) + L\,(g s_z + LL')\nabla_1^2 \psi = = \frac{g \varrho_0 e^{\sigma z/2}}{c_p T_0} L \nabla_1^2 Q \tag{3.4}$$

For the earth's atmosphere the value of $\sigma^2/4$ is about 2.5×10^{-9} m^{-2} which is much smaller than $\partial^2/\partial z^2$, therefore the term with σ^2 in (3.4) may be neglected.

In this study we shall be concerned only with the addition of latent heat due to condensation, which is considered to be determined by vertical motion alone. Thus we shall assume

$$\begin{cases} Q = -L_v \dfrac{dm_s}{dt} = -L_v \dfrac{\partial m_s}{\partial z} \text{ for saturated upward motion,} \\ Q = 0 \text{ for unsaturated or descending motion.} \end{cases} \tag{3.5}$$

Here m_s is the saturation mixing ratio of the air and L_v is the latent heat of condensation. With this Q we may combine the term on the right side of eq. (3.4) with the last term on the left by introducing the stability factor S, defined by

$$S = -s_z - \frac{L_v}{c_p \Theta_0}\frac{\partial m_s}{\partial z} = \frac{T_0}{\Theta_0}(\gamma - \gamma_m) = -S_E \text{ for saturated ascending motion,} \tag{3.6}$$

$$S = -s_z = \frac{T_0}{\Theta_0}(\gamma - \gamma_d) \text{ for unsaturated or descending motion,}$$

where $S_E = \Theta_0^{-1}\partial\Theta_E/\partial z$, Θ_E is the equivalent potential temperature and γ_m and γ_d are the moist and dry adiabatic lapse rate. It may be mentioned that according to this assumption, S is always discontinuous across the plane separating the ascending and the descending motions, if the descending air is unsaturated.

With this simplification, equation (3.4) reduces to the following

$$L'(f^2 + L^2)\left(\frac{\partial^2 \psi}{\partial z^2} - \frac{\sigma^2}{4}\psi\right) + L\text{-}(gS + LL')\nabla_1^2 \psi = 0 \tag{3.7}$$

The vertical distribution of s_z and S_E in the West Indies region is given in figure 1, which are deduced from the mean soundings given by JORDAN (1958, 1959).

We mention that if the hydrostatic approximation has been used, the second L^2L' term will disappear from this equation whereas if a geostrophic balance is assumed for the x-equation of motion, then the first L^2L' term will disappear from this equation. It has been shown in another paper (KUO, 1960) that in discussing the stability problem for motions produced by unstable stratification we must not use these two approximations simultaneously; otherwise the possibility of instability will be completely lost.

In order to avoid too much mathematical complexity, we shall only consider two-dimensional systems, either in the form of long rolls or with circular symmetry. In either of these two cases the variables depend only on one horizontal coordinate. The long rolls may be used to represent the long cloud strips, while the circular system may be used to represent an isolated system such as a single cloud, a tropical storm or a tornado. It may also be considered as an approximation to the hexagonal or even square patterns. This latter interpretation is particularly desirable because of the extreme complexity of the three-

[1] The equation for nonviscous flow in a shearing mean current will be given in section 7.

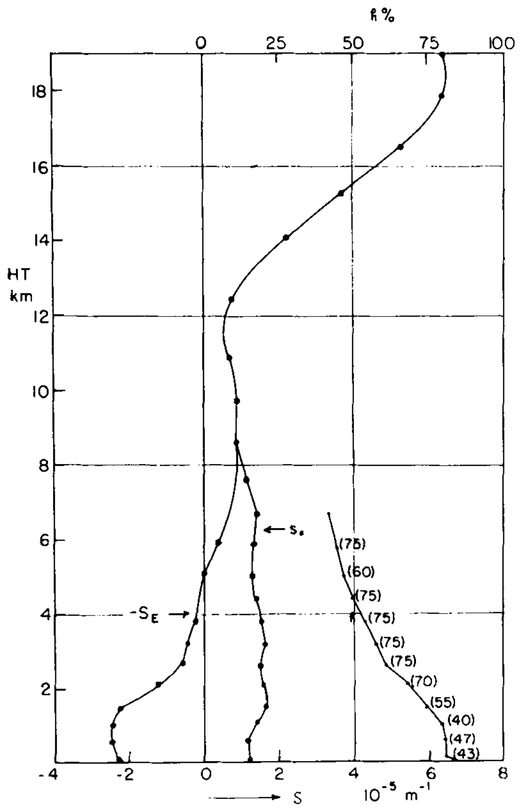

Fig. 1. Vertical distributions of the stability factors S and the relative humidity h in West Indies during the hurricane season (July—October). The numbers in the parentheses along the h-curve indicate the lift (in mb) needed to produce saturation.

dimensional solutions in case when the descending region is stable.

Equation (3.7) must be solved with respect to certain specific boundary conditions. The external boundaries of the system under consideration are horizontal planes at $z = 0$ and $z = \infty$. Since $\psi = \varrho_0 w e^{\sigma z/2}$ and $\varrho_0 = \varrho_{00} e^{-\sigma z}$, where ϱ_{00} is the mean sea level density, we shall require the vanishing of ψ both at the lower and at the upper boundaries. The latter is assumed in order to have the kinetic energy per unit volume approach zero[1] as z approaches infinity. Therefore two of the boundary conditions that must be satisfied by ψ are

$$\psi = 0 \text{ at } z = 0 \text{ and as } z \to \infty \qquad (3.8)$$

[1] For problems concerning motions in the upper atmosphere, it will be more appropriate to require ψ to remain finite as $z \to \infty$.

For simplicity, we shall use $\psi = 0$ at $z = 0$ and $z = h$ instead.

When the viscous effect is taken into consideration in full, additional conditions must be imposed because the differential equation is of higher order. For simplicity, we shall consider the lower boundary as very smooth and therefore require the vanishing of tangential stresses and of w and s (the so called free boundary conditions). These conditions can be satisfied by harmonic solutions.

In the horizontal directions the fluid is assumed to extend to infinity. The conditions we impose in the horizontal direction is either finiteness or periodicity of the solution. Specifically, for the cases of circular symmetry or long rolls we shall require the vanishing of the radial velocity u at the axis of symmetry. We shall also require u to vanish at $r = r_2$, which may be finite or infinite. In addition, $\partial w/\partial r$ is also assumed to vanish at these two radii. Thus the external boundary conditions in the r-direction are:

$$\psi = 0 \text{ and } \nabla_1^2 \psi = 0 \text{ at } r = 0 \text{ and } r = r_2 \qquad (3.9)$$

For inviscid flow these two conditions are identical and therefore there is only one condition for each given value of r.

Since in the problems treated below the stability factor varies discontinuously (either) in the horizontal (or in the vertical) directions, certain conditions must be imposed on the internal boundaries. The necessary requirements are the continuity of the normal velocity v_n and of the pressure across these boundaries. In terms of ψ, the first requirement is equivalent to the continuity of ψ, while the second depends upon the S variation, and may be derived from the third equation of (2.4 a) and eq. (2.6).

Thus for inviscid flow the continuity of p' across a vertical surface requires the continuity of $(\varrho s - q^2)w$, therefore the condition is given by (from eq. [3.1]).

$$\frac{d\psi_1}{dx} = -l'^2 \frac{d\psi_2}{dx}, \; l'^2 = \frac{-gS_2 + q^2}{gS_1 - q^2} \qquad (3.10)$$

where q is the time rate of amplification. Therefore w changes its sign abruptly across the surface, as has been found by HAQUE (1952).

On the other hand, for steady state viscous flow this requirement is given by

$$\nu_1 \nabla_1^2 w_1 + g s_1 = \nu \nabla_1^2 w_2 + g S_2 \text{ for } x = x_1 \quad (3.11)$$

In the stable descending region the motion is usually very weak, therefore the hydrostatic approximation should hold in this region. The first term on the right then drops out and we have

$$\nu \nabla_1^2 w_1 + g s_1 = g s_2 \quad (3.11\text{ a})$$

Since the velocity must be continuous in viscous flow, we must also require w to vanish on this surface. Hence we must require

$$u_1 = u_2,\ w_1 = w_2 = 0 \text{ at } x = x_1 \quad (3.11\text{ b})$$

It should be pointed out that the internal boundary $x = x_1$ which separates the ascending from the descending current is a free boundary. The position of this boundary is not given, but must be determined by the solution. This makes the problem somewhat more difficult as compared with the other problems which involve only fixed internal boundaries.

When S changes discontinuously in the vertical direction, the inviscid equations require the continuity of both the horizontal and the vertical velocity components, therefore continuity of both ψ and $\partial\psi/\partial z$.

4. The growth rate and most preferred scale in an unstable layer of constant S.

The nature of convective motion in a single layer of constant S is relatively wellknown from the previous studies of HAQUE (1952), CHANDRASEKHAR (1953), LILLY (1960), and KUO (1960). Here we wish to recapitulate some already known results to facilitate the discussion of the convective motions for a variable S. For this case, the solution of eq. (3.7) that satisfies the free boundary conditions is of the form

$$\psi = A e^{qt} \sin \frac{k\pi z}{h} F(x, y) \quad (4.1)$$

where $F(x, y)$ is a harmonic function satisfying the equation

$$\nabla_1^2 F = -\alpha^2 \frac{\pi_2}{h^2} F \quad (4.1\text{ a})$$

The length scale being used is a depth h, and α is the horizontal wave number and k is the vertical wave number. In dealing with atmospheric motions we should use the eddy coefficient of viscosity, which in general has different values in different directions. We shall denote the eddy viscosity coefficient in the horizontal directions by ν_h and that in the vertical direction by ν_z.

Different solutions of (4.1 a) can be obtained to represent different plane forms of the convection cells. For example, the solutions that represent long cloud rolls and circular cells and rectangular cells are given by, respectively

$$F_1 = \sin \frac{\alpha\pi x}{h}, \qquad F_2 = r J_1\left(\frac{\alpha\pi r}{h}\right)$$

$$F_3 = \sin \frac{\alpha_1 \pi x}{h} \sin \frac{\alpha_2 \pi y}{h} \quad (4.1\text{ b})$$

When the solution (4.1) is substituted in eq. (3.7) we find

$$q = \left(\frac{g\alpha^2 S - k^2 f^2}{k^2 + \alpha^2}\right)^{1/2} - \frac{\pi^2}{h^2}(\alpha^2 \nu_h + k^2 \nu_z) \quad (4.2)$$

Thus q has its maximum for $k = 1$, corresponding to the first mode in the vertical. We shall use this value of k only. Then q is given by

$$q = \left(\frac{g\alpha^2 S - f^2}{1 + \alpha^2}\right)^{1/2} - \frac{\pi^2}{h^2}(\nu_z + \alpha^2 \nu_h) \quad (4.2\text{ a})$$

Eq. (4.2 a) shows that for the smaller disturbances with a horizontal wave number α larger than $f/\sqrt{gS}$, q is real and therefore the disturbance will grow if the first term of eq. (4.2 a) is larger than the frictional effect, and will be damped out gradually if the second term is numerically larger than the first term, which is the case for the very small disturbances. On the other hand, q is complex but has a negative real part for the larger disturbances whose horizontal wave number α is smaller than $f/\sqrt{gS}$. Hence these larger disturbances behave like the stable gravitational-internal long waves in a stable atmosphere, and may propagate with a certain speed. To demonstrate this let us put

$$\psi = A e^{-bt} \sin \frac{k\pi z}{h} \sin \frac{\pi}{h}(\alpha_1 x + \alpha_2 y - nt) \quad (4.3)$$

Substituting in (3.7) we find

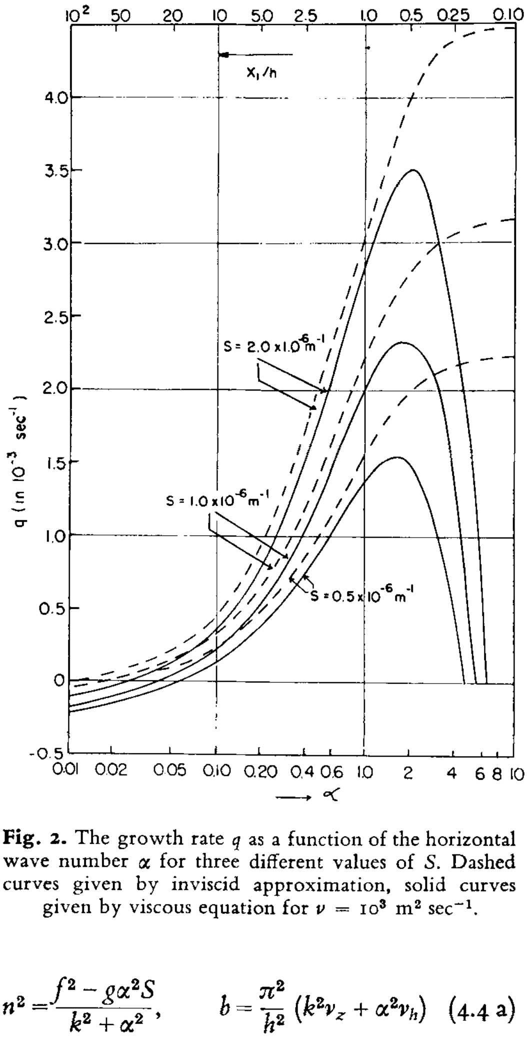

Fig. 2. The growth rate q as a function of the horizontal wave number α for three different values of S. Dashed curves given by inviscid approximation, solid curves given by viscous equation for $\nu = 10^3$ m^2 sec^{-1}.

$$n^2 = \frac{f^2 - g\alpha^2 S}{k^2 + \alpha^2}, \qquad b = \frac{\pi^2}{h^2}(k^2\nu_z + \alpha^2\nu_h) \quad (4.4\text{ a})$$

where $\alpha^2 = \alpha_1^2 + \alpha_2^2$. Therefore the phase velocity is given by

$$c_{ph} = \frac{n}{(k^2 + \alpha^2)^{1/2}} = \frac{(f^2 - g\alpha^2 S)^{1/2}}{k^2 + \alpha^2} \quad (4.4\text{ b})$$

Since the phase velocity depends on both frequency and direction, the propagation is both dispersive and anisotropic.

From equations (4.2 a) and (4.4 b), we see that for any given positive S there exists a limiting horizontal wave number α_0 given by $\alpha_0^2 = f^2/gS$ which separates the amplifying shorter waves from the decaying long waves.

The physical interpretation of this limiting wave number is that for the perturbations with α smaller than α_0, the kinetic energy of the horizontal motion produced by the Coriolis effect through a displacement is greater than the potential energy released during this displacement, while the reverse is true for perturbations with α greater than α_0.

Many meteorologists have interpreted this limiting wave number as representing the preferred scale of motion for the given S, and applied it directly to the hurricane formation (see HAQUE 1952, SYONO 1953, and LILLY 1960). However, since the perturbations with $\alpha > \alpha_0$ are growing with time while the limiting perturbation is not, there seems to be no reason to expect this limiting perturbation to become the dominant one, at least not according to the linearized equations. So far as linearized equations are concerned, the most preferred scale of motion must be the one whose growth rate q is a maximum, or the one with a lowest critical stability factor $S = S_m$ for marginal stability ($q = 0$), if we consider S has been raised gradually from below to above S_m. In this section we shall examine the preferred scale from these two different interpretations.

The values of q as given by eq. (4.2 a) for $f = 5 \times 10^{-5}$ sec^{-1} (corresponding to $\phi = 20$ deg. lat.), $h = 10$ km and for three different values of S which often exist in the tropics are plotted in fig. 2. The solid curves correspond to $\nu_z = \nu_h = 10^3$ m^2 sec^{-1}, while the dashed curves correspond to inviscid motions. It is seen that when friction is neglected, q has its maximum when $\alpha = \infty$ ($L = 0$), whereas with friction the maximum occurs at about $\alpha = 1.6$, corresponding to a horizontal scale $r_1 = 0.76\ h$ for long rolls and $r_1 = 0.48\ h$ for a circular cell. This preferred wave number shifts very slowly toward a higher value as S increases. Thus the higher is S, the smaller is the preferred scale of motion.

Comparing the dashed curves with the corresponding full curves, we see that the effect of eddy viscosity on q is small for $\alpha < 1$. We also mention that except for very small values of α, the effect of the Coriolis parameter f is also small.

To obtain the lowest value of S for the maintenance of convection we set $q = 0$, $k = 1$, $\nu_z = \nu_h = \nu$ in eq. (4.2).

Introducing the nondimensional quantities R and T, defined by

$$R = \frac{gSh^4}{k\nu\pi^4}, \qquad T = \frac{f^2h^4}{\nu^2\pi^4} \tag{4.5}$$

we then find the following relation

$$R\,\alpha^2 = (1+\alpha^2)^3 + T \tag{4.6}$$

which is identical to the equation obtained by CHANDRASEKHAR (1953). From this equation we find that for large T the critical value of R and the most preferred α are given by

$$\begin{aligned} R_m &\to 3\left(\frac{T}{2}\right)^{2/3} \\ \alpha_m &\to \left(\frac{T}{2}\right)^{1/6} \end{aligned} \tag{4.7}$$

Numerical values of R_m and α_m for this case have been computed by CHANDRASEKHAR (1953). We note that for fixed ν, the critical temperature gradient S_m increases with $f^{4/3}$ while for fixed f, S_m is proportional to $\nu^{2/3}$ if we keep the Prandtl number constant. Thus in this case, the horizontal scale of the motion will shrink indefinitely as the value of T increases.

5. Convection in conditionally unstable atmosphere

In order to avoid mathematical complications in this section we shall assume that the convection cells are of the form of long rolls, with their axis in the direction of y, so that all the variables are independent of y. Circular cells shall be treated in the next section.

The stability factor S is supposed to be independent of height while its horizontal variation is given by

$$S = S_1 > 0 \text{ for } 0 < x < x_1$$

$$S = S_2 = -l^2 S_1 \text{ for } x_1 < x < x_2$$

where $x = x_1$ is the boundary between the ascending and the descending branches and $2x_2$ is the width of the cell. The function ψ is assumed to be given by $\psi = \Phi(x) \sin \frac{\pi z}{h}$, therefore it is only necessary to solve for the horizontal function $\Phi(x)$.

A. *Nonviscous solution*

The nonviscous solution for the case with a stable descending region and $q = 0$ has been discussed by HAQUE (1952) and by LILLY (1960). The two most important results obtained by these authors are (*i*), the vertical velocity has a finite discontinuity across the surface which separates the ascending from the descending region, and (*ii*), the ratio of the area of the ascending motion to that of the descending motion descreases as the ratio $l^2 = -S_2/S_1$ increases. These conclusions also hold approximately when the viscous effect is taken into consideration. Because of the importance of these conclusions, we shall redevelop the nonviscous solution briefly and obtain some additional information.

When $\nu = 0$, the solution $\Phi(x)$ is then given by

$$\Phi_1 = A \sin \alpha\xi \qquad \text{for } 0 \leqslant \xi \leqslant \xi_1 \tag{5.1}$$

$$\Phi_2 = A\,\frac{\sin\alpha\xi_1 \sinh\beta(\xi_2-\xi)}{\sinh\beta(\xi_2-\xi_1)} \qquad \text{for } \xi_1 \leqslant \xi \leqslant \xi_2$$

where $\xi = \frac{\pi x}{h}$, α and β are defined by

$$\alpha^2 = \frac{f^2+q^2}{gS_1-q^2}, \quad \beta = \frac{\alpha}{l'}, \quad l'^2 = \frac{gS_1l^2+q^2}{gS_1-q^2} \tag{5.1 a}$$

Substitution of (5.1) in the boundary condition (3.9) yields the compatibility equation

$$\tanh\beta\Delta\xi = l' \tan\alpha\xi_1, \quad (\Delta\xi = \xi_2-\xi_1) \tag{5.2}$$

which is similar to the one obtained by LILLY (1960), except that here l', α and β all involve q and that ξ_1 appears explicitly. A similar equation holds for circular disturbances.

Equation (5.2) has an infinite number of roots of which only the first is of interest to us. This root can easily be computed for given values of $\beta\Delta\xi$. The results show that as $\beta\Delta\xi$ increases from 0.5 to 1.5, the value of $\alpha\xi_1$ (which measures the area of the ascending motion) is doubled, whereas for $\beta\Delta\xi > 1.5$, $\alpha\xi_1$ changes only slightly.

When $\beta\Delta\xi$ is large, i.e., when $\beta\Delta\xi > 1.5$, eq. (5.2) reduces to

$$\frac{1-x}{l^2+x} = \tan^2\alpha\xi_1 \tag{5.2 a}$$

where $x = q^2/gS_1$. On the other hand, when $\beta\Delta\xi$ is large but $\alpha\xi_1$ is small, we obtain

$$x \simeq 1-(l^2+1)\,\alpha^2\xi_1^2 \tag{5.2 b}$$

which shows that the amplification rate decreases with increasing l and increasing $\alpha\xi_1$.

When both $\beta\Delta\xi$ and $\alpha\xi_1$ are small, eq. (5.2) reduces to

$$x = 1 - l^2/(\Delta\xi/\xi_1) \qquad (5.2\text{ c})$$

which shows that the area ratio $\Delta\xi/\xi_1$ must be greater than l^2 in order to have amplification and that the larger is the ratio $\Delta\xi/\xi_1$, the larger is x.

Since eq. (5.2) contains α and ξ_1, we need another relation between q and the other parameters in order to determine q and ξ_1 as functions of α. Such an equation can be obtained by substituting the quantities u, v, w and s corresponding to the solution (5.1) in the kinetic energy integral (2.9), taken over the volume of a single cell. For the case of large $\beta\Delta\xi$ this relation is given by

$$x\ (1+\alpha^2) - \alpha^2 + \alpha_0^2 = \frac{\sin 2\alpha\xi_1}{2\,\alpha\xi_1} \times \{\alpha^2 (1-x) - \beta^2 (l^2 + x)\} \qquad (5.3)$$

where $\alpha_0^2 = f^2/gS_1$. According to (5.1 a) the right side member of this equation is zero. Therefore we have the remarkable result that q depends neither on l nor on ξ_1 and that x is a function of the horizontal wave number α of the ascending region alone, given by

$$x = \frac{\alpha^2 - \alpha_0^2}{1 + \alpha^2} \qquad (5.3\text{ a})$$

This equation is identical with eq. (4.2 a), except that the frictional effect has been neglected. Even though this equation is obtained under the assumption that $\beta\Delta\xi$ is large such that tanh $\beta\Delta\xi$ is nearly equal to 1, it is believed that it also holds approximately under more general conditions.

Substituting (5.3 a) in (5.2 a), we obtain

$$\tan \alpha\xi_1 = \left\{\frac{1 + \alpha_0^2}{(1 + l^2)\ \alpha^2 + l^2 - \alpha_0^2}\right\}^{1/2} \qquad (5.4)$$

This equation determines ξ_1 uniquely as a function of α for given values of l. The values of ξ_1 as given by this equation for various values of α and for $l = 1$ and $l = 2$ are given in the following table. It is seen that for the same α, ξ_1 decreases with increasing l, indicating that the influence of the descending motion in the stable region is to reduce the area of the ascending motion.

Table 1. Values of $\xi_1 \left(= \frac{\pi x_1}{h}\right)$ as given by (5.4) for different values of α and of l.

l \ α	0.01	0.02	0.05	0.1	0.2	0.5	1.0	1.6	2
1	78.5	39.3	15.7	7.8	3.83	1.37	0.52	0.24	0.16
2	46.4	23.2	9.3	4.6	2.32	0.83	0.32	0.15	0.10

B. *Viscous solution and stability criterion*

When the viscous effect is taken into consideration by the Navier-Stokes equations, it is difficult to obtain the growth-rate explicitly. On the other hand, the concept of marginal stability becomes meaningful, we shall therefore derive the stability criterion according to this concept. For this case, the function $\Phi_1(\xi)$ for the ascending region is the solution of the following equation

$$(D^2 - 1)^3\, \Phi_1 - R\, D^2\, \Phi_1 - T\, \Phi_1 = 0 \qquad (5.4)$$

where R is the Rayleigh number for the unstable ascending region, and is defined by (4.5), and $D^2 = d^2/d\xi^2$ for long rolls and $D^2 = \xi \frac{d}{d\xi} \frac{1}{\xi} \frac{d}{d\xi}$ for circular cells.

In the descending region it is reasonable to assume the validity of the hydrostatic approximation. The equation for Φ_2 is then given by

$$(D^2 - 1)^2\, \Phi_2 + T\, \Phi_2 = R_2 D^2\, \Phi_2 \qquad (5.5)$$

where $R_2 = g s_{z2}\ h^4/k\nu\pi^4$. For convenience sake, we set $R_2 = l^2 R$ so that only one stability parameter R occurs in the solution.

Exact solutions of these two equations can be obtained, and when the external and internal boundary conditions are applied and the arbitrary constants have been eliminated, we shall find a transcendental equation relating R to the scale of the motion. However, such an approach is too complicated for the present problem and therefore will not be attempted here. Instead, we shall only find approximate solutions of the equations (5.4) and (5.5) and make use of the kinetic energy integral to determine the critical value of R and the preferred scale of motion. Thus, this method is similar to the variation method.

In this section we deal only with cloud rolls. The function Φ_1 is assumed to be given by

$$\Phi_1 = \sum_{s=1}^{3} A_s \sin \alpha_s\, \xi \qquad (5.6)$$

If Φ_1 is to satisfy eq. (5.5) exactly, then α_s must be a root of eq. (4.6). However, since only an approximate solution is to be used, we do not take eq. (4.6) exactly but merely as a guide for choosing the three α_s in Φ_1. Therefore we write the equation for α_s as

$$(1 + \alpha_s^2)^3 + T - R\,\alpha_s^2 \simeq 0 \qquad (5.4\,\text{a})$$

Let us arrange the three roots of this equation according to their order of magnitude. For relatively large R and not too large T we then have

$$\alpha_1^2 \simeq \frac{T+1}{R}, \quad \alpha_2^2 \equiv -\mu^2 \cong -R^{1/2}, \quad \alpha_3^2 \simeq R^{1/2} \qquad (5.7)$$

Hence the magnitudes of $-\alpha_2^2$ and of α_3^2 are much larger than that of α_1^2. For example, for atmospheric conditions we find R is of the order of 10^4 to 10^5, while T is from 1 to 100. Under these conditions the approximations in (5.7) are certainly satisfied. We shall assume this to be true even for near critical values of R.

Since α_3 is real and is much larger than α_1 when R is large, A_3/A_1 must be very small in order to have Φ_1 remain positive for $0 < \xi < \xi_1$. For simplicity we take $A_3 = 0$. On the other hand, α_2 is imaginary and $\sin \alpha_2 \xi$ increases exponentially with ξ. Therefore the solution in the ascending region is given by

$$\Phi_1 = A \left\{ \sin \alpha\xi - \frac{\alpha}{\mu} \frac{\cos \alpha\xi_1}{\cosh \mu\xi_1} \sinh \mu\xi \right\} \qquad (5.8)$$

where α is used to represent α_1. This solution satisfies $\Phi_1(0) = 0$ and $w_1(\xi_1) = 0$.

It may be mentioned that the hydrostatic approximation cannot be used in this region when friction is included because it gives a rapidly oscillating solution. Similarly, we find the solution of eq. (5.5) that satisfies the conditions (3.9) at $\xi = 0$ and $\xi = \xi_2$ and the condition (3.11 b) at $\xi = \xi_1$ is given by

$$\Phi_2 = B \left\{ \sinh \beta_1 (\xi - \xi_2) - \frac{\beta_1}{\beta_2} \frac{\cosh \beta_1 \Delta\xi}{\cosh \beta_2 \Delta\xi} \times \sinh \beta_2 (\xi - \xi_2) \right\}$$

$$B = A \frac{\cos \alpha\xi_1}{\cosh \beta_1 \Delta\xi} \frac{\beta_2}{\mu} \frac{\mu \tan \alpha\xi_1 - \alpha \tanh \beta_1 \Delta\xi}{\beta_1 \tanh \beta_2 \Delta\xi - \beta_2 \tanh \beta_2 \Delta\xi} \qquad (5.9)$$

where $\Delta\xi = \xi_2 - \xi_1$. Here β_1 and β_2 are the roots of the following equation

$$(\beta_{1,2}^2 - 1)^2 - l^2 R \beta_{1,2}^2 + T = 0 \qquad (5.10)$$

Since R_2 is defined by the normal stability factor s_{z2} of the dry air, it is usually of the order of 10^5. Therefore the two roots are approximately given by

$$\beta_1^2 \simeq \frac{T+1}{l^2 R + 2} \simeq \frac{\alpha^2}{l^2} \ll 1, \quad \beta_2^2 \simeq l^2 R + 2 \gg 1 \qquad (5.10\,\text{a})$$

On the other hand, if R_2 is very small, we shall have

$$\beta_1^2 \simeq 1 + i\,T^{1/2}, \quad \beta_2^2 \simeq 1 - i\,T^{1/2} \qquad (5.10\,\text{b})$$

This case will be excluded from the present consideration.

Note that when R is large, both α/u and β_1/β_2 are much smaller than 1. Then the second terms of the solutions (5.8) and (5.9) contribute very little except very near to the surface of discontinuity. Since the first terms of these solutions are the same as the nonviscous solution (5.1) except that in (5.1) α and β involve the amplification rate q, we may call these first terms the nonviscous approximations.

The variations of w and s at the level $z = h/2$ in the two regions are illustrated in fig. 3 by the dashed lines for the cases of $l = 1$ and $l = 2$, for a large R and an infinite $\Delta\xi$. The fully drawn curves represent the nonviscous solution. This example represents a circular symmetrical disturbance, which we shall discuss in section 6. It is seen that for $l = 1$, strong descending motion takes place close to the internal boundary and decreases rapidly with increasing ξ, whereas for larger l (more stable surroundings), the descending motion is much weaker but more wide spread. It is also seen that as l increases, the ascending area diminishes. The contribution by the second terms of (5.8) and (5.9) are very small except very close to the internal boundary $\xi = \xi_1$. However, as will be shown in the next section, the viscous terms are important when R is relatively small and that they contribute significantly to the dissipation.

For comparison, the variations of w, s' and p' as given by the nonviscous solution for the case of unstable stratification everywhere has also been represented by the dashed curve w_1

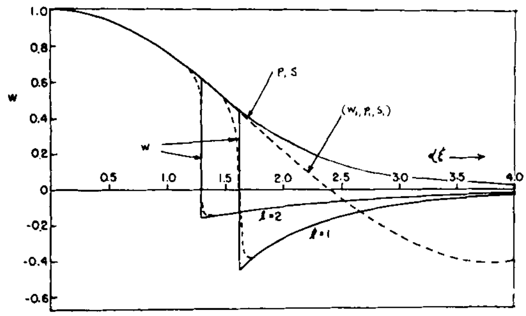

Fig. 3. Variation of w with ξ for $l = 1$ and $l = 2$ in conditionally unstable atmosphere. Solid curves given by inviscid solution. Dashed curves given by viscous solution.

in fig. 3. It is seen that for this case s' is negative in the descending region whereas in the conditionally unstable atmosphere s' is positive everywhere.

In order to apply the boundary conditions (3.11 a) we must obtain s_1 and s_2 by solving eq. (2.6). When the variation of ϱ_0 is disregarded and β_1 and β_2 differ from 1, these solutions are given by

$$s_1 = -\frac{A h s_{z1} \alpha}{K\pi} \sin \frac{\pi z}{h} \left\{\frac{\cos \alpha\xi}{1+\alpha^2} + \frac{\cosh \mu\xi}{\mu^2 - 1} \frac{\cos \alpha\xi_1}{\cosh \mu\xi_1}\right\}$$

$$s_2 = -\frac{B h s_{z2} \beta_1}{K\pi} \sin \frac{\pi z}{h} \left\{\frac{\cosh \beta_1(\xi - \xi_2)}{1 - \beta_1^2} + \frac{1}{\beta_2^2 - 1} \frac{\cosh \beta_1 \Delta\xi}{\cosh \beta_2 \Delta\xi} \cosh \beta_2 (\xi - \xi_2)\right\} \quad (5.11)$$

Substituting these functions in (3.11 a) and neglecting the rather small terms involving α/u and β_1/β_2 we obtain

$$\frac{(1-\beta_1^2)(\beta_2^2 - 1)}{\beta_2^2 - \beta_1^2} \left\{\frac{1}{1+\alpha^2} + \frac{1}{\mu^2 - 1} - \frac{\alpha^2 + \mu^2}{R}\right\} = l \frac{\beta_2}{\mu} \frac{\mu \tan \alpha\xi_1 - \alpha \tanh \mu\xi_1}{\beta_2 \tanh \beta_1 \Delta\xi - \beta_1 \tanh \beta_2 \Delta\xi} \quad (5.12)$$

We are particularly interested in the cases when R is relatively large so that the approximations (5.7) and (5.10 a) are valid. Eq. (5.12) can then be simplified to

$$\frac{1-\beta_1^2}{1+\alpha^2} \tanh \beta_1 \Delta\xi = l \left(\tan \alpha\xi_1 - \frac{\alpha}{\mu} \tanh \mu\xi_1\right) \quad (5.12\text{a})$$

This equation differs from eq. (5.2) by the factor $(1-\beta_1^2)/1 + \alpha^2$. Note that this equation involves the two unknowns α and ξ_1 besides the ratio l which is supposed to be known. Therefore another relation is needed for their determination. However, since both α^2 and β_1^2 are usually much smaller than 1, a first approximation can be obtained by omitting them against 1, we then have

$$\tanh \beta_1 \Delta\xi = l \tan \alpha\xi_1 \quad (5.12\text{ b})$$

The first root of this approximate equation are given in table 2a for various values of $\beta_1 \Delta\xi$ and l. It is seen that for a given l, $\alpha\xi_1$ decreases with decreasing $\beta_1\Delta\xi$, indicating that the placing of a fixed wall at a finite distance reduces the area of the ascending motion. However, this effect is prominent only when $\beta_1\Delta\xi$ is less than 1.5. For $\beta_1\Delta\xi$ greater than 1.5, $\alpha\xi_1$ is almost equal to its optimum value corresponding to $\beta_1\Delta\xi = \infty$. It is seen that $\alpha\xi_1$ decreases rapidly with increasing l.

Table 2a. Values of $\alpha\xi_1$ as given by (5.12 b) for different values of $\beta_1\Delta\xi$ and of l.

$\beta_1\Delta\xi$ \ l	1	2	3	5	10
∞	0.786	0.464	0.322	0.197	0.100
2.5	.780	.460	.318	.196	.099
2.0	.765	.450	.310	.190	.096
1.5	.735	.425	.295	.180	.090
1.0	.650	.360	.250	.150	.076
0.5	.431	.230	.155	.093	.047

In order to make use of the kinetic energy integral (2.9) we must obtain v_1 and v_2 by solving the y-component equation of (2.4 a). The results are

$$v_1 = \frac{A\pi\, T^{1/2}}{h} \cos \frac{\pi z}{h} \left\{\frac{\sin \alpha\xi}{1+\alpha^2} + \frac{\alpha \sinh \mu\xi}{\mu(\mu^2 - 1)} \frac{\cos \alpha\xi_1}{\cosh \mu\xi_1}\right\} \quad (5.13)$$

$$v_2 = \frac{B\pi\, T^{1/2}}{h} \cos \frac{\pi z}{h} \left\{\frac{\sinh \beta_1(\xi - \xi_2)}{1 - \beta_1^2} + \frac{\beta_1 \sinh \beta_2(\xi - \xi_2)}{\beta_2(\beta_2^2 - 1)} \frac{\cosh \beta_1 \Delta\xi}{\cosh \beta_2 \Delta\xi}\right\}$$

Substituting the velocity components u, v, w and the quantity s given by these solutions in eq. (2.9) and setting $\partial K/\partial t$ to zero we obtain a rather complicated equation relating R to the various parameters. Only a much simplified version of this equation shall be given here, which is based on the assumptions that $\mu \gg \alpha$, $\beta_2 \gg \beta_1$, and $\beta_1 \Delta\xi$ is larger than 1.5, so that we may put $\tanh \beta_1 \Delta\xi = \tanh \beta_2 \Delta\xi = 1$. Under these circumstances the ratio B/A of (5.9) reduces to

$$\frac{B}{A} \simeq -\frac{\sin \alpha\xi_1}{\sinh \beta_1 \Delta\xi} \simeq -\frac{1+\alpha^2}{l(1-\beta_1^2)} \frac{\cos \alpha\xi_1}{\cosh \beta_1 \Delta\xi} \quad \text{by (5.12 b)}$$

and the equation resulting from the kinetic energy integral takes the simple form

$$\alpha^2 R = (1+\alpha^2)^3 + T + F \frac{\sin 2\alpha\xi_1}{2\alpha\xi_1} \qquad (5.14)$$

where $F = (3\alpha^4 + 5\alpha^2 + 1)\,\alpha^2 + (1+\alpha^2)\alpha^2 \mu\, l + 2\,\beta_1\beta_2 - \beta_1^2 - \beta_1^4$

According to the equations (5.7) and (5.10 a), we have

$$\alpha\mu \simeq \frac{(T+1)^{1/2}}{R^{1/4}}, \quad \beta_1\beta_2 \simeq (T+1)^{1/2}$$

Therefore eq. (5.14) is the other equation relating R to the quantities α and $\alpha\xi_1$, besides the known quantities T and l. Together with eq. (5.12), we may solve for R in terms of α, and the preferred horizontal wave number α_m and the minimum R_m can be obtained for every given set of T and l.

Because of the complexity of the computation, only the first approximations of α_m and R_m have been obtained. The procedure adopted is to compute the first approximation of $\alpha\xi_1$ from eq. (5.12 b) for $\beta_1\Delta\xi = \infty$. These are given in table 2 b below, together with the corresponding function $I \equiv \sin 2\,\alpha\xi_1/2\alpha\xi_1$.

Table 2 b. Values of $\alpha\xi_1$ for $\beta_1\Delta\xi > 1 \cdot 5$ and various l as given by eq. (5.12 b), and the corresponding I.

l	1	2	3	5	10
$\alpha\xi_1$	0.785	0.462	0.320	0.198	0.100
I	0.637	0.863	0.933	0.975	1.000

The approximate values of α_m and R_m have been obtained and are given in table 3 for various values of l and T. The α_m values change only slightly with l, therefore only the mean is given in the last line. These results show that R_m increases both with T and with l. For $l \gg 1$, the R_m values are about twice as that required by a layer of fluid of constant R.

Table 3. Values of R_m as functions of T and l.

l \ T	0	5	10	50	100	500	1 000
1.	12.5	23.0	29.5	62.1	92.2	225	343
2.	15.2	28.1	35.5	74.7	111.0	270	412
5.	18.6	35.6	46.2	97.0	139.0	349	540
α_m	0.707	0.95	1.00	1.30	1.53	1.93	2.19

Because of the presence of the factor $(1-\beta_1^2)/1+\alpha^2$ on the left hand side of eq. (5.12 a), the true values of $\alpha\xi_1$ are somewhat smaller than that given by eq. (5.12 b) in table 2 a. On the other hand, due to the precence of the second term on the right side of eq. (5.12 a), the correction is actually small in most cases.

When l is greater than 3, the function I is very nearly equal to 1. The R_m and α_m values for $l = 5$ in table 3 are computed from this approximation.

For very large values of T and of l, we have the asymptotic formula

$$R \simeq 4\,(1+\alpha^2)^2 + \frac{T}{\alpha^2} \qquad (5.15\text{ a})$$

From this we find for large T and large l:

$$R_m \to 3\,T^{2/3}$$

$$\alpha_m \to \frac{1}{\sqrt{2}}\,T^{1/6}$$

Thus these asymptotic relations are similar to that given by eq. (4.7), but with different coefficients, therefore giving a larger R_m and a smaller α_m.

It may be mentioned that if the approximate relation (5.12 b) is used in the kinetic energy integral, it will result in the following equation (KUO, 1960)

$$R\alpha^2\left(I - \frac{\alpha^2+\beta_1^2}{1-\beta_1^2} I\right) = (1+\alpha^2)^3 + T\left(1 + \frac{\alpha^2+\beta_1^2}{1-\beta_1^2} I\right) + (1+\alpha^2)\, I\,\{3\alpha^4 + 2\alpha^2 - \beta^4 + l\alpha^2\mu + 2\beta_1\beta_2\} \qquad (5.17)$$

This equation differs from eq. (5.14) in two important aspects. First, the horizontal wave number α must be greater than zero but less than 1, secondly the asymptotic behavior for large T and large l is $R_m \to 5.83\ T$, $\alpha_m \to 0.643$.

We mention that the critical values of R given in table 3 are rather small when compared with the R values observed in the atmosphere when the unstable layer is more than one kilometer deep. Under those conditions perturbations of different scales will be able to grow simultaneously and the motion must be turbulent and the linear theory will not be quite valid.

6. Circular disturbances in conditionally unstable atmosphere

The next mathematically simple plan form of convection is that of circular symmetry. This form may be used either as representing an isolated convection cell or as an approximation to the system of hexagonal cells, therefore it is of interest to discuss this case in more detail, especially in regards to the space distribution of the various quantities. For cells of this form, the function Φ for the two regions are given by

$$\Phi_1 = A\xi \left\{\tau_1(\alpha\xi) + \frac{\alpha}{\mu}\frac{\tau_0(\alpha\xi_1)}{\tau_0(i\mu\xi_1)}\, i\,\tau_1(i\mu\xi)\right\}$$

$$\Phi_2 = B\xi \left\{Z_1(i\beta_1\xi) - \frac{\beta_1}{\beta_2}\frac{Z_0(i\beta_1\xi_1)}{Z_0(i\beta_2\xi_1)}\, Z_1(i\beta_2\xi)\right\} \qquad (6.1)$$

where τ_0 and τ_1 are Bessel functions of order zero and order one, respectively, while α and μ and β_1 and β_2 are given by eqs. (5.7) and (5.10 a). The functions Z_1 is given by

$$Z_1(x) = H_1^1(x) - \frac{H_1^1(x_2)}{H_1^2(x_2)}\, H_1^2(x) \qquad (6.1\text{ a})$$

where H_1^1 and H_1^2 are the two Hankel functions of order one. Z_0 is obtained from Z_1 by differentiation. These solutions satisfy the conditions $\Phi_1(0) = \Phi_2(\xi_2) = 0$ and $w_1(\xi_1) = w_2(\xi_1) = 0$. The ratio B/A is determined by the condition $\Phi_1 = \Phi_2$ at $\xi = \xi_1$. When ξ_2 is very large, Z_1 reduces to H_1^1 alone and $Z_0 = i\,H_0^1$.

Following the same procedure as employed in the preceding section, we obtain s and the tangential velocity v in the two regions by solving eq. (2.6) and the tangential component of eq. (2.4 a). The perturbation pressure is then obtained from the vertical equation of motion. The results are

$$\left\{\begin{aligned} s_1 &= -A\alpha\frac{\partial s_{01}}{\partial z}F_1\left\{\frac{\tau_0(\alpha\xi)}{1+\alpha^2} + \frac{1}{\mu^2-1}\frac{\tau_0(\alpha\xi_1)}{\tau_0(i\mu\xi_1)}\times \tau_0(i\mu\xi)\right\} \\ s_2 &= -Bi\beta_1\frac{\partial s_{02}}{\partial z}F_1\left\{\frac{Z_0(i\beta_1\xi)}{1-\beta_1^2} + \frac{1}{\beta_2^2-1}\times \frac{Z_0(i\beta_1\xi_1)}{Z_0(i\beta_2\xi_1)}\, Z_0(i\beta_2\xi)\right\} \end{aligned}\right. \qquad (6.2\text{ a})$$

$$\left\{\begin{aligned} v_1 &= A\,F_2\left\{\frac{\tau_1(\alpha\xi)}{1+\alpha_2} - \frac{\alpha}{\mu(\mu^2-1)}\frac{\tau_0(\alpha\xi_1)}{\tau_0(i\mu\xi_1)}\times i\tau_1(i\mu\xi)\right\} \\ v_2 &= B\,F_2\left\{\frac{Z_1(i\beta_1\xi)}{1-\beta_1^2} + \frac{\beta_1}{\beta_2(\beta_2^2-1)}\frac{Z_0(i\beta_1\xi_1)}{Z_0(i\beta_2\xi_1)}\times Z_1(i\beta_2\xi) + C\,Z_1(i\xi)\right\} \end{aligned}\right. \qquad (6.2\text{ b})$$

$$F_1 = \frac{h}{K\pi}\sin\frac{\pi z}{h}, \qquad F_2 = \frac{\pi T^{1/2}}{h}\cos\frac{\pi z}{h} \qquad (6.2\text{ c})$$

The as yet unspecified constant C in v_2 is determined by the continuity of v at $\xi = \xi_1$. Note that this term does not contribute to the dissipation.

Substituting Φ_1, Φ_2, s_1 and s_2 in eq. (3.11 a), neglecting the small terms involving β_1/β_2 and assuming that ξ_2 is large, we obtain

$$-\frac{1-\beta_1^2}{1+\alpha^2}\frac{H_1^1(i\beta_1\xi_1)}{iH_0^1(i\beta_1\xi_1)}\tanh\beta_1\Delta\xi = l \left\{\frac{\tau_1(\alpha\xi_1)}{\tau_0(\alpha\xi_1)} + \frac{\alpha i\tau_1(i\mu\xi_1)}{\mu\tau_0(i\mu\xi_1)}\right\} \qquad (6.3)$$

which is the counterpart of eq. (5.12 a).

The values of the first root of this equation, with the factor $(1-\beta_1^2)/1+\alpha^2$ omitted, are given in table 4 for various values of $\beta_1\Delta\xi$ and l. These results again show that $\alpha\xi_1$ decreases both with increasing $\beta_1\Delta\xi$ and with increasing l. Comparing with the values in table 2a we notice that here $\alpha\xi_1$ decreases more slowly as l increases. However, since for this case the area is proportional to ξ^2, we find that the ratio of the area of the ascending region to that of the descending region is descreasing more rapidly with l than for long rolls. For example, for $l = 3$ and $\beta_1 \Delta \xi = 1.5$, this ratio is 15 for long rolls and is 26 for circular cells.

Table 4. Values of $\alpha \xi_1$ as given by (6.2) for different values of $\beta_1 \Delta \xi$ and l.

$\beta_1 \Delta \xi$ \ l	1	2	3	5	10
∞	1.615	1.28	1.12	0.97	0.83
2.5	1.605	1.27	1.11	0.96	0.83
2.0	1.59	1.26	1.10	0.95	0.82
1.5	1.57	1.25	1.09	0.94	0.81
1.0	1.45	1.10	0.96	0.84	0.71
0.5	1.10	0.82	0.72	0.65	0.54

Another relation between R, α and ξ can be obtained by an application of the kinetic energy integral. The form of this equation is similar to that for the long rolls and it yields nearly the same numerical results and therefore will not be given here.

As has been mentioned at the end of the preceding section, when a deep unstable layer exists in the atmosphere, the actual R is usually many orders of magnitude larger than the critical value. Under such conditions not only the disturbance with wave number α_m, but a large band of disturbances of widely different scales may grow simultaneously; the motion is more likely to be turbulent. The upper limit of the horizontal scale of this band of amplifying perturbations is approximately given by $\alpha_0^2 = f^2/gS_1$. For $f = 5 \times 10^{-5}$ sec^{-1}, $S_1 = 5 \times 10^{-6}$ m^{-1}, we find $\alpha_0 = 0.007$. The radial dimension of the ascending region of this limiting disturbance is given in table 5 for various values of l. These values compare favorably with that of the rain areas of the tropical storms, which suggests that convective process is relevant to the dynamics of these storms.

Table 5. Radius of ascending region of the limiting perturbation.

l	1	2	3	5	10
$r_{1m}(km)$	362	288	252	218	187

Now we shall discuss the distribution of the various quantities in space as given by the solutions. For this purpose we shall choose two different examples, one is super-critical ($R \gg R_m$) and the second is nearly critical.

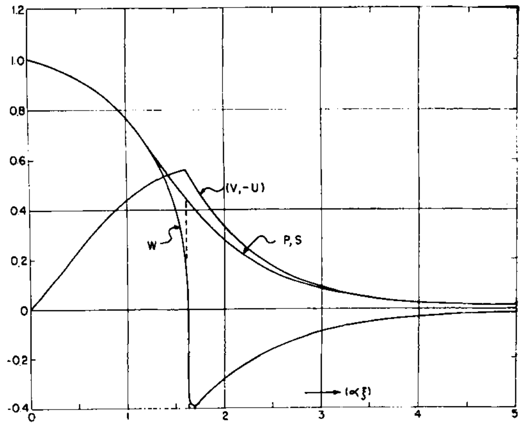

Fig. 4. Radial variations of the mid-level vertical velocity (W), the surface tangential wind (V), the surface radial wind ($-U$), the midlevel entropy (S) and surface pressure (P) perturbations for a disturbance in a super-critical stratification.

Example 1. $R = 1000$, $T = 100$, $\alpha = 0.31$,
$$\mu = 5.62$$
$$R_2 = R_1,\ l = 1,\ \beta_1 = 0.31,$$
$$\beta_2 = 31.66.$$

Since R is super-critical, α and μ are simply given by eq. (5.7) whereas β_1 and β_2 are given by eq. (5.10 a). For this case α/μ and β_1/β_2 are small and the second terms of the equations (6.1) and (6.2) are much smaller than the corresponding first terms except near the surface of discontinuity $\xi = \xi_1$. This is also true for all the large scale disturbances, particularly for the limiting disturbance $\alpha = \alpha_0 = f/\sqrt{gs_1}$. Since α^2 and β_1^2 are small, the root of eq. (5.12 a) is nearly the same as that of eq. (5.12 b), viz, $\alpha \xi_1 = 1.614$ for $\beta_1 \Delta \xi \gg 1.5$.

The radial variation of the mid-level ($z = h/2$) vertical velocity for this case is represented by the curve W in fig. 4, while the curve ($-U$, V) represents the ground level inward radial and tangential velocities. If we take $W_{(\xi=0)}$ as unity, then the unit of the radial velocity u is α^{-1} while that of the tangential velocity v is $T^{1/2}/\alpha$.

These curves show that for this case w changes rapidly across the surface $\xi = \xi_1$ while u and v are continuous, have their maxima occuring here and their radial derivatives are discontinuous. Since both u and v are proportional to $\cos \dfrac{\pi z}{h}$, the lower part

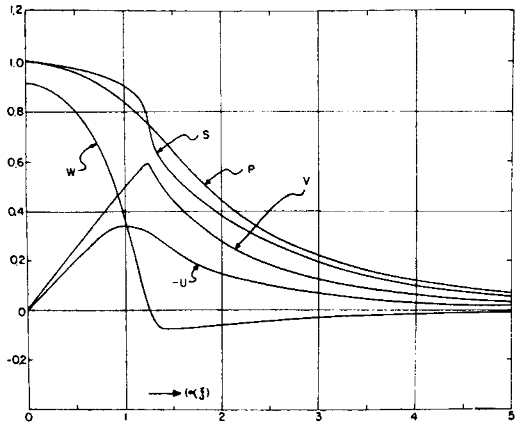

Fig. 5. Radial variations of W, V, $-U$, P and S for a disturbance in nearly critical stratification.

of the cell has an inward flow and a counter clockwise motion while the upper part has an outward flow and clockwise tangential motion. The vorticity is anticyclonic at all levels for large r.

The radial variation of s at the mid-level and that of the perturbation pressure at the top are represented by the curve marked with (P, S). Because of the smallness of α^2 and β^2, the pressure distribution is nearly hydrostatic and the two curves P, S coincide. Since p' is proportional to $\cos\frac{\pi z}{h}$, we have low pressure in the lower part and higher pressure in the upper part. On the other hand, the temperature departure is positive at all levels and all radial distances, with its maximum occurring at the mid-level and at the center.

It may be mentioned that when α and β_1 are small, the radial distributions are nearly the same as that given by the nonviscous solutions except very near the surface of discontinuity.

Example 2. $R = 16$, $T = 0$, $\alpha = \alpha_m = 0.707$; $\mu = 2$.

$R_2 = 64$, $l = 2$, $\beta_1 = 0.3535$, $\beta_2 = 8$.

For this case the root of eq. (5.12 b) is $\alpha\xi_1 = 1.28$. On recomputing from eq. (5.12 a) we find the corrected value of $\alpha\xi_1$ is 1.24, therefore the correction is quite small.

The radial variations of the various quantities are represented in fig. 5. Comparing with the fully drawn curve in fig. 3 for $l = 2$, we see that the present solutions deviate very much from the corresponding non-viscous solutions. For this case w decreases more gradually across the surface $\xi = \xi_1$, and U and V are represented by two different curves. The maximum radial velocity occurs inside $\xi = \xi_1$ while that of the tangential velocity occurs just on this surface. The pressure deviates appreciably from the hydrostatic pressure in the ascending region, therefore p and s are represented by two different curves. Note that s varies only slightly in the ascending region.

It may be pointed out that even though the solutions give different space variations for different α, all of them have the same overall character, that is the temperature departure is positive everywhere and the low level pressure is low compared to the undisturbed surroundings. Because these features bear so much resemblance to that of tropical storms, one is tempted to consider that the origin of the tropical storms is simply conditional instability, and to identify them with the "limiting perturbation" discussed above. However, since the most preferred wave number is α_m, which is much larger than α_0, some other physical processes must be operative in order to produce a distinct large scale system. In section 8 we shall discuss some such factors which favor a large scale circulation.

7. Plane forms and horizontal spacing

When the stratification is unstable everywhere, both the critical value R_m and the preferred horizontal wave number α_m are given by the vertical solution while the horizontal solution determines the plane form and the spacing of the cells. For the case of smooth bottom and top boundaries and very small T, $\alpha_m = 0.707$.

For long cloud rolls, the width λ of a cell, which is composed of two opposite vortex rolls, as is drawn in fig. 6 a, is

$$\lambda = 2\sqrt{2}\, h \tag{7.1}$$

It can easily be shown that the distance between the centers of two adjacent square cells is given by the same expression. For this case we have $F = \sin\frac{l\pi x}{h}\sin\frac{l\pi y}{h}$ and the arrangement of the cells are as shown in fig.

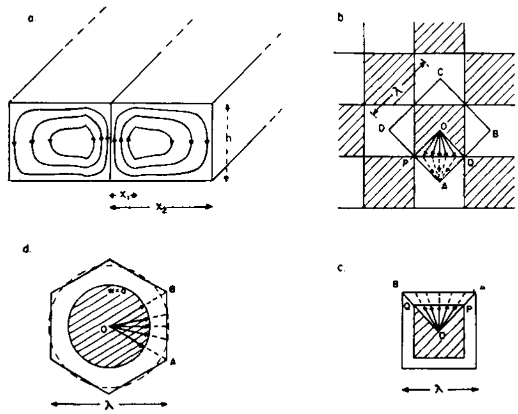

Fig. 6. a) Schematic illustration of streamlines in a cross-section of a long roll convection cell. b) Regions of ascending and descending motions in a square cell as given by the simple solution $F(x, y)$. c) Ascending and descending regions of a physically more probable square cell. d) Ascending and descending regions of a hexagonal cell.

6 b. It is seen that a unit cell ABCD as represented by this expression is composed of a square central region and four equilateral triangles. Since $l = {}^1/_2$ and $\lambda^\star = 2\ h/l = \sqrt{2}\lambda$, λ is again given by (7.1).

It may be remarked that even though the mathematical expressions for the square or rectangular cells represented in fig. 6 b are simple, this form of the cells seems to be dynamically inefficient, judging from the fact that the radial motion in the vertical planes *OP* and *OQ* must be very weak because the vertical velocity does not change sign.

On the other hand, the arrangement in fig. 6 c seems to represent a more efficient system. Here the counter motion takes place in the region between the two parallel squares whose sides are $\lambda/\sqrt{2}$ and λ. As is indicated in this figure, strong radial motion can take place in any meridional plane in this system. It is a little surprising to find that this cell form is not represented by any simple solution of eq. (4.1 a).

The other regular equilateral polygons that can fill up the whole space between them are triangles and hexagons. The mathematical expressions of these polygons have been obtained by CHRISTOPHERSON (1940) as solutions of eq. (4.1 a). From fig. 6 d it would seem that hexagon is a dynamically efficient configuration because the cell wall is nearly at the same distance from the boundary of the Central Core which separates the ascending and the descending regions. It is seen that the boundary of this central core is practically circular and that the cell wall itself never depart very much from a mean circle.

The distance between two hexagonal cells is

$$\lambda = \sqrt{3}\, L = 3.226\ h \qquad (7.2)$$

where L is the length of a side of the hexagon.

As to the question of the relative stability of the various plane forms in the presence of a mean current with vertical shear, the answer may be obtained by an investigation of the differential equation for nonviscous flow without Coriolis effect, which is given by

$$L^2 \left(\frac{\partial^2 w}{\partial z^2} - \sigma \frac{\partial w}{\partial z}\right) - (U_{zz} + \sigma U_z)\, L \frac{\partial w}{\partial x} + + (L^2 + g s_z) \nabla_1^2 w = 0 \qquad (7.3)$$

It can be shown that for this case the long rolls ($\partial/\partial x = 0$) are the most preferred form of the perturbation. Detailed discussion of this problem will be given in another paper.

Now we shall turn to the cases when the stratification is unstable for ascending motion only. Our problem is to determine the dimension of the preferred convection cells and of the ascending region from the results obtained in the two preceding sections. For these convection cells the effect of earth's rotation is negligible, we shall therefore take $T = 0$, $\alpha_m = 0.707$.

In determining R_m and α_m, it has been assumed that $\beta_1 \triangle \xi$ is large enough such that a further increase of its value has little effect on ξ_1. The results in tables 2a and 4 indicate that $\beta \triangle \xi = 2.0$ may be taken as the proper value for long roll clouds and $\beta \triangle \xi = 1.5$ for circular cells. Using these values for $\beta \triangle \xi$ we can then determine the dimensions of the ascending region and that of a unit cell from the values of $\alpha \xi_1$. The results for these two cases are given in tables 6 and 7, together with the respective area ratios of the descending to the ascending regions.

We note that while the area of the ascending region diminishes as l increases, the dimension of the cell as a whole becomes larger when the descending region is more stable. Thus for

$l = 5$, the distance between two cloud rolls is about 9 h. Take 4 km as the effective depth[1] of the unstable layer, we then obtain $2x_2 =$ 36 km. On the other hand, if the depth is only about 1 km, the horizontal spacing will be less than 10 km. These values are comparable with the spacing of the long roll clouds obtained from satellite observations (see KUETTNER, 1958, WINSTON, 1960). However, more detailed comparison with observations awaits detailed measurements of both h and l.

Table 6. Values of $2x_1/h$, A_2/A_1, and $2x_2/h$ for long rolls. $\alpha_m = 0.707$, $\beta\triangle\xi \geq 2.0$.

l	1	2	3	5	10
$\alpha\xi_1$	0.765	0.450	0.310	0.190	0.096
$2x_1/h$	0.69	0.41	0.28	0.17	0.087
$2x_2/h$	2.49	4.00	5.68	9.17	18.1
A_2/A_1	2.6	8.9	19.4	52.6	208.0

Table 7. Values of r_1/h, r_2/h and A_2/A_1, for circular cells. $\alpha_m = 0.707$, $\beta\triangle\xi = 1.5$.

l	1	2	3	5	10
$\alpha\xi_1$	1.57	1.25	1.09	0.94	0.81
r_1/h	0.71	0.56	0.49	0.42	0.36
$2r_2/h$	2.77	3.83	5.04	7.61	14.23
A_2/A_1	2.8	10.6	25.3	79.6	380.0

8. Factors favorable to development of large scale motions

As has been pointed out in section 6, the ascending area of the "limiting disturbance" is of the same order of magnitude as the rain area of the tropical storms and that the distributions of the various quantities given by our solutions bear many resemblances to that observed in tropical cyclones. It is therefore very tempting to identify these tropical storms with the limiting perturbation and to attribute their origin to simple convection. However, since the most preferred scale of motion in an unstable atmosphere is that of cumulus cloud, corresponding to $\alpha = \alpha_m$, the motion must be dominated by cumulus cloud scale convection if it evolves from random small-amplitude perturbations. Therefore the initiations of the much larger and very distinctive disturbances such as easterly waves and tropical depressions must be attributed to other physical factors other than simple convective instability, such as barotropic or baroclinic instability of the mean wind, or shearing instability. The evidence that the central parts of many tropical disturbances are cold before they transform into hurricanes (see RIEHL, 1954) lends support to the supposition that these perturbations are barotropic in origin, therefore they are totally unrelated to the present study.

However, some of these small amplitude traveling tropical disturbances do transform into deep tropical cyclones, and the energy required for such transformations are definitely supplied by the latent heat of condensation. In order that there shall be a gain of kinetic energy during ascent, it is evident that the stratification must be conditionally unstable. The question before us is: what are the conditions that must be satisfied in order these large scale convective systems shall develop and shall not be destroyed by the much stronger smaller scale systems which are most likely to develop in the unstable atmosphere? The following few points and conditions seem to be very significant in this respect:

(i) A deep layer of moist but unsaturated, conditionally unstable atmosphere ($\gamma_m < \gamma < \gamma_d$), which is stable with respect to small perturbations, but possesses enormous amount of energy that can be released by organized, finite amplitude perturbations.

(ii) Rapid removal of moisture by condensation and precipitation from the ascending air, so that the descending air is relatively dry, drier than the normal surface air.

(iii) A warm ocean surface which supplies heat and moisture to the lower atmosphere continuously but slowly. The lower atmosphere will remain unsaturated unless it has stayed a long time or traveled a long distance over the ocean. This condition will insure the growth of the large scale convective motion and avoid to have the cumulus scale convection from developing spontaneously. Furthermore, the small scale convection systems which do develop in the general ascending area of the large scale motion will cut off their ow nenergy

[1] The effective depth is usually larger than the depth of the unstable layer because of the penetration of convection into the stable layers.

supply by their descending motion, therefore will be short-lifed.

(iv) Pre-existence of a large-scale disturbance with weak convergence and divergence and weak vertical motion, and some kind of instability which initiates large scale convection. For example, when the absolute vorticity of the mean motion is negative, i.e., when $\left.\frac{\partial vr}{\gamma \partial r}\right)_\theta < -f$, it will definitely initiate large scale perturbations.

(v) Modification of the stratification in the general ascending region through upward transports of heat and water vapor by both large and small scale convections makes the lapse-rate in this region to approach the saturated adiabatic gradually. After this has happened, the sporadic cumulus scale convection will stop growing too rapidly. On the other hand the large scale motion will be maintained by the horizontal temperature gradient set up by the motion.

Since a quantitative study of the nature of the convective motion under these varying circumstances requires numerical integration, it must be left for further investigation.

REFERENCES

BJERKNES, J., 1938: Saturated ascent of air through a dry-adiabatically descending environment. *Quart. J. R. Meteor. Soc.*, **64**, pp. 325—330.

CHANDRASEKHAR, S., 1953: The instability of a layer of fluid heated below and subject to coriolis forces. *Proc. Roy. Soc. A*, **217**, pp. 306—327.

CHRISTOPHERSON, D. G., 1940: Note on the vibration of membranes. *Quart. J. Math.* **11**, pp. 63—65.

FRITZ, S., and WEXLER, H., 1960: Cloud pictures from Satellite Tiros 1. *Mon. Wea. Rev. U. S. W. B.* **88**, pp. 79—87.

HAQUE, S. M. A., 1952: The initiation of cyclonic circulation in a vertically unstable stagnant air mass. *Quart. J. R. Meteor. Soc.*, **78**, pp. 394—406.

JORDAN, C. L., 1958: Mean soundings for West Indies area. *J. Meteor.* **15**, pp. 91—97.

— 1959: *Mean soundings for the Gulf of Mexico area.* Rep. No. 20, National Hurricane Research Project, Miami, Florida.

KUETTNER, J., 1959: The band structure of the atmosphere. *Tellus* **11**, pp. 267—294.

KUO, H. L., 1960: *On convection and heat transfer.* Proc. I. C. N. W. P., Tokyo.

LILLY, D. K., 1960: *On the theory of disturbances in a conditionally unstable atmosphere. Mon. Wea. Rev., U. S. W. B.*, 88 (1), 1—17.

PETTERSSEN, S., 1939: Contributions to the theory of convection, *Geof. Pub. Vol.* **XII**, Oslo.

RIEHL, H., 1954: *Tropical meteorology.* McGraw Hill, New York.

SOBHAG, MAL, 1931: Forms of stratified clouds, *Beitr. Phys. fr. Atm.* **17**, pp. 40—68.

WINSTON, JAY S., 1960: Satellite pictures of a cut-off cyclone over the eastern Pacific. *Mon. Wea. Rev., U. S. W. B.* **88**, pp. 295—326.