On the problem of truncation error

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Abstract

It is argued that a more systematic study of truncation error is highly desirable. A number of integrations are commuted from the same partial differential equation using a variety of discretization schema. In particular, the difficulty of advecting sharp gradients of a conservative quantity is emphasized.

1. Introduction

This paper suggests the desirability of a systematic and, if possible, unified study of the finite-difference approximations to the partial differential equations commonly used in meteorological dynamics. Investigations of stability and accuracy of solutions of such difference equations are numerous in the literature, but most are studies of a particular approximation to a particular equation, with no attempt to contribute more general theorems, or even insights which might be applicable in other cases.

With a single exception, all examples computed in this paper are approximations to the one-dimensional linear conservative equation

$$\dot{\zeta} = \frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} = 0 \qquad U = \text{constant}, \qquad (1)$$

selected because of its very frequent occurrence in meteorological integrations. ζ may, of course, be any conservative quantity, but for convenience may be referred to as the vorticity. The solution of this equation is obviously

$$\zeta = f(x - Ut) \qquad (2)$$

with the initial condition

$$\zeta = f(x), \qquad t = 0$$

so that any initial ζ-pattern moves along with speed U without change in shape. It is the fact that the solution is exact for arbitrary initial conditions that is important here. The usual study of truncation error places emphasis upon the behavior of a single eigenfunction of a linear system, and although arbitrary initial states can be represented, the computations are seldom carried out. The techniques of the present paper are applied to the very important problem of forecasting the movement of sharp maxima and strong gradients. Such configurations are representable only by a large number of eigen-components, so that the conventional approach would require considerable computation. It will be noted that the accuracy of the results is well below the level one might anticipate from articles which emphasize only the stability of the computational schema and the preservation of integral invariants of the motion.

Boundary conditions are not considered, the solutions being valid on the infinite line. This seems justified, since the problems associated with boundaries have become less important with the larger grids used in most integrations.

It would be highly desirable to have comparable results in two space dimensions, but the techniques which would permit this are unknown to the author.

2. Differential-difference approximations

As a beginning, some insight is gained by the integration of differential-difference approximations, but owing to the sparcity of references to this technique in the meteorological literature, a few preliminary remarks are necessary. The use of differential-difference equations to approximate partial differential equations is in fact a classical procedure. A recent study by PINNEY (1958) contains a thirteen-page bibliography of references dating back to a paper by James Bernoulli in the early part of the 18th century. BATEMAN (1943) in a characteristically learned and elegant survey lists a variety of physical problems to which differential-difference equations have been applied. Among meteorologists, the first to bring the powerful classical results to bear on our current problems were apparently certain Soviet researchers, the most convenient reference in this connection being a paper by OBUKHOV (1957) which has been translated into English.

The simplest possible difference equation arises from the approximation of a derivative by a finite difference, say

$$\dot{u} \equiv \frac{du}{dt} = \frac{1}{2\tau}\left[u\,(t+\tau) - u\,(t-\tau)\right]. \qquad (3)$$

If this approximation is considered as an identity, what restrictions does this place on the function $u(t)$, or in other words, what class of functions is represented by all possible solutions of (3), treated as a differential-difference equation?

We note first that any second degree polynomial

$$u = a_0 + a_1 t + a_2 t^2$$

satisfies (1) identically. Further, an exponential

$$u = be^{st}$$

is also a solution provided that s is a root of the transcendental equation

$$\sinh s\tau = s\tau \qquad (4)$$

Thus the series

$$u(t) = a_0 + a_1 t + a_2 t^2 + \sum b_n e^{s_n t} \qquad (5)$$

is a general solution, where the s_n satisfy (4), and the b_n may be arbitrarily chosen. The class of functions which may be represented by such a series as (5) is an interesting and difficult mathematical problem. If the s_n are imaginary, this class corresponds to the almost-periodic functions; if they are real and negative, the series is a special case of a Dirichlet series. About these two classes of functions certain informative theorems exist, but in the present case, where the s_n are all complex, the problem is much more difficult and beyond the scope of this paper. Thus the simplest possible question one can ask concerning truncation of derivatives—what class of functions is represented by the differencing? — leads to profound and unsolved mathematical problems. For our purposes it is sufficient to note that if s_n is a solution of equation (4), so also is $-s_n$. Thus the series in (5) represents in general a function which is exponentially unbounded for $t \to \pm\infty$, although the original function u approximated by (3) may have been bounded over the entire t-axis. If half of the coefficients b_n are chosen zero, the solution u is then exponentially unbounded on one half-axis and exponentially vanishing on the other. If this type of result characterizes even the simplest finite difference approximation, it must be expected that in more complex finite-difference schemes, singularities and other undesirable properties of solutions may be introduced which are not present in the continuous representation of the infinitesimal calculus.

Another problem familiar to meteorologists is illustrated by a simple differential-difference equation: the dependence of the properties of the solution on the scheme of difference employed. If (3) is replaced by a forward difference,

$$\tau\dot{u}\,(t) = u(t+\tau) - u(t),$$

the equation corresponding to (4) is

$$1 + \tau s = e^{\tau s}$$

In this case there is only one root s_n with negative real part, all the rest having positive real parts. Thus a bounded function $u(t)$ can in general be represented only on the negative t-axis. The converse is true for solutions of the differential-difference equations arising from a backward difference, for which the equation corresponding to (5) is

$$1 - \tau s = e^{-\tau s}$$

An infinity of roots s_n exist with negative real parts, a single root with positive real part. Thus we may generalize, admittedly rather vaguely, that as approximation to $\dot{u}$ the forward and backward differences are mirror images of each other; as one solution becomes unbounded, the other approaches zero, and vice versa. The centered difference is symmetric in $\pm t$, and therefore is the more suitable approximation for solutions which should oscillate or maintain constant amplitude. Stated otherwise, the centered difference is a second order difference and thus exhibits a symmetry not derivable with one-sides, first order differencing. Perhaps it is possible to establish precise general principles as to when each approximation is best, but this has not been done, even for such simple equations as the above.

The example studied by Obukhov is the one-dimensional conservation equation (1). It is noted that if a centered difference be taken for the x-derivative,

$$\frac{\partial \zeta_k}{\partial x} = \frac{1}{2h}(\zeta_{k+1} - \zeta_{k-1}),$$

where

$$\zeta_k = \zeta(kh, t), \qquad k = 0, \pm 1 \pm 2, \ldots$$

the conservation equation becomes

$$\frac{d}{d\tau}\zeta_k = \frac{1}{2}(\zeta_{k-1} - \zeta_{k+1}), \tag{6}$$

where $\tau = Ut/h$ is a non-dimensional time variable. The fundamental solution for this differential-difference equation is

$$\zeta_k(\tau) = J_k(\tau),$$

where J_k is a Bessel function of the first kind. Since this solution at the initial time $t = 0$ is zero for every value of k except $k = 0$, for which it is unity, the fundamental solution may be generalized by superposition:

$$\zeta_k(\tau) = \sum_{k=-\infty}^{\infty} a_p J_{k-p}(\tau). \tag{7}$$

This satisfies the arbitrary initial field

$$\zeta_k(0) = a_k.$$

The solution $J_{k-p}(\tau)$ thus acts as a Green's or influence function. In so far as the basic requirement of conservation of total vorticity is concerned, this solution is satisfactory. If we write it as

$$\zeta_k(\tau) = \sum_{q=\infty}^{\infty} a_{k-q} J_q(\tau),$$

we have the total vorticity

$$\begin{aligned} Z(\tau) \equiv \sum_{k=-\infty}^{\infty} \zeta_k(\tau) &= \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a_{k-q} J_q(\tau) \\ &= \sum_{q=-\infty}^{\infty} J_q(\tau) \sum_{k=-\infty}^{\infty} a_{k-q} \\ &= Z_0 \sum_{q=-\infty}^{\infty} J_q(\tau), \end{aligned}$$

where Z_0 is the total initial vorticity. The well-known Laurent expansion for the Bessel coefficients is

$$e^{\frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)} = \sum_{q=-\infty}^{\infty} \alpha_q J_q(\tau),$$

and the result $Z \equiv Z_0$ follows upon setting $\alpha = 1$.

However, the vorticity *distribution* is not maintained as given initially, that is, the vorticity on each parcel is not advected with constant velocity, and it is this failure which was first pointed out by Obukhov. Obukhov computes the solution (7) for a given initial state represented by a unit step function at $t = 0$ and finds that a train of spurious waves follows the propagation of the disturbance. We may see this phenomenon, as well as another which suggests that Obukhov's conclusions are questionable, by taking the simpler initial field

$$\begin{aligned} \zeta(x, 0) &= 1, \qquad & |x| \le 3/2 \\ &= 0 & \text{otherwise.} \end{aligned}$$

This corresponds to a single maximum of vorticity being advected with velocity U, the continuous solution being

$$\begin{aligned} \zeta(x, t) &= 1, \qquad & |x - Ut| \le 3h/2 \\ &= 0, & \text{otherwise.} \end{aligned} \tag{7a}$$

The corresponding discrete initial field is

$$\begin{aligned} \zeta_k(0) &= a_k = 1, \qquad & k = -1, 0, 1 \\ &= 0 & \text{otherwise.} \end{aligned} \tag{7b}$$

It is evident that the discrete field (7 b) corresponds not only to the two step-functions (7 a) but to any distribution which satisfies the discrete values. Thus (7 b) may be thought of as a general representation of a sharp maximum with steep gradients on either side. Since the continuous conservation equation (1) merely requires the initial pattern to move without change in shape, no difficulties are encountered in the true solution. For simplicity, the figures are all drawn for the particular continuous initial condition (7 a).

Thus the solution (7) takes the form

$$\zeta_k(\tau) = \sum_{p=-1}^{1} J_{k-p}(\tau). \tag{8}$$

It is to be emphasized that although the particular initial field chosen makes for clarity in the discussion of the solution, all remarks are quite general and apply equally to more complex initial conditons.

We first note that the speed of propagation of the disturbance is approximately correct, since $J_k(\tau)$ has a maximum for k approximately equal to τ. Next we evaluate the solution (8) far behind the disturbance after a long time, through use of the asymptotic expansion of the Bessel function for large argument:

$$J_k(\tau) \cong \left(\frac{2}{\pi\tau}\right)^{\frac{1}{2}} \cos\left[\tau - \left(k + \frac{1}{2}\right)\pi\right] \tag{9}$$

In the solution (8), substituting from (9), the first and third Bessel functions cancel, giving the simple asymptotic solution (9), $\zeta_k(\tau) \cong J_k(\tau)$. This is clearly oscillatory in both k and τ, an example of Obukhov's "parasitic waves". Note that the amplitude falls off like $\tau^{-1/2}$. The existence of the oscillations we may associate with use of the centered difference.

However, this asymptotic solution is valid only in the region $k \ll \tau$, well behind the disturbance. To study the propagation of the disturbance itself one must effect an expansion of $J_k(\tau)$ in the region $k \doteq \tau$. This result was derived by WATSON (1948, pp. 245—248):

$$J_k(\tau) \cong \frac{1}{3\pi} \sin\left(\frac{\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(\frac{1}{6}\tau\right)^{-1/3},$$

$$|1 - k/\tau| \ll 1$$
$$6 \ll \tau$$

For this value we obtain the result

$$\zeta_k(\tau) \cong 0.775\,\tau^{-1/3}$$

The region of maximum vorticity is advected with the proper speed and its amplitude, instead of remaining constant, falls off like $\tau^{-1/3}$.

Consider now the same equation with a backward difference for $\partial\zeta/\partial x$:

$$\frac{d\zeta_k}{d\tau} = \zeta_{k-1} - \zeta_k. \tag{10}$$

Obukhov shows that this is also an approximation for the advection equation with diffusion,

$$\frac{\partial\zeta}{\partial t} = -U\frac{\partial\zeta}{\partial x} + \alpha\frac{\partial^2\zeta}{\partial x^2}$$

with a centered difference in x and $\alpha = hU/2$. We may note that (10) is the differential equation for the Poisson process, a time-dependent stochastic process which assumes that a constant probability Pdt exists for the occurence of a single event in the time interval dt. In the equation (10), $\zeta_k(\tau)$ is interpreted as the probability that exactly k events occur in the time interval τ, with probability $P = 1$ that exactly one event occurs in the time interval dt. (See for example, FELLER, 1957, pp. 400—402.)

Stochastic processes and diffusion are, of course, intimately related, but it is a little difficult to see their connection with conservation equation. The fact is that although we are entirely familiar with the use of a single partial differential equation to represent a variety of physical phenomena, the introduction of finite differences immensely expands this variety. It presumably behooves meteorologists to become familiar also with these new representations if they are to employ finite differences with maximum effectiveness.

The solution of (10), namely the Poisson distribution, is well known:

$$\zeta_k(\tau) = \frac{e^{-\tau}\tau^k}{k!} \qquad k \geq 0$$
$$= 0 \qquad k < 0,$$

with the initial value

$$\zeta_k(0) = 1, \qquad k = 0$$
$$= 0, \qquad k \neq 0.$$

The corresponding influence function is

$$e^{-\tau}\tau^{k-p}/(k-p)!$$

so that the solution for the arbitrary initial field

$$\zeta_k(0) = a_k$$

is

$$\zeta_k(\tau) = \sum_{p=-\infty}^{k} a_p e^{-\tau} \frac{\tau^{k-p}}{(k-p)!}$$

$$= e^{-\tau} \sum_{q=0}^{\infty} a_{k-q} \frac{\tau^q}{q!}.$$

As before, it can be shown that the total vorticity is conserved, and that the speed of propagation is approximately correct, since $e^{-\tau}\tau^q/q!$ has a maximum for $\tau \doteq q$.

For the single initial maximum of vorticity (7 b) this becomes

$$\zeta_k(\tau) = e^{-\tau}\left[\frac{\tau^{k-1}}{(k-1)!} + \frac{\tau^k}{k!} + \frac{\tau^{k+1}}{(k+1)!}\right]$$

$$= \frac{e^{-\tau}\tau^{k+1}}{(k+1)!}\left[1 + \frac{k+1}{\tau} + \frac{k(k+1)}{\tau^2}\right]. \quad (11)$$

As might have been expected with the use of a backward difference, the solution is monotonic in x, thus eliminating the "parasitic waves". On this basis, Obukhov finds it superior to the solution (8). However, it is equally important to note that for large τ the asymptotic value is $\frac{3e^{-\tau}\tau^{k+1}}{(k+1)!}$. This is $< \tau^{-1/2}$ where $k = \tau$, in contrast to $\tau^{-1/3}$ of the centered-difference solution. As a comparison we may compute the *maximum* value of the two solutions. For a wind speed $U = 20$ m/sec and a grid distance $h = 300$ km, the value $\tau = 6$ corresponds to 9.0×10^4 seconds, or a little more than one day. At the end of this time we have (from Poisson distribution tables)

$$[\zeta_k(6)]_{\max} \sim 0.48 \qquad \text{(backward difference)}$$

and (from tables of Bessel functions)

$$[\zeta_k(6)]_{\max} \sim 0.86 \qquad \text{(centered difference)}$$

Instead of conserving and advecting the vorticity maximum strictly with the velocity U, the centered difference approximation has dispersed a small portion of it into the spurious waves, but the backward-difference approximation has diffused away 52 per cent, both up- and down-stream. The exact solutions for three different times, $\tau = 5$ (about 21 hours), $\tau = 10$, and $\tau = 15$ are represented in Figure 1. Curves have been sketched through the discrete points as an aid to perspicuity.

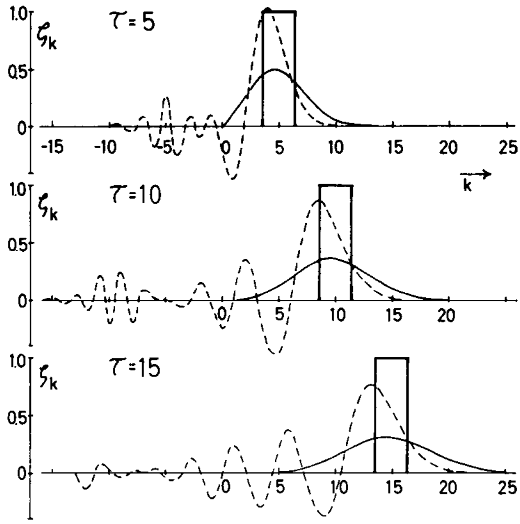

Fig. 1. Three solutions of the advection equation for (non-dimensional) times $\tau = 5, 10, 15$.
——— (exact) solution of continuous equation (4)
— — — — solution (8) of centered differential-difference equation
———— solution (11) of backward differential-difference equation
For typical meteorological values, ten units of non-dimensional time correspond to about 42 hours.

A glance suffices to show that neither finite-difference solution is a satisfactory approximation to the continuous solution. In Obukhov's example, the effect of the diffusion on reducing the maximum vorticity was not apparent, so that the choice between the differencing schemes seemed easier. Perhaps a more extended comparison is indicated, using more realistic wave-patterns.

The most frequently integrated equation of meteorology, the barotropic vorticity equation, expresses a conservation law, the final output being not the vorticity but the height of the pressure surface. The height is a twice-integrated function of the vorticity field, and as such may be expected to be a much smoother field.

This smoothing is particularly effective in eliminating the spurious oscillations contained in the solution (8). To see this it is convenient to take an initial vorticity pattern representing

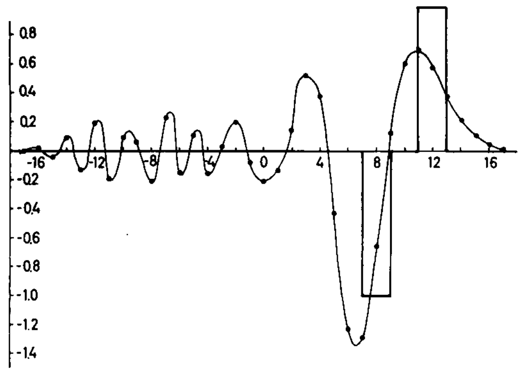

Fig. 2 a. Solution of the centered differential-difference equation contrasted with exact solution. The "vorticity" pattern.

exact solution
differential-difference solution

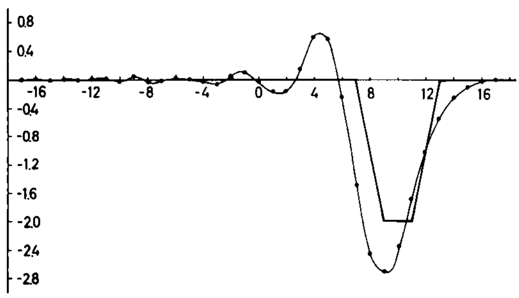

Fig. 2 b. The integrals of the solutions of Figure 2 a. The "velocity" pattern.

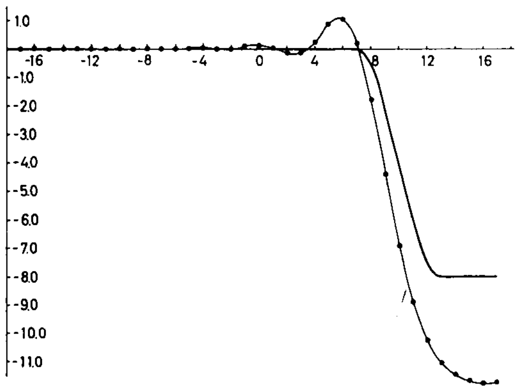

Fig. 2 c. The integrals of the function of Figure 2 b. The "height" pattern.

a sharp minimum. The solution was then integrated according to the simple formula

$$V_k = \sum_{p=\infty}^{k} \zeta_p,$$

and this in turn was integrated,

$$\psi_k = \sum_{p=-\infty}^{k} V_p.$$

The ζ, V, and ψ fields are plotted in Figures 2 a, b, c. As expected, the accuracy of the approximation increases markedly. If the finite difference approximation produces oscillatory errors, it is fortunate to be primarily interested in quantities which are space integrals of the quantities governed by the dynamic equations.

3. An application to the wave equation

To emphasize the importance of choosing a finite-difference scheme which conserves conservative quantities, we may examine an approximation to the wave equation,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}. \tag{12}$$

Since the wave equation is derivable from two equations similar to the advection equation,

$$\frac{\partial \xi}{\partial t} = c \frac{\partial \eta}{\partial x} \qquad \frac{\partial \eta}{\partial t} = c \frac{\partial \xi}{\partial x}$$

by elimination of either dependent variable, it is natural to expect solutions similar to the ones with which we have been dealing. This is the case. A centred difference for x yields

$$\frac{d^3 \xi_k}{d\tau^2} = \xi_{k+1} + \xi_{k-1} - 2\xi_k, \tag{13}$$

where $\tau = ct/h$, h being the x-difference as before. The fundamental solution of (13) is

$$\zeta_k(\tau) = J_{2k}(2t),$$

but the general solution is complicated by the fact that if the medium is unbounded in the x-direction, two initial fields are required to specify a unique solution, both $\dot{\xi}$ and its first time derivative. The corresponding solution of the wave equation (that of D'Alembert) is

$$\xi(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) + \frac{1}{2}\int_{x-ct}^{x+ct} g(s)\,ds \qquad (14)$$

where $\xi(x,0) = f(x)$ may be interpreted as the initial displacement field and $\dot{\xi}(x,0) = g(x)$ as the initial velocity field. If we assign the arbitrary conditions

$$\xi_k(0) = a_k$$
$$\dot{\xi}_k(0) = b_k,$$

the required solution of the differential-difference equation (13) is obtained as before by use of an influence function:

$$\xi_k(\tau) = \sum_{p=-\infty}^{\infty} a_p\, J_{2(k-p)}(2\tau) + \sum_{p=-\infty}^{\infty} b_p \int_0^{\tau} J_{2(k-p)}(2s)\,ds. \qquad (15)$$

This solution has apparently been known for half a century. It is attributed by BATEMAN to KOPPE (1901) in an obscure reference not easily available. The first term on the right hand side behaves like the advective solution (7), except as is easily seen, since $J_{2n}(\tau) = J_{-2n}(\tau)$ if a_k and b_k are even functions of k, then ξ_k is also an even function of k, This represents the propagation of the wave in both directions, corresponding to the continuous solution (14). It is however, the second term that concerns us now. Koppe noted that for large values of the time τ[1],

$$\int_0^{\tau} J_{2n}(2s)\,ds \cong 1,$$

which means that whereas the initial displacements a_k die out like $\tau^{-1/2}$, the initial velocities b_k are conserved in magnitude. From the point of view of the meteorologist this is a little frightening. A quantity such as the height of a constant pressure surface is much more easily and accurately measured than the vertical velocity of the parcels on the pressure surface. Thus it would seem possible for the solution after a long time to contain little other than error, the accurate displacement observations having been dispersed into spurious waves and the erroneous initial velocities carefully preserved.

[1] See Appendix for a proof of this expansion.

It is difficult to see how this source of truncation error could be eliminated. That it is a property of the finite differencing, not of the continuous solution, can be shown. Let us consider an example. The initial displacement field we maintain as before:

$$f(x) = 1 \qquad |x| \leq 3/2$$
$$= 0 \qquad \text{otherwise,}$$

with the corresponding discrete field

$$a_k = 1 \qquad k = -1, 0, 1$$
$$= 0 \qquad \text{otherwise.}$$

Now let the initial velocity field be identically zero, except in a small region of width 2ε at the origin:

$$g(x) = E \qquad |x| \leq \varepsilon$$
$$= 0 \qquad \text{otherwise.}$$

This may be interpreted as an instrumental error of magnitude E in the instrument or instruments measuring velocity at the observing station at the origin (assuming these stations to be evenly distributed at $k = 0, \pm 1, \pm 1, \pm 2, \ldots$):

$$b_k = E \qquad k = 0$$
$$= 0 \qquad \text{otherwise.}$$

D'Alembert's solution (14) gives for the displacement term (when $ct > 2$)

$$f(x-ct) + f(x+ct) = 1 \qquad |x \pm ct| \leq 3/2$$
$$= 0 \qquad \text{otherwise;} \qquad (16a)$$

and for the velocity term

$$\frac{1}{2}\int_{x+ct}^{x+ct} g(s)\,ds = \varepsilon E \qquad \varepsilon - ct < x < ct - \varepsilon$$
$$= 0 \qquad \text{otherwise,} \qquad (16b)$$

The continuous solution thus propagates an error proportional to ε; so that isolated errors do not contribute appreciably in the integration. But the discrete system is at the mercy of the errors of the isolated observing stations, and here the least accurate initial conditions are asymptotically dominating.

The solution (15) is easily computed for the above initial conditions. Since k takes only

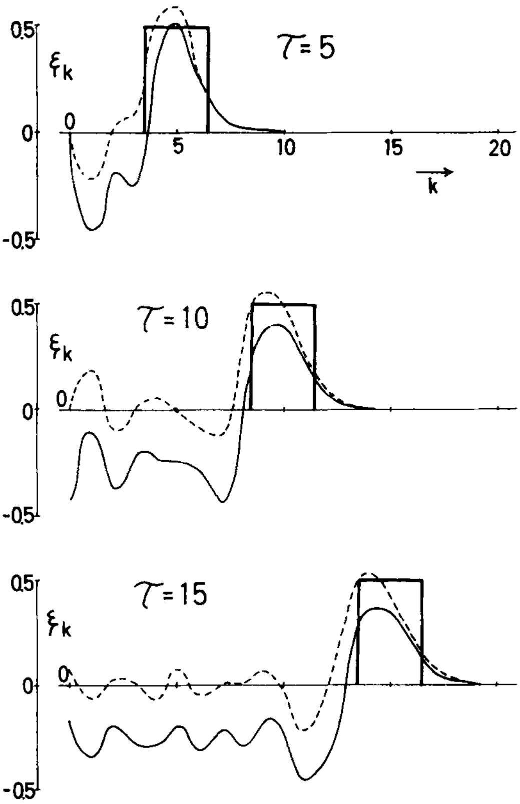

Fig. 3. Solutions of the wave equation for (non-dimensional times $\tau = 5, 10, 15$.
▬▬▬ (exact) solution (16) of continuous equation
— — — solution (17) of differential-difference equation without initial error
——— solution (17) of differential-difference equation with initial error at single isolated point $k = 0$.

integral values we have (WATSON, 1948, p. 545)

$$\int_0^\tau J_{2(k-p)}(2s)\,ds = \Big[\sum_{q=0}^{\infty} J_{2(k-p)+2q+1}(s)\Big]_0^{2\tau}$$

$$= \sum_{q=0}^{\infty} J_{2(k-p+q)+1}(2\tau).$$

The final solution is then

$$\xi_\tau(\tau) = \sum_{q=k-1}^{k+1} J_{2q}(2\tau) + E\sum_{q=0}^{\infty} J_{2(k+q)+1}(2\tau). \tag{17}$$

The continuous solution (16 a, b) and the solution (17) for $E=0$ and for $E={}^1/_2$ are plotted in Figure 3. The increasing relative contribution of the error E is apparent. In systems of equations such as those of atmospheric motion, which contain both conservation of some quantity and wave-propagation of another, this effect could be disastrous. It is a little difficult to say precisely what time scale is involved here. For the long external gravity waves with phase speeds of hundreds of meters per second, the time scale of, say, 10 non-dimensional units is only a few hours. If internal gravity waves are considered, with phase speeds of the order of the wind speed, the time scale is comparable to that of Figure 1.

4. Partial difference formulations

The more conventional approach to partial differential equations in meteorological computations is to approximate all derivatives by finite differences, a method yielding apparently superior results to that of the previous sections. It is well known that if the conservation equation (1) is approximated by a forward difference in time and a centered difference in x, the solution is unbounded as time increases. However, an interesting result is obtained by taking one-sided differences for each derivative forward in time and backward in space. If $x = k\,\Delta x$, $t = n\,\Delta t$, we have

$$\frac{\zeta_{k,n+1} - \zeta_{k,n}}{\Delta t} = -U\frac{\zeta_{k,n} - \zeta_{k-1,n}}{\Delta x}$$

or

$$\zeta_{k,n+1} = (1-\alpha)\zeta_{k,n} + \alpha\zeta_{k-1,n} \tag{18}$$

where $\alpha = U\,\Delta t/\Delta x$. As in example (10) of Section 2, in which a onesided derivative was used, the equation (18) may be given a probabilistic interpretation. If α is the probability of winning one unit at each trial, and $(1-\alpha)$ is the probability of winning no units, then $\zeta_{k,n}$ is the probability of possessing k units after n independent trials. It is to be expected, then, that the solutions of (18), viewed as an approximation to (1), will be bounded and stable for $0 < \alpha < 1$. This is easily shown to be true by conventional stability analysis. Further, since a probability cannot be negative, no unwanted oscillations will be present.

The classical solution to this Bernoulli problem is (e.g. FELLER, 1957, pp. 136—137)

$$\zeta_{k,n} = \binom{n}{k}(1-\alpha)^{n-k}\alpha^k,$$

which may be generalized for arbitrary initial conditions (that is, say, for various initial capitals of the gambler). If

$$\zeta_{k,0} = a_k$$

we have

$$\zeta_{k,n} = \sum_{j=-p}^{p} a_{-j}\binom{n}{k+j}(1-\alpha)^{n-k-j}\alpha^{1},$$

where p may be made as large as desired. For the particular initial condition (7b) the solution is

$$\zeta_{k,n} = \sum_{j=-1}^{1}\binom{n}{k+j}(1-\alpha)^{n-k-j}\alpha^{j}. \tag{19}$$

In a study of a similar solution in BLAIR et al. (1957), posthumously published (Appendix 3, by von Neumann, pp. 77—81), it is pointed out that for large n the DeMoivre-Laplace limit theorem is applicable here. The $\zeta_{k,n}$ of equation (19) may be interpreted as the probability that the number of wins lies between $k-1$ and $k+1$, and the solution is approximated by the normal distribution function (FELLER, 1957, p. 172)

$$\zeta_{k,n} \cong \Phi\left\{\frac{k+\frac{3}{2}-\alpha n}{[2n\alpha(1-\alpha)]^{1/2}}\right\} - \Phi\left\{\frac{k-\frac{3}{2}-\alpha n}{[2n\alpha(1-\alpha)]^{1/2}}\right\}, \tag{20}$$

where

$$\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

It is evident that this solution $\zeta_{k,n}$ has a maximum in the neighborhood of $k = n\alpha$, that is at $k \triangle x = U \triangle t$. The speed of propagation of the disturbance is thus correct. However, as von Neumann pointed out, the factor $[2n\alpha(1—\alpha)]^{1/2}$ is a measure of the dispersion of a sharp maximum after n time intervals, or a real time interval proportional to $n\alpha$. Therefore, the spurious diffusion is greater for α small.

These results seem consistent with those of Section 2 in which a one-sided differential-difference equation, amenable to interpretation as representing a stochastic process, diffused away some of the sharp maximum of the initial state. The analogy seems even closer if we regard equation (18) as representing a sort of random walk, with α the probability of a unit-step forward and $(1—\alpha)$ the probability of remaining stationary. This is evident if (18) is compared with the conventional random walk equation (e.g. FELLER, 1957, p. 318). Whereas the conventional random walk is, in the limit, a classical diffusion process, the solution of equation (18) is a forward translation and a backward diffusion.

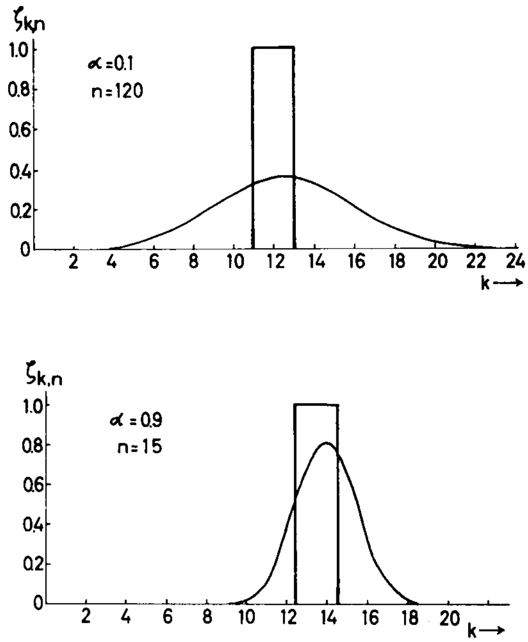

Fig. 4. Approximations according to (14) of solutions to the conservation equation with a forward time difference and backward space difference (18). (a) $\alpha = 0.1$, $n = 120$; (b) $\alpha = 0.9$, $n = 12$.

Two examples have been computed from (20) and are pictured in Figure 4. If U is taken as 20 m/sec and $\triangle x$ as 300 km, then time intervals $\triangle t$ of twenty-five minutes and 225 minutes, respectively correspond to $\alpha = 0.1$ and 0.9. In order to make the elapsed time comparable with the cases $\tau = 10$ in the previous figures (about 42 hours), the respective values $n = 120$ and $n = 12$ were selected. The marked difference in the extent of the diffusion is apparent. The parameter α must be less than but close to unity if the solution of the difference equation is to be a satisfactory approximation to the solution of the conversation equation.

We may now consider the more usual approximation to equation (1), which uses

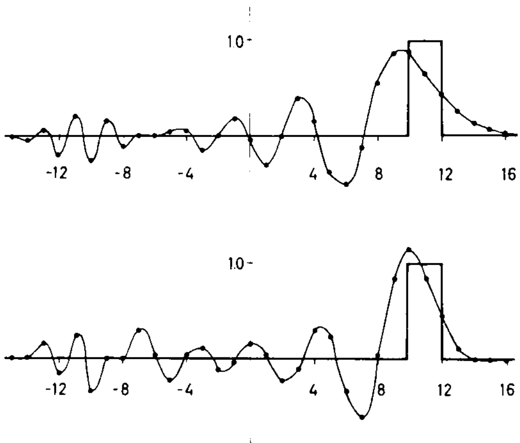

Fig. 5. Computed solutions of (21) with a forward difference for the first step. (a) $\alpha = 0.5$, $n = 23$; (b) $\alpha = 0.9$, $n = 13$.

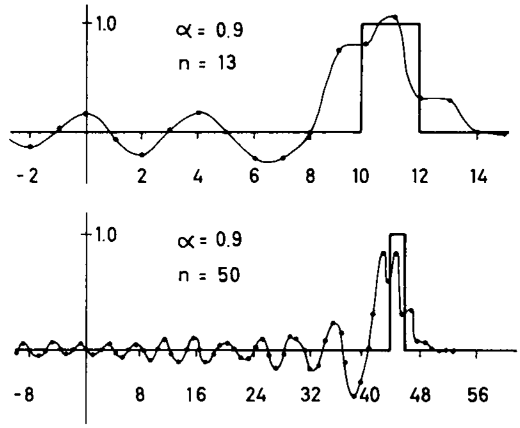

Fig. 7. Computed solution of (21), assigning true values at the first time step as a second initial condition. (a) $\alpha = 0.9$, $n = 13$. Cf. Figure 5 b. (b) $\alpha = 0.9$, $n = 50$. Cf. Figure 6 b.

centered differences for both time and space. The second-order partial difference equation is of the familiar form

$$\zeta_{k,n+1} = \zeta_{k,n-1} - \alpha(\zeta_{k+1,n} - \zeta_{k-1,n}), \quad (21)$$

where α has the same definition as before. As is well known, the solution is stable for $\alpha < 1$. A number of problems in connection with (21) have been discussed by GATES (1959, 1960), to which detailed reference will be made.

As with the centered differential-difference formulation, the solution admits both positive and negative values, and thus cannot represent the probability of a stochastic event. In fact,

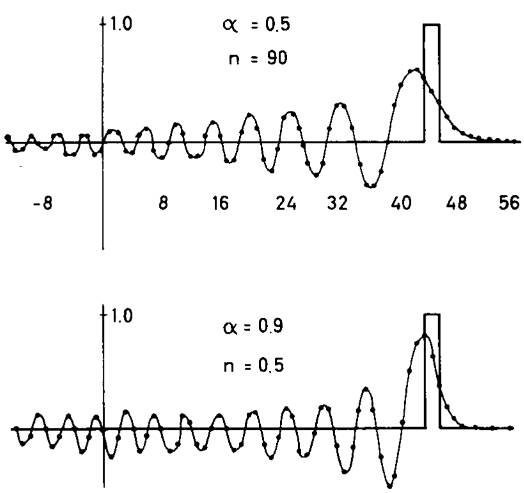

Fig. 6. The same as Figure 5 for (a) $\alpha = 0.5$, $n = 90$; $\alpha = 0.9$, $n = 50$.

(21) is formally the meaningless special case of the equation governing the stochastic birth-and-death process (FELLER, 1957, p. 407) if the probability of death is set equal to -1.

The fact that (21) is of an order higher than (18) in the time derivative requires an extra initial condition. The procedure used in numerical forecasting, dictated by the realities of synoptic observtions, has been to apply a forward time difference for the first time step, and to use equation (21) from the second step onward. Another possibility available to us here is to assign true values for the first time step as a second initial condition.

Unfortunately, an analytical expression for the solution of (21) is not possible without more computation than is required for a direct stepwise integration, and therefore the latter procedure has been followed. In Figure 5 are pictured the solutions corresponding to a total elapsed time comparable to previous examples, for the cases $\alpha = 0.5$ and $\alpha = 0.9$. In Figure 6 are the same solutions after almost four times as many time steps. Both of these computations used the conventional forward difference for the first time step. As an example of the result of using assigned true values for a second initial condition at the first step, Figure 7 pictures the solution corresponding to $\alpha = 0.9$, $n = 13$.

The spurious oscillations characterizing all solutions have been analyzed in detail by GATES (1959, pp. 12—15). The sharp maximum

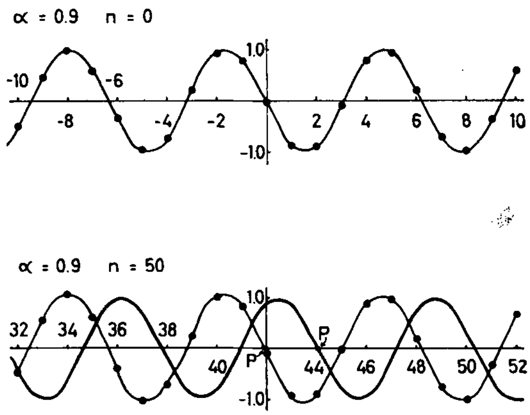

Fig. 8. The propagation according to (21) of an initially sinusoidally-distributed field. (a) $n = 0$. (b) $n = 50$; true solution, heavy line; computed solution, light line. The zeros labeled P and P' coincided initially.

of the present examples-although by no means unrealistic meteorologically, given conventional values of the grid distance—is composed of a large number of eigen-solutions of appreciable amplitude, and the spurious oscillations distort the solution much more seriously than would be the case with a single sine wave of wave length comparable in scale to that of the sharp maximum. It should be noted that this distortion is not appreciably reduced either by varying α nor by taking a second initial condition in place of the forward difference.

The satsifactory character of computations according to (21) when applied to a single eigen-function may be illustrated by a field the initial distribution of which is exactly sinusoidal. The boundary conditions become immediately operative in this case; but the results up to 50 time steps can be consulted at the center of the pattern, as yet uncontaminated by boundary effects. Figure 8 shows such a section. At five time steps the distortion in amplitude and phase speed would be still negligible on the scale of the diagram, and after 50 time steps the computation remains quite satisfactory, although the computed wave has fallen a little more than one-quarter wave length behind the true one.

The result is consistent with the analytic formulas derived by GATES (1959, pp. 12—13, 19—21) for the wave selected.

We may gain further insight by inspection of the actual Fourier series for a periodic

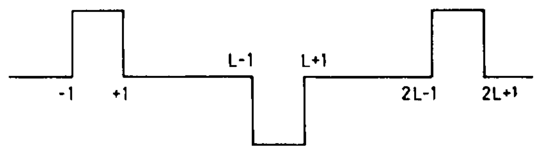

Fig. 9.

pattern of sharp maxima and minima, arranged as in Figure 9, in which the extremal regions are two units in width and are separated by L units. The series for this distribution is

$$\zeta(x) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\left[\frac{(2m+1)\pi}{L}\right] \cos\left[\frac{(2m+1)\pi}{L}x\right].$$

Now consider the case in which the separation L becomes large compared to the width of the maxima. For, say, $L = 20$, certainly not an unusual separation of sharp gradients on a cyclonic scale, the amplitude of the wave of length $L/5$ is about two-thirds that of the wave of length L, and the amplitude of the wave of length $L/10$ is one-third of the latter value. These short waves are approaching grid-mesh dimensions, and accuracy in their numerical treatment is not to be expected.

Mr. G. E. Birchfield of the University of Chicago has kindly allowed me to exhibit a computation (unpublished) of his which contains a spurious oscillation, apparently due to this effect. The computation was a two-dimensional conservation of vorticity, using centered time and space differences, from a hurricane forecast. Figure 10 represents the distribution

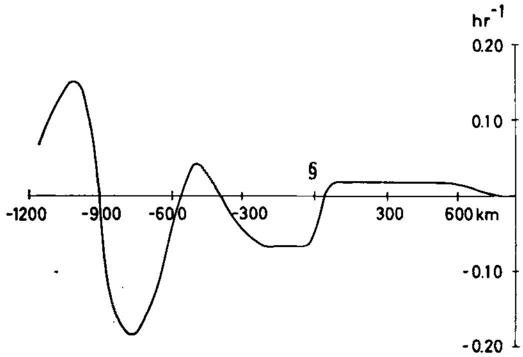

Fig. 10. Forecast vorticity distribution along direction of motion of a hurricane (after G. E. Birchfield) exhibiting spurious oscillations.

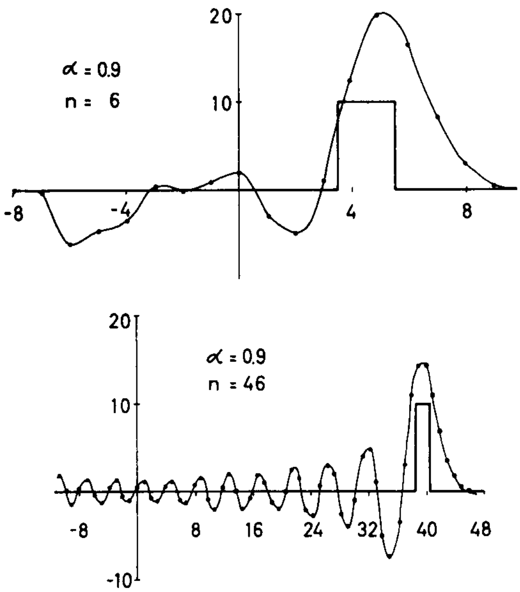

Fig. 11. Computed solutions of (21) using special starting procedure of Gates. (a) $n = 6$; (b) $n = 46$.

of vorticity along a line which is roughly the direction of movement of the system.

GATES (1960, pp. 4—12) has suggested a technique for reduction of this contaminating effect which calls for a special starting procedure. The first forward time difference is taken over a smaller interval $\Delta t/4$ the second step (a centered difference) over interval $\Delta t/2$, the third over Δt, and the fourth and all subsequent steps over $2\Delta t$. Applied to the example of the present paper, the results are pictured in Figure 11. The accuracy is not appreciably greater than that achieved by the previous scheme. Gates' conclusions are, of course, unexceptionable, but we see again the effect of a pattern which incorporates a large number of Fourier components of appreciable amplitude.

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Appendix

Let

$$I \equiv \int_0^\tau J_{2k}(2s)\,ds \qquad \text{(i)}$$

To obtain the asymptotic expansion for large τ, we express the Bessel function as an integral,

$$J_{2k}(2s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp.\, 2i(s \sin\varphi - k\varphi)\, d\varphi$$

substitute into (i), setting $\sigma = s/\tau$, and obtain

$$I = \frac{\tau}{2\pi}\int_0^1 \int_{-\pi}^{\pi} \exp.\, 2if(\sigma, \varphi)\, d\varphi\, d\sigma$$

where

$$f(\sigma, \varphi) = \tau\sigma \sin\varphi - k\varphi$$

We now apply the method of steepest descent:

$$\frac{\partial f}{\partial \sigma} = \tau \sin\varphi = 0 \qquad \text{for } \varphi = \varphi_0 = 0$$

$$\frac{\partial f}{\partial \varphi} = \tau\sigma\cos\varphi - k = 0 \text{ for } \sigma = \sigma_0 = k/\tau\cos\varphi_0 = k/\tau$$

Thus for $0 < k < \tau$ the saddle-point is on the path of integration, and for $\tau^{1/2} \gg 1$,

$$I \sim \frac{\tau}{2\pi}\int_0^1 \int_{-\pi}^{\pi} \exp\, 2i\tau(\sigma - \sigma_0)\varphi\, d\varphi\, d\sigma$$

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp.\, 2i\sigma'\varphi'\, d\varphi'\, d\sigma'$$

$$\sim \frac{1}{\pi}\left(\int_{-\infty}^{\infty} \exp. iu^2 du\right)^2$$

$$\sim \frac{1}{\pi}(\sqrt{\pi})^2$$

$$\sim 1$$

For $-\tau < k < 0$, the saddle-point $\varphi_0 = \pi$ may be used to obtain the same result. As $|k| \to \tau$ the value of the integral begins to fall off rapidly, and for $|k| > \tau$, the integral receives no contribution from the neighborhood of the saddle-point, so that its order of magnitude is $k^{-1/2}$.

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