On the Effect of Friction in Baroclinic Waves[1]

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(Manuscript received July 10, 1961)

Abstract

Pertubation equations for a two-layer model are used to study the effect of the frictiona boundary layer in waves on a baroclinic current. It is found, that the change of the propagation speed of waves due to friction is significant only within the waveband, for which amplification occurs. Compared with the frictionless case, this waveband is broader when friction is included. The frictional instability is explained by kinetic energy considerations.

1. Introduction

The atmosphere does work against the surface of the earth and thereby loses kinetic energy. In addition internal friction transforms kinetic energy into internal energy.

This dissipative effect of friction together with heating are factors which determine the long-term behaviour of the atmosphere. However, for short period numerical prediction a more important effect of friction is probably that it generally causes convergence and divergence near the earth surface, where the friction force is comparable with the pressure and coriolis forces. Through the vertical velocities thus generated, the frictional influence can be transferred to those parts of the atmosphere where local friction forces are negligible.

CHARNEY and ELIASSEN (1949) have studied the effects of the inclusion of the frictionally created vertical velocities into an equivalent barotropic model. In such a model the only influence of friction is to dissipate kinetic energy. However, in a baroclinic atmosphere this need not be true. The frictionally-created vertical motions can be associated with a thermal field in such a way that kinetic energy is released. This is the case for example in a tropical cyclone.

In this note, perturbation equations for a two-layer model are used to study the effect of friction in waves on a baroclinic current. The intention is to get a qualitative picture of the effect of incorporating friction into this model in some reasonable way.

2. The Model

Two adjacent layers of the atmosphere are considered, lying between pressure levels p_0, p_2 and p_4,

where $$p_0 - p_2 = p_2 - p_4 > 0.$$

The mean pressure levels in these ayers are indicated by indices 1 and 3. The pressure surface p_0 is assumed to coincide approximately with the top of the frictional layer, and p_4 must be such that at that level $\omega = \frac{dp}{dt}$ can be assumed zero. The equations for this model are (e.g. THOMPSON 1961)

[1] The research reported here has been sponsered in part by the Geophysics Research Directorate, AFCRL, AFRD, of the Air Research and Development Command, USAF, through its European Office.

$$\frac{\partial}{\partial t}\nabla_p^2 z + \frac{g}{f}J(z,\nabla^2 z) + \beta\frac{\partial z}{\partial x} - \frac{f^2}{g}\frac{\partial\omega}{\partial p} = 0 \qquad (1a)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial p}\right) + \frac{g}{f}J\left(z, \frac{\partial z}{\partial p}\right) - \sigma\omega = 0. \qquad (1b)$$

$$\sigma = \frac{\alpha}{g}\frac{\partial \ln\Theta}{\partial p}; J(z, z') = \frac{\partial z}{\partial x}\frac{\partial z'}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial z'}{\partial x}$$

As usual z is the height of the isobaric surface, g is the acceleration due to gravity, f is the coriolis parameter and $\beta = \frac{\partial f}{\partial y}$, which is assumed constant. Θ is the potential temperature and α the specific volume.

Eqs. (1a) and (1b) comprise a complete system of equations with two dependent variables z and ω.

3. Derivation of the phase-speed equation

The motion is assumed to consist of small perturbations superimposed on a basic current $U(p)$, a function of p alone. The perturbation field is assumed independent of y. Since it is implicitly assumed that the effect of dissipation on the basic current is exactly balanced by some external energy input, consideration of the effect of friction in our model may be confined to the perturbation field.

If z and ω now refer to the corresponding perturbation quantities, the following equations are obtained from (1):

$$\frac{\partial}{\partial t}\frac{\partial^2 z}{\partial x^2} + U\frac{\partial^3 z}{\partial x^3} + \beta\frac{\partial z}{\partial x} - \frac{f^2}{g}\frac{\partial\omega}{\partial p} = 0. \quad (2a)$$

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial p} + U\frac{\partial}{\partial x}\frac{\partial z}{\partial p} - \frac{dU}{dp}\frac{\partial z}{\partial x} - \sigma\omega = 0. \quad (2b)$$

The vorticity equation (2a) is applied to the levels 1 and 3, the thermodynamic equation (2b) at the level 2. The vertical derivatives $\frac{\partial z}{\partial p}$ and $\frac{\partial \omega}{\partial p}$ are estimated by finite differences. The stability factor σ is taken to be constant. At the level p_0, ω is assumed to have some value ω_0 dependent on the mass convergence or divergence in the frictional layer.

The boundary conditions are thus:

$\omega = \omega_0$	at the level 1.
$\omega = 0$	at the level 4.

The finite difference approximations to the p-derivative of ω become:

$$\frac{\partial\omega_3}{\partial p} = \frac{\omega_2}{\Delta p}; \frac{\partial\omega_1}{\partial p} = \frac{\omega_0 - \omega_2}{\Delta p},$$

where Δp is the pressure thickness of each of the two layers.

Instead of the heights z_1, z_3 and the velocities U_1, U_3 new variables are introduced, defined by:

$$\bar{z} = \frac{1}{2}(z_3 + z_1); \quad \bar{U} = \frac{1}{2}(U_3 + U_1)$$

$$z^\star = \frac{1}{2}(z_3 - z_1); \quad U^\star = \frac{1}{2}(U_3 - U_1).$$

$\bar{z}$ and $\bar{U}$ can be considered as the height and the basic current, respectively, at the pressure level p_2. To the same order of approximation, $z^\star$ is proportional to the temperature of the perturbation field and $U^\star$ to the vertical shear of the basic current, at this level.

With this notation the following equations are obtained:

$$\frac{\partial}{\partial t}\frac{\partial^2\bar{z}}{\partial x^2} = \bar{U}\frac{\partial}{\partial x}\frac{\partial^2\bar{z}}{\partial x^2} + U^\star\frac{\partial^2 z^\star}{\partial x^2} + \beta\frac{\partial\bar{z}}{\partial x} - \frac{1}{2}\frac{f^2}{g}\frac{\omega_0}{\Delta p} = 0 \qquad (3a)$$

$$\frac{\partial}{\partial t}\frac{\partial^2 z^\star}{\partial x^2} + \bar{U}\frac{\partial}{\partial x}\frac{\partial^2 z^\star}{\partial x^2} + U^\star\frac{\partial}{\partial x}\frac{\partial^2\bar{z}}{\partial x^2} + \beta\frac{\partial z^\star}{\partial x} - \frac{f^2}{g}\frac{\omega_2}{\Delta p} + \frac{1}{2}\frac{f^2}{g}\frac{\omega_0}{\Delta p} = 0. \qquad (3b)$$

$$\frac{\partial z^\star}{\partial t} + \bar{U}\frac{\partial z^\star}{\partial x} - U^\star\frac{\partial\bar{z}}{\partial x} + \sigma\omega_2\frac{\Delta p}{2} = 0. \quad (3c)$$

With the assumptions of the Ekman layer theory, the vertical velocity at the top of the friction layer has been found (e.g. CHARNEY and ELIASSEN, 1949) to be proportional to the geostrophic vorticity at that level:

$$w_0 = F\frac{\partial^2 z_0}{\partial x^2}, \text{ where } w = \frac{dz}{dt} \text{ and}$$

$$F = \frac{g}{f}\sin 2\gamma\sqrt{\frac{K}{2f}}$$

γ is the angle between the surface wind and isobars, and K is the eddy viscosity.

Since the total rate of change of pressure arises mainly from vertical motion, we may write:

$$\omega_0 = -\varrho_0 g w_0 = -\varrho_0 g F \frac{\partial^2 z_0}{\partial x^2}.$$

The quantity $\frac{\partial^2 z_0}{\partial x^2}$ is obtained by linear extrapolation:

$$\frac{\partial^2 z_0}{\partial x^2} = \frac{\partial^2 \bar{z}}{\partial x^2} - 2\frac{\partial^2 z^\star}{\partial x^2},$$

When the expression for ω_0 is introduced into eqs. (3) and ω_2 eliminated between (3b) and (3c), the following equations result

$$\frac{\partial}{\partial t}\frac{\partial^2 \bar{z}}{\partial x^2} + \bar{U}\frac{\partial^3 \bar{z}}{\partial x^3} + U^\star \frac{\partial^3 z^\star}{\partial x^3} + \beta\frac{\partial \bar{z}}{\partial x} + \\ + \frac{k}{2}\frac{\partial^2 \bar{z}}{\partial x^2} - k\frac{\partial^2 z^\star}{\partial x^2} = 0 \qquad (4a)$$

$$\frac{\partial}{\partial t}\frac{\partial^2 z^\star}{\partial x^2} + \bar{U}\frac{\partial^3 z^\star}{\partial x^3} + U^\star\frac{\partial^3 \bar{z}}{\partial x^3} + \beta\frac{\partial z^\star}{\partial x} - \\ - \mu^2\left(\frac{\partial z^\star}{\partial t} + \bar{U}\frac{\partial z^\star}{\partial x} - U^\star\frac{\partial^2 \bar{z}}{\partial z}\right) - \\ - \frac{k}{2}\frac{\partial^2 \bar{z}}{\partial x^2} + k\frac{\partial^2 z^\star}{\partial x^2} = 0, \qquad (4b)$$

where $\mu^2 = -\frac{2f^2}{\sigma g(\Delta p)^2}$ and the constant k contains F.

These equations are homogeneous linear differential equations in $\bar{z}$ and $z^\star$. An elementary solution of the form

$$\bar{z} = \bar{M}\cdot e^{i\alpha(x-ct)}$$

$$z^\star = M^\star \cdot e^{i\alpha(x-ct)}$$ may be sought.

α is a constant wave number and c is the phase-speed. Introducing these relationships into eqs. (4), we obtain:

$$\left\{i[\alpha^2(c-\bar{U})+\beta] - \frac{k\alpha}{2}\right\}\bar{z} - \{i\alpha^2 U^\star - k\alpha\}z^\star = 0 \qquad (5a)$$

$$\left\{iU^\star(\mu^2-\alpha^2) + \frac{k\alpha}{2}\right\}\bar{z} + \\ + \{i[(\alpha^2+\mu^2)(c-\bar{U})+\beta] - k\alpha\}z^\star = 0. \qquad (5b)$$

A non-zero solution for $\bar{z}$ and $z^\star$ exists only if the determinant of the coefficients vanishes. This condition gives the following quadratic equation for $c-\bar{U}$:

$$A_0\,(c-\bar{U})^2 + (A_1+iB_1)(c-\bar{U}) + \\ + (A_2+iB_2) = 0, \qquad (6)$$

where

$$A_0 = \alpha^2(\alpha^2+\mu^2)$$
$$A_1 = \beta(2\alpha^2+\mu^2)$$
$$A_2 = \frac{k\alpha}{2}(3\alpha^2+\mu^2)$$
$$B_1 = \alpha^2 U^{\star 2}(\mu^2-\alpha^2)+\beta^2$$
$$B_2 = -\frac{k\alpha}{2}[U^\star(2\mu^2-3\alpha^2)+3\beta]$$

The general solution to (4) can then be written as

$$\bar{z} = \sum_{\alpha=0}^{\infty}(\bar{M}_1 e^{ic_1\alpha t} + \bar{M}_2 e^{ic_2\alpha t})e^{i\alpha x} \qquad (7a)$$

$$z^\star = \sum_{\alpha=0}^{\infty}(M_1^\star e^{ic_1\alpha t} + M_2^\star e^{ic_2\alpha t})e^{i\alpha x} \qquad (7b)$$

Equation (6) may be written in a non-dimensional form in terms of $x = \frac{\alpha}{\mu}$, $S = \frac{U^\star}{\beta/\mu^2}$ and $\varkappa = \frac{k}{\beta/\mu}$. The folowing form results

$$\frac{C-\bar{U}}{\beta/\mu^2} = -a - ib \pm \sqrt{a^2 - d - b^2 + i(e-f)}, \qquad (8)$$

where

$$a = \frac{x^2}{2}\frac{2+x^2}{1+x^2}$$

$$b = \frac{\varkappa}{4}\frac{x(3+x^2)}{1+x^2}$$

$$d = \frac{x^4 + S^2(x^2-1)}{1+x^2}$$

$$e = \frac{\varkappa}{4}\frac{x^3(2+x^2)(3+x^2)}{(1+x^2)^2}$$

$$f = \frac{\varkappa}{2}x\cdot\frac{(2S+3)x^2-3S}{1+x^2}.$$

The real part of the phase velocity (8) is the speed of propagation of the perturbations, the imaginary part multiplied by the corresponding wave number is the growth rate of the wave.

4. The effect of friction

In Figure 1, the speed of propagation of waves, corresponding to $S=4$ and $\varkappa=1$, are given as a function of the wavelength X. This value of S has been chosen in order to demonstrate all the characteristic features of the frictionless case. This case has been discussed, for example, by ELIASSEN (1952).

The chosen value of $\varkappa$ corresponds to a typical eddy viscosity of the order of 10^5 cm^2 sec^{-1}.

If the imaginary part c_i of the phase speed is positive, the waves are growing. In this case, c_i is shown in Figure 1.

The characteristic features of the solution are:

a) *Frictionless case*

The waves consist of two components, which outside a specified band of wave numbers are neutral (constant amplitude) and have different speeds of propagation. Within the wavelength band, both components have the same velocity relative to the basic current, but one component is amplifying while the other is damped. This waveband is determined by the vertical shear of the basic current and the stability of the atmosphere.

It can be shown, that in the case of neutral waves, the geopotential and thermal fields are in phase or 180° out of phase, but for amplifying waves the thermal field lags behind the geopotential field. The physical explanation for such an amplification is that kinetic energy is produced by the solenoidal circulation set up by the associated thermal and ω-fields.

b) *Friction included*

The effect of friction on the propagation speed of waves is only markedly significant within the waveband, for which amplification occurs. For the longer amplifying waves in this band, friction results in propagation speeds closer to those for barotropic Rossby waves.

Compared with the frictionless case, the effect of friction reduces the amplitude of the perturbation field except within two narrow wavebands. These remarkable exceptions arise from a broadening of the waveband, for which amplification occurs. To wholly explain this phenomenon one has to study the three-dimensional structure of the waves described by the general solution (7) subject to certain initial conditions. However, a qualitative explanation can be obtained more easily by considering changes in the kinetic energy, as given by the model.

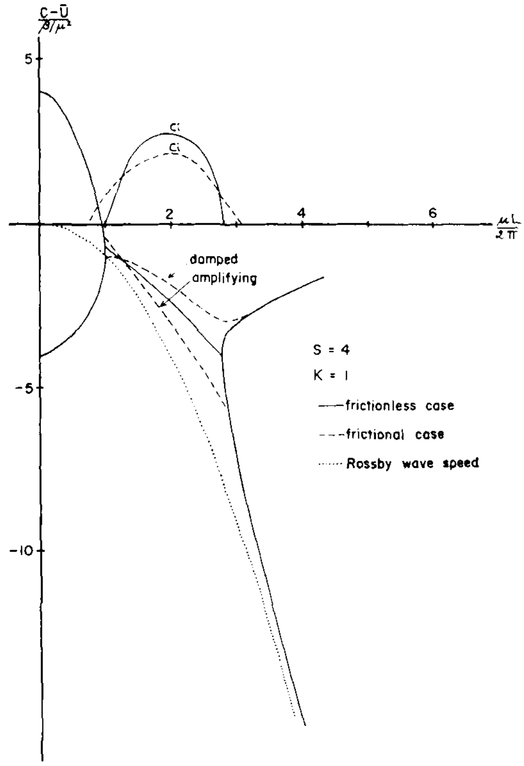

Fig. 1. The real part and the positive imaginary part of the phase speed $\frac{c-\bar{U}}{\beta/\mu^2}$ as a function of a dimensionless wavelength $\frac{\mu L}{2\pi}$.

5. Kinetic energy considerations

From the physical point of view, the instability of perturbations may be regarded as a consequence of energy conversion in which internal and potential energy of the atmosphere is transformed into kinetic energy of the perturbations.

The perturbation kinetic energy in our model can be expressed as

$$K = \frac{1}{2}(v_3^2 + v_1^2) = \bar{v}^2 + v^{\star 2} =$$

$$= \frac{g^2}{f^2}\left[\left(\frac{\partial \bar{z}}{\partial x}\right)^2 + \left(\frac{\partial z^\star}{\partial x}\right)^2\right],$$

where v is the perturbation (meridional) wind and $\bar{v} = {}^1/_2\,(v_3 + v_1)$, $v^\star = {}^1/_2\,(v_3 - v_1)$.

An expression for the change of the average kinetic energy will be derived. Firstly:

$$\frac{\partial K}{\partial t} = \frac{2g^2}{f^2}\left[\frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \bar{z}}{\partial x}\right) + \frac{\partial z^\star}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial z^\star}{\partial x}\right)\right].$$

Averaging over one whole wavelength yields

$$\widetilde{\frac{\partial K}{\partial t}} = -\frac{2g^2}{f^2}\left[\widetilde{\bar{z}\frac{\partial}{\partial t}\frac{\partial^2 \bar{z}}{\partial x^2}} + \widetilde{z^\star \frac{\partial}{\partial t}\frac{\partial^2 z^\star}{\partial x^2}}\right],$$

$$\text{where } \widetilde{(\quad)} = \frac{1}{L}\int_0^L (\quad)\, dx.$$

Application of eqs. (3) finally gives

$$\widetilde{\frac{\partial K}{\partial t}} = -4\frac{g^2}{f^2}U^\star \frac{\alpha^2\mu^2}{\alpha^2+\mu^2}\widetilde{\bar{z}\frac{\partial z^\star}{\partial x}} - \frac{g^2}{f^2}\alpha^2 k \cdot \widetilde{\bar{z}\,(\bar{z} - 2z^\star)}. \qquad (9)$$

The first term on the right hand side indicates the general condition for the rate of change of the average kinetic energy of the perturbations in a frictionless case. The term is positive when $z^\star$-field lags behind the z-field, and has a maximum when this lag is 90° (THOMPSON, 1959).

Considering the second term, the vorticity at the pressure surface p_0 was obtained by a linear extrapolation

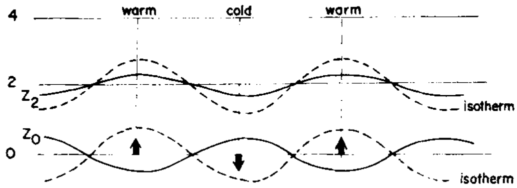

Fig. 2. Maximal growth of the kinetic energy due to the friction.

$$\frac{\partial^2 z_0}{\partial x^2} = \frac{\partial^2 \bar{z}}{\partial x^2} - 2\frac{\partial^2 z^\star}{\partial x^2}.$$

Consequently the z_0 and $(\bar{z} - 2z^\star)$ fields must be in phase. From (9) it follows that friction causes a decrease in the kinetic energy in the model, if the correlation between $\bar{z}$ and z_0 is positive, and a growth of the kinetic energy, if it is negative.

When using linear extrapolation, the quantity $\frac{\partial}{\partial p}\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2}{\partial x^2}\frac{\partial z}{\partial p} = -\frac{R}{pg}\frac{\partial^2 T}{\partial x^2}$ is assumed to be constant in the vertical. In other words the temperature field is assumed to have no tilt in the vertical.

The condition for maximum growth of kinetic energy due to friction is schematically shown in Figure 2. The phase difference between $\bar{z}$-field and z_0-field is 180°. The arrows indicate the direction of the vertical movement at the level p_0.

The figure explains qualitatively what structure of waves is required for frictional instability. Because the $\bar{z}$ and $z^\star$-fields are in-phase, the first term on the right hand side of equation (9) is zero. The frictionally created vertical motions cause an extra positive solenoidal circulation and the mean kinetic energy can grow.

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