A Study of the Dynamics of a Stratified Fluid in Relation to Atmospheric Motions and Physical Weather Prediction

By GEIRMUNDUR 'ARNASON

Bureau of Naval Weapons, Project NANWEP, U.S. Navy, Monterey, California, U.S.A.

(Manuscript received December 5, 1960, revised version February 28, 1961)

Abstract

The persistent skill of the 500 mb barotropic forecasts in routine numerical weather prediction is emphasized in contrast to the failure of baroclinic prediction models to furnish useful low-level forecasts. In the belief that this failure is first of all a result of lack of control mechanism capable of ensuring well-behaved development, the author suggests the possibility of designing a multi-level prediction model capable of realistic displacements of pressure systems but containing no mechanism for development.

For this purpose, a one-layer model consisting of a stratified fluid is exposed to theoretical analysis by means of the technique of small perturbations. This analysis shows that owing to the stratification of the fluid, the perturbations have a vertical structure resembling that of the atmosphere. The dynamics of the perturbations is studied in some detail. One of the results is an expression for wind divergence applicable to non-linear prediction. It is shown that the stratified model is capable of more realistic displacement of planetary waves than either version of the barotropic models (Rossby, 1939; Cressman, 1958).

More important than this improved control of the planetary waves is the model's apparent capability of realistic displacements of pressure systems at low levels. In application, however, there is the problem of how to best interpret concepts from the perturbation theory such as basic current, thickness of the layer, and the static stability of the basic flow.

1. Introduction

Since the inception of routine numerical weather prediction in 1955 by the Joint Numerical Weather Prediction Unit in Washington, D.C., the barotropic model, conceived by Rossby more than two decades ago, has played a role far beyond expectations. In its present form (Cressman, 1958), it yields wind forecasts for the 500 mb surface which in skill equal or surpass forecasts made routinely by any other method.

Baroclinic models that have been tested at JNWP and elsewhere have on the whole been disappointing, in that they have not produced forecasts for levels below 500 mb that could compete favorably with subjective predictions made by skilled forecasters. The lack of fulfillment of initial promises of some of the baroclinic models stands out sharply against the persistent skill of the barotropic model.

It is paradoxal that the usefulness of the barotropic model appears to hinge on the very fact that it makes no attempt to cope with the central problem in weather prediction – *the problem of cyclogenesis.* One of the difficulties, so far, with the use of baroclinic models has been the lack of a control mechanism capable of ensuring a well-behaved development. Predicted displacements of large-scale disturbances by baroclinic models are usually realistic, but erratic developments and distortion of existing pressure systems have, in many cases, made the forecasts of doubtful value.

In view of the considerable practical value of 500 mb barotropic forecasts and the obvious desire to produce forecasts of comparable skill for lower levels, we pose the question whether it would be feasible to design a prediction model *capable of realistic displacements of pressure systems at a number of levels but containing no mechanism for development.* It is assumed that displacement prediction alone is a useful product, and that a model of the type suggested may produce multi-level forecasts of a skill comparable to that of forecasts made by conventional methods.

An attempt is made in this paper to answer this question by proposing that a one-layer, stratified model may be adequate for the stated purpose. In the following sections we present a theoretical study of such a model by means of the perturbation technique and then discuss its applicability to hemispheric prediction of wind and pressure.

In section two of this paper we define the model as a single-layer, compressible fluid of table density stratification and bounded by two quasi-horizontal surfaces of constant pressure. The eigenvalue problem generated by the perturbation equations is solved in section three and the resulting frequency equation is discussed in section four. In section five we study the vertical structure and the dynamics of two-dimensional (x and p) perturbations and extend the discussion to three-dimensional (x, y, and p) perturbations; in section six the model's possible use for a multi-level, non-linear prediction is discussed.

Below we summarize the main results of the perturbation analysis and the tentative conclusions reached about the usefulness of the model for short-range (up to 36 hours) forecasting:

1) The frequency equation for the stratified model is cubic in the wave speed, c, and is essentially of the same form as that of the autobarotropic model with a free surface (Rossby, 1939). There are, however, two significant differences:
   a) Because of the inclusion of static stability, the speed of the internal gravity waves is of the same order of magnitude as the speed of large-scale pressure systems and is, therefore, much less than the speed of the external gravity waves of the autobarotropic model.
   b) The "synoptic" wave velocity (see eq. (6.1) p. 165) may have a sign opposite hat of non-divergent waves, depending on the static stability, the strength of the basic current, latitude, and the scale of the perturbation. This is contrary to the "synoptic" wave velocity (see eq. (4.4) p. 161) of the autobarotropic model with a free surface which always has the same sign as the velocity of non-divergent waves.
2) Because of the difference between thees two models, described above, the stratified model is better equipped to predict the observed movement of planetary waves than both the divergent (Cressman, 1958) and the non-divergent barotropic models. When applied to hemispheric 500 mb forecasts, this may result in increased usefulness for latitudes below approximately 35° N.
3) The basic vertical structure of the vertical velocity of the perturbations is nearly sinusoidal with m internal nodal points ($m=0, 1, 2, \ldots$). Superposition gives any required distribution compatible with the imposed vanishing of vertical speed at the lower and upper boundaries. This suggests that one may, with some confidence, apply the stratified model at a number of levels by properly interpreting such parameters as basic current, static stability and thickness of the stratified layer. This implies, for instance, that if one attempts a 1,000 mb forecast by this model, the basic current used would be a vertically integrated mean of the zonal wind, or of a wind obtained by horizontal averaging, i.e., a "steering" current.
4) The theoretical phase angle between the temperature and pressure waves is not typical of that most commonly observed in the atmosphere. The main reason for this is

the preclusion of baroclinicity from the basic flow, which prevents the occurrence of the synoptically important temperature advection behind and ahead of a pressure low. We have concluded, however, that correct phase angle is crucial first of all for prediction of developments and to a much lesser degree for the prediction of displacements.

5) The structure and dynamics of the perturbations of the model resemble the waves in the Easterlies as described by RIEHL (1954). We suggest that displacements of these disturbances might be predicted by the model described in this paper, possibly with some modifications necessitated by the low latitudes.

2. The model

We consider a single layer of a compressible, inviscid fluid of infinite lateral extent and in hydrostatic equilibrium. Contrary to the well known one-layer, autobarotropic models described by Rossby in the late thirties (ROSSBY 1939), we shall impose no restriction on the fluid's density distribution except for a stable stratification. As a result of the introduction of a variable density, the simple vertical structure of the autobarotropic one-layer models, with or without a free surface, is lost.

In describing the physics of the stratified model,[1] we prefer to replace the two equations for horizontal motion (the primitive equations of motion) by the vorticity and the divergence equations. The former is a prognostic equation for the vertical component of vorticity, ζ, and the latter a prognostic equation for the horizontal divergence, div $\mathbf{v}$; both are derived from the primitive equations of motion by differentiation. Using a x, y, p coordinate system, the five dependent variables u, v, ω, ϕ, and ϑ are determined by the following system of equations and appropriate boundary conditions:

Vorticity equation:

$$\frac{\partial \zeta}{\partial t} + \operatorname{div}(\eta \mathbf{v}) + \mathbf{k} \cdot \nabla \times \left(\omega \frac{\partial \mathbf{v}}{\partial p}\right) = 0 \quad (2.1)$$

[1] For the sake of easy reference we shall use the notation "the stratified model" when needed.

Divergence equation:

$$f\zeta - 2J(v, u) - \nabla^2\phi + v\frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y} - \frac{d}{dt}(\operatorname{div}\mathbf{v}) - (\operatorname{div}\mathbf{v})^2 - \operatorname{div}\left(\omega\frac{\partial \mathbf{v}}{\partial p}\right) = 0 \quad (2.2)$$

Continuity equation:

$$\frac{\partial \omega}{\partial p} + \operatorname{div}\mathbf{v} = 0 \quad (2.3)$$

Adiabatic equation:

$$\frac{\partial \vartheta}{\partial t} + \mathbf{v} \cdot \nabla\vartheta + \omega\frac{\partial \vartheta}{\partial p} = 0 \quad (2.4)$$

Hydrostatic equation:

$$\frac{\partial \phi}{\partial p} + F\vartheta = 0; \qquad F = \frac{R}{p}\left(\frac{p}{p_0}\right)^{R/c_p} \quad (2.5)$$

where u, v, ω are the three cartesianvelocity components, ϕ and ϑ geopotential and potential temperature respectively, f the coriolis parameter, and $\mathbf{k}$ a unit vector directed upward along the vertical. Moreover,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \text{ and } \operatorname{div}\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (2.6)$$

$$J(v, u) = \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} \quad (2.7)$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}, \qquad \frac{d}{dt} = \text{the total derivative}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors directed along the x- and y-axis, respectively; R is the gas constant and c_p the specific heat for constant pressure.

The specific purpose of this paper is to study the significance of a variable density alone as a single added feature to the above-mentioned autobarotropic models. We shall, however, not attempt a study of the stratified model in its most general form, as described by the system (2.1) to (2.7) above, but confine ourselves to the application of the theory of small harmonic perturbations superimposed upon a stationary and simple basic flow. To this basic flow we ascribe a uniform zonal current U[2] which is in a geostrophic and hydrostatic balance, i.e.,

$$fU + \frac{\partial \bar{\phi}}{\partial y} = 0 \qquad (2.8)$$

$$\frac{\partial \bar{\phi}}{\partial p} + F\bar{\vartheta} = 0 \qquad (2.9)$$

Since U is independent of pressure, it follows from (2.8) and (2.9) that $\frac{\partial \bar{\vartheta}}{\partial y} = 0$; i.e., the basic flow is *barotropic*.

The theory for small perturbations and its application is so well known from the meteorological literature (see for instance Bjerknes, et al., 1933), that the mathematical manipulations needed for deriving the perturbation equations are relegated to an appendix. We shall point out here that the perturbation amplitudes will depend on the vertical coordinate p, and that owing to the stratification of the basic flow, the small disturbances themselves are *baroclinic*.

The mathematical treatment is simplified for the time being by assuming independence of the meridional component y. Later on the effect of meridional dependence of the perturbations will be included.

3. The eigenvalue problem and its solution

The following differential equation for the amplitude $D(Z)$ of the vertical velocity $\omega\left(=\frac{dp}{dt}\right)$ was derived in the appendix:

$$D'' + D' - QD = 0 \qquad (3.1)$$

where a prime means differentiation with respect to $Z = \ln \frac{p_0}{p}$ and

$$Q = \frac{RH_m s\Delta}{(c-U)(C_i^2 - \Delta^2)} \qquad (3.2)$$

[2] A basic current U which is not zero is strictly incompatible with the assumptions that it be geostrophic and with the boundary conditions imposed on the fluid later on. This criticism applies, of course, equally well to the autobarotropic models. Inconsistency of this type is common when dealing with simple models and does not invalidate the essence of the results obtained.

Notations are explained in the appendix. The quantity Q is a constant for an isothermal atmosphere; its variation with height is slight in the real atmosphere and will therefore for the sake of convenience be omitted in solving (3.1).

For Q constant, the fundamental solutions of (3.1) are trigonometric or exponential; the general solution may be written as

$$D = e^{-\frac{1}{2}Z}\left[a_1 e^{r_1 Z} + a_2 e^{-r_1 Z}\right] = \left(\frac{p}{p_0}\right)^{1/2}\left[a_1\left(\frac{p_0}{p}\right)^{r_1} + a_2\left(\frac{p}{p_0}\right)^{r_1}\right] \qquad (3.3)$$

where a_1 and a_2 are arbitrary constants, and r_1 is the positive root of

$$r_1^2 = \frac{1}{4} + Q \qquad (3.4)$$

We distinguish between two possibilities: a) r_1 is real; b) r_1 is purely imaginary. In the latter case the solution (3.3) may be written:

$$D = e^{-\frac{1}{2}Z}\left[b_1 \sin rZ + b_2 \cos rZ\right] \qquad (3.5)$$

where

$$r^2 = -\left[\frac{1}{4} + Q\right] > 0 \qquad (3.6)$$

A general treatment of (3.3) is outside the scope of this paper. Proper boundary conditions allowing correctly for the kinematic constraints at the earth's surface and including an inner discontinuity (tropopause) lead to a complicated frequency equation which includes both external and intenal gravity waves in addition to inertia and Rossby waves. For the case of pure gravity waves, we refer to Bjerknes et al. (1933), Physikalishe Hydrodynamik, pp. 364—367, which deals with a double-layer isothermal atmosphere with a free surface.

We shall here confine ourselves to a single-layer fluid and preclude the external gravity waves by imposing the boundary condition $D = 0$ (i.e., $\omega = 0$) at $Z = 0$ and at $Z = Z_1$, where Z_1 is finite. The perturbations are therefore restricted to a layer of finite thickness. The solution to (3.1) compatible with these boundary conditions is

$$D = b_1 e^{-\frac{1}{2}Z} \sin rZ \tag{3.7}$$

with the side condition

$$(rZ_1)^2 = -\left[\frac{1}{4} + Q\right] Z_1^2 = (m\pi)^2; \quad m = 1, 2, 3, \ldots \tag{3.8}$$

Substitution for Q from (3.2) and rearrangement of terms in (3.8) results in the frequency equation

$$\Delta[g'H - (c-U)\Delta] + (c-U)C_i^2 = 0 \tag{3.9}$$

where

$$g' = \frac{RsZ_1}{g\left(m^2\pi^2 + \frac{1}{4}Z_1^2\right)} g = \frac{\varepsilon}{(m\pi)^2} \frac{1}{1 + \frac{Z_1^2}{4m^2\pi^2}} g \tag{3.10}$$

where

$$\varepsilon = \frac{RsZ_1}{g} \tag{3.11}$$

and

$$H = H_m Z_1 = \frac{RT_m}{g} Z_1 \tag{3.12}$$

In the atmosphere g' is very much smaller than g; for $s = 0.5 \; 10^{-2}$ degr. C/m and $Z_1 = 1.65$ ($p_1 = 200$ mb), we obtain $\varepsilon = \frac{RsZ_1}{g} = 0.23$, and for $m = 1$, $g' = 0.023$ g.

4. The frequency equation

Equation (3.9) is of third degree in c, i.e., for each eigenvalue, m, there are three possible solutions for the phase velocity of a given wave. It is of interest to compare (3.9) with the frequency equation for the one-layer, autobarotropic model with a free surface, so well known from the works of ROSSBY (1939) and subsequent writers who have found it useful in theoretical analysis. When the vorticity and divergence equations are used in deriving the frequency equation for this autobarotropic model, we obtain:

$$\Delta[gH - (c-U)\,\Delta] + cC_i^2 = 0 \text{ [3]} \tag{4.1}$$

which has the same form as (3.9) except that the term $(c-U)C_i^2$ is replaced by cC_i^2. As we shall see later on, this implies a significant difference in terms of the dynamics of the two models and their applicability to forecasting the movements of atmospheric pressure systems. Another important difference is that of g' in (3.9) replacing g in eq. (4.1). This means that internal gravity waves whose speed is $C'_g = \sqrt{g'H} =$ in the stratified model replace external gravity waves of the speed $C_g = \sqrt{gH}$ in the autobarotropic model. The expression for the speed of internal gravity waves

$$C'_g = \sqrt{g'H} = \frac{C_g}{m\pi}\sqrt{\frac{\varepsilon}{1 + \frac{Z_1^2}{4m\pi^2}}} \tag{4.3}$$

is, except for the factor $1 + \frac{Z_1^2}{4m^2\pi^2}$, of the same form as the one derived by BOLIN (1953) in his paper on "The Adjustment of a Non-balanced Velocity Field towards Geostrophic Equilibrium in a Stratified Fluid." As we have seen, 0.23 is a typical atmospheric value for the stability parameter ε. Since corresponding value for the oceans is about $2 \cdot 10^{-3}$ internal gravity waves travel 10 times faster in the troposphere than in the oceans.

It is customary, cf. CHARNEY (1947), to assume only one of the three roots of (4.1) to

[3] The frequency equation (4.1), see for instance CHARNEY (1947) eq. (31), is usually given as

$$\Delta[gH - (c-U)^2] + cC_i^2 = 0 \tag{4.2}$$

which differs from (4.1). This discrepancy owes its explanation to different ways of applying the equations of motion to the derivation of a frequency equation. Equation (4, 2) will, for instance, result when the vorticity equation is used in conjunction with one of the undifferentiated (primitive) equations of motion, whereas (4.1) is obtained when the latter is replaced by the divergence my equation. When discussing this apparent paradox with my colleague, George J. Haltiner, U.S. Naval Postgraduate School, California, he pointed out that the explanation must be sought in the inconsistency of allowing latitudinal variation in the coriolis parameter and at the same time imposing the restriction that both the basic current and the perturbations be independent of latitude. This inconsistency can be remedied only by extending the eigenvalue problem to two dimensions which, however, is outside the scope of this paper.

be meteorologically meaningful; this root is obtained by omitting $(c-U)\Delta$ compared with gH in (4.1) which gives,

$$c_a = \frac{gH}{gH + C_1^2}(U - \beta/k^2) \tag{4.4}$$

where the subscript "a" signifies an approximate solution to (4, 1). Since g' is much less than g, the validity of an analogous approximation for (3.9) is less certain but if we assume $\Delta/C_i \ll 1$, which means replacing the linearized divergence equation by the geostrophic wind law in the derivation of the frequency equation (3.9), the approximate solution is in this case

$$c'_a = U - \frac{g'H}{g'H + C_i^2}\beta/k^2 \tag{4.5}$$

We emphasize the difference in form between (4.4) and (4.5), and that the expression for c'_a is essentially the same as that for the real part of the phase velocity of amplified baroclinic waves, cf. THOMPSON (1956). For $U=0$, the frequency equations (3.9) and (4.1) are of the same form; this may be interpreted as follows: *the propagation of gravity waves in the two models is the same if the autobarotropic, one-layer model with a free surface has a thickness H_1 defined by*

$$H_1 = \frac{g'}{g}H \tag{4.6}$$

where H is the thickness of the stratified model.

As to the question of the meteorological significance of the roots of equations (3.9) and (4.1), and to what extent one of these roots is approximated by (4.5) and (4.4) respectively, the author has solved (4.1) exactly for the first three wave numbers. The wave number n is related to the wave length L and latitude φ as follows:

$$nL = 2\pi R\cos\varphi \tag{4.7}$$

where R is the radius of the earth. Fig. 1 shows the three roots c_1, c_2, and c_3 of (4.1) for $n=1$, $U=0$, and $H=10^4$, and for comparison the speed of non-divergent waves, c_R, has been entered together with the approximate solution c_a (given for $H=10_4$ and $H=10^2$ m). In view of the observation that waves corresponding to $n=1$ move so slowly that they appear quasi-stationary on successive weather maps, WOLFF (1958); MARTIN (1958), all three solutions to (4.1) are *meteorologically unrealistic.* The approximate solution, c_a, corresponding to $H=10^2$, appears more realistic save for very low latitudes. For the limiting case $H=0$, c_a approaches zero except at the equator which constitutes a mathematical singularity in that the value of c_a depends on the manner in which C_i and H approach zero.

When H in eq. (4.1) goes to zero, this equation becomes

$$c[C_i^2 - \Delta^2] = 0$$ [4]

which has the three roots $c=0$, $c=-\beta/k^2 \pm C_i$. The behavior of the three roots of (4.1) for different values of H is shown in Fig. 2. The solution c_1 approaches the value $-\beta/k^2 - C_i$ when H goes to zero and increases numerically with increasing H. The solution c_2 is positive and increases with increasing H; for $H\to 0$, it is zero for $0 \le \varphi \le \varphi_0$ and equal to $-\beta/k^2 + C_i$ for $\varphi_0 \le \varphi \le \frac{\pi}{2}$, where φ_0 is the latitude at which $C_i = \beta/k^2$. Thus the solution $c_2(\varphi)$ has a discontinuous first derivative for $\varphi=\varphi_0$. The third root, c_3, of eq. (4.1) is limited for all values of H; when H approaches infinity, c_3 approaches c_R, the speed of non-divergent waves, and when H goes to zero it approaches the value $-\beta/k^2 + C_i$ in the interval $0 \le \varphi \le \varphi_0$ and zero in the interval $\varphi_0 \le \varphi \le \frac{\pi}{2}$. In the point $\varphi=\varphi_0$, c_3 has therefore a discontinuous first derivative just as c_2 did. Fig. 2 gives c_3 as a function of latitude for $H=0$, 10^2, 10^3, $5\cdot 10^3$, and 10^4 m as well as for H infinite.

For increasing wave number, the approximate solution, c_a, approaches c_3 more and more, and for $n \ge 3$ these two are identical for all practical purposes if $H \ge 10^2$ m. In his paper "Der Mechanismus des Meteorologisches Lärmes", HINKELMANN (1951), discussed briefly the possibility of solving for c_3 by means of the continuous fraction

[4] The solution $\Delta = \pm C_i$ is possible only when $H=0$ in (4.1) or $g'=0$ in (3.9). It follows from the perturbation equations (see the appendix) that $C=0$, i.e., no pressure perturbations, and $B=\pm iA$. This is the case of combined inertia oscillations and vorticity oscillations.

$$\Delta_q = \frac{-C_i^2}{gH + C_i^2 - \Delta^2_{q-1} + \Delta_{q-1}\beta/k^2}(U - \beta/k^2) \tag{4.8}$$

where $\Delta_q = c_q - U + \beta/k^2$ and $\Delta_0 = 0$

Equation (4.8) may also be written as

$$c_q = \frac{gH - \Delta^2_{q-1} + \Delta_{q-1}\beta/k^2}{gH + C_i^2 - \Delta^2_{q-1} + \Delta_{q-1}\beta/k^2}(U - \beta/k^2) \tag{4.9}$$

which for $\Delta_{q-1} = 0$ reduces to (4.4). Computations show that for increasing q, c_q will approach c_3 but the convergence is slow for $n = 1$, in particular for low latitudes and small H.

As the wave number increases, the solution c_3 and its approximation c_a, appear to agree better with the wave speeds one observes by studying successive weather maps. The two remaining solutions c_1 and c_2, on the other hand, approach more and more the speed of pure gravity waves. The main features of Figures 1 and 2 apply in essence to any wave length. With decreasing wave length the point of intersection between the curves $c = 0$ and $c = -\beta/k^2 + C_i$ (Fig. 2) will move to the left and c_3 will approach $c = 0$, whereas c_2 and c_1 approach the horizontal curves $C_g = \pm\sqrt{gH}$. Table 1 gives the latitude φ_0 (where $\beta/k^2 = C_i$) in relation to the wave number n.

Table 1. Shows the latitude, φ_0, at which $\beta/k^2 = C_i$, in relation to the wave number n.

φ_0	n
38.2	1
24.5	2
17.6	3
13.7	4
11.0	5
9.3	6
8.0	7
7,1	8
6.3	9
5.7	10

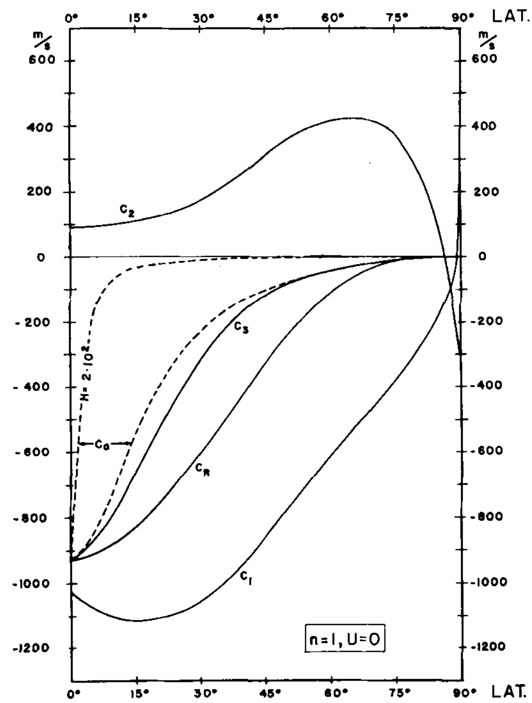

Fig. 1. Shows the three roots c_1, c_2, and c_3 of equation (4.1) for $H = 10^4$ m together with the speed c_R of non-divergent waves and the approximate solution c_a for $H = 10^4$ and $H = 10^2$ m.

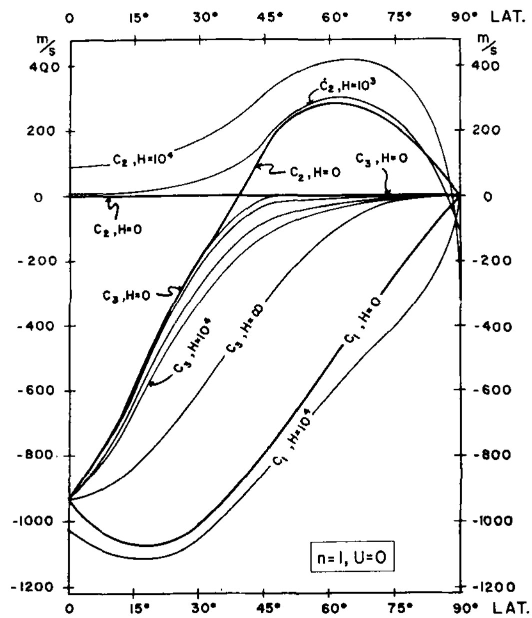

Fig. 2. The three solutions to eq. (4.1), c_1, c_2, and c_3 are shown as functions of latitude for selected values of H. The three thick, solid lines portray the solutions for the limiting case $H = 0$. In this case the point φ_0, 0, where φ_0 is the latitude at which $C_i = \beta/k^2$, is a branch point for c_2 and c_3 both of which have discontinuous first derivative. The H values that go with the unlabelled c_3 curves are: 10^2, 10^3, $5 \cdot 10^3$ m.

We emphasize at this point that the incapability of the ondimensional, barotropic modeleto move the very longest waves at speed comparable with those observed in the atmosphere, is closely related to the omission of lateral variation of the perturbations. As we shall see, however, this weakness of the model remains when it is extended to two dimensions. This extension is most easily accomplished by assuming the perturbations periodic in both the x and y directions. If we denote the lateral wave number by k_1, it is readily shown that previous results hold if we replace k^2 by $k^2+k_1^2$ in the expressions for Δ and C_i.[5] Thus,

$$\Delta = c - U + \beta/k^2 + k_1^2;\; C_1^2 = \frac{f^2}{k^2+k_1^2} \qquad (4.10)$$

$$c_a = \left(U - \frac{\beta}{k^2+k_1^2}\right)\frac{gH}{gH+C_i^2} \qquad (4.11)$$

A few case studies of wave $n=1$ for 500 mb show that the corresponding number in the North—South direction is about four. For $U=0$ and $H=10^4$ this corresponds to a westward speed c_3 of 55 m/sec at the Equator, decreasing to 37 m/sec at latitude 45°. *This speed is still excessive as compared with observed displacement in the atmosphere.* A further consequence of including the second dimension is that the approximate solution c_a is essentially identical with one of the roots (c_3) of (4.1) for all wave numbers.

What has been said above about the roots of equation (4.1) applies directly to the roots of equation (3.9) when we observe that

$$c' = U + c_{u=0} \qquad (4.12)$$

where $c_{u=0}$ is any of the three roots of (4, 1) for $U=0$, and c' is the corresponding root of (3.9). The limiting case $H=0$ in the previous discussion of the autobarotropic model corresponds to the condition $g'=0$ in the stratified model, i.e., indifferent stratification, a condition which, of course, is physically more meaningful than $H=0$. It is clear that for a typical atmospheric value of the static stability, the speed of planetary waves in the stratified mode is comparable to their observed speed in the atmosphere.

5. The structure and dynamics of the perturbations

In section 3 we found the following solution for the amplitude function of the vertical velocity:

$$D = b_1 e^{-\frac{1}{2}Z}\sin rZ \qquad (5.1)$$

where $r = m\pi/Z_1$, and $m = 1, 2, 3, \ldots$

Since D is known, the remaining amplitudes are readily derived by means of equations (8) to (12) in the appendix; we arrive at the following expressions for C and E, respectively:

$$C = b_2\left[-\frac{1}{2}\sin rZ + r\cos rZ\right]e^{\frac{1}{2}Z} \qquad (5.2)$$

$$E = -\frac{b_2}{R}\left[\frac{1}{4}+r^2\right]\sin rZ e^{\left(\frac{1}{2}+\frac{R}{c_p}\right)Z} \qquad (5.3)$$

where,

$$b_2 = -i\frac{g'H}{kp_0(c-U)}b_1 \quad (i^2=-1) \qquad (5.4)$$

It follows from (5.1), (5.2), and (5.4) that the ϕ and ω waves are 90° out of phase in the x-direction and that the ω wave is ahead when $c-U<0$, which is the case of greatest synoptic interest. Since E is proportional to the temperature amplitude, it follows from (5.2), (5.3), and (5.4) that the temperature wave is exactly out of phase with the wave for the geopotential. *Thus, near the lower boundary, pressure lows are warm and highs are cold; this is reversed in the upper half of the layer because of the change in sign of C with height.* This rather unrealistic pressure-temperature relationship has to be viewed against the absence of a meridional temperature gradient in the basic flow and against the lack of temperature advection that would otherwise be associated with such a gradient.

In order to get an expression for the horizontal wind divergence we differentiate (5.1) with respect to p and utilize the relation

$$\frac{d}{dp} = -\frac{1}{p_0}e^Z\frac{d}{dZ}; \text{ this gives}$$

[5] The derivation is particularly easy if we introduce a stream function and velocity potential into the perturbation equations instead of u and v.

If not $f/k \gg \beta/k^2 \frac{k_1/k}{1+(k_1/k)^2}$, the expansion to the second dimension is less simple than stated above.

$$\frac{dD}{dp} = -\frac{b_1}{p_0}\left[-\frac{1}{2}\sin rZ + r\cos rZ\right] e^{\frac{1}{2}Z} \qquad (5.5)$$

which is proportional to the wind divergence as follows from the continuity equation. We infer from (5, 5) that div $\mathbf{v}$ has a zero point below the midpoint of the layer bounded by two successive nodal points for ω. For $m=1$ this zero point is at 0.4 Z_1; for a layer extending from 1,000 to 200 mb, this point is close to 500 mb. Because of the exponential factor in (5.5), div $\mathbf{v}$ is numerically larger at the top than at the bottom. It follows from (5.2) and (5.5) that the amplitude for the geopotential reverses its sign at the same pressure div $\mathbf{v}$ does. Fig. 3 shows the vertical distribution of ω, div $\mathbf{v}$, ϕ, and the temperature T for $m=1$.

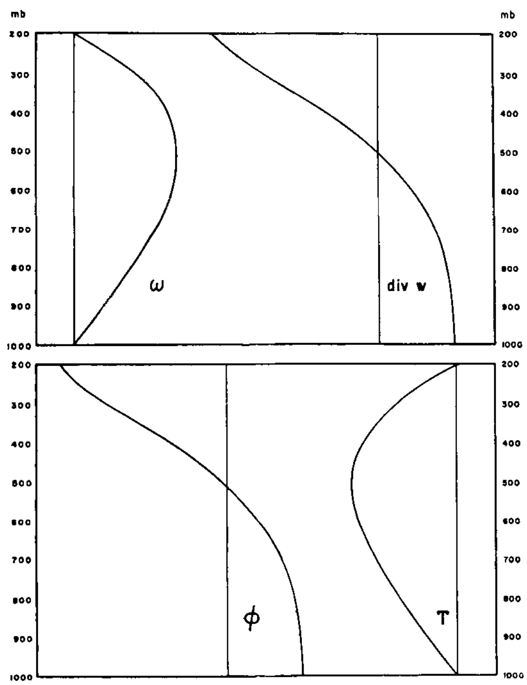

Fig. 3. Shows the vertical structure of ω, div $\mathbf{v}$, ϕ, and the temperature T in arbitrary units for $m=1$.

The amplitudes A and B for the wind components u and v are easily derived from (10) and (11) of the appendix; expressed in terms of C they are

$$A = -\frac{\Delta}{C_i^2 - \Delta^2} C = \frac{c-U}{g'H} C,$$
$$\text{or } u = \frac{c-U}{g'H}\phi \qquad (5.6)$$

$$B = \frac{C_i^2}{C_i^2-\Delta^2}\frac{ikC}{f}, \quad \text{or } v = \frac{C_i^2}{C_i^2-\Delta^2} v_g \qquad (5.7)$$

where v_g is the geostrophic counterpart to v.[6]

We have made use of eq. (3.9) in deriving the second expression for A in (5.6). Differentiation of (5.6) and (5.7) with respect to x gives

$$\frac{\partial u}{\partial x} = \operatorname{div} \mathbf{v} = \frac{c-U}{g'H}\frac{\partial \phi}{\partial x} \qquad (5.8)$$

and

$$\frac{\partial v}{\partial x} = \zeta = \frac{C_i^2}{C_i^2-\Delta^2}\zeta_g \qquad (5.9)$$

It is clear from the equations above that the extent to which the wind is geostrophic depends crucially on Δ, where,

$$\Delta = c - U + \beta/k^2 \qquad (5.10)$$

the smaller Δ is, the more geostrophic is the perturbation wind. By dividing (5.7) into (5.6) we obtain

$$\frac{\text{div}}{\zeta} = i\,\frac{\Delta}{C_i} \qquad (5.11)$$

The ratio Δ/C_i is dynamically an important number in that it expresses the degree to which the perturbation flow is solenoidal and to which degree it is potential and, which is much the same, the degree to which the perturbation wind is geostrophic. Figure 4 shows this ratio in relation to c_R and the three solutions c_1, c_2, c_3 to (4.1) corresponding to $H=0$

[6] It follows from (5.7) that the v-component is supergeostrophic and appears to approach infinity when Δ approaches C_i. We recall, however, that $\Delta = C_i$ (inertia motion) corresponds to $C=0$, i.e., there is no pressure wave associated with the field of motion. Consequently the u and v components must have indefinite amplitude if expressed in terms of pressure in accordance with (5.6) and (5.7). When one of the primitive equations of motion is used instead of the divergence equation in deriving the perturbation equations, v corresponding to the solution c_3 is subgeostrophic and (5.7) is replaced by

$$v = \frac{v_g}{1 - \frac{\Delta(c-U)}{C_i^2}}$$

see footnote on p. 160 and "Lectures on Physical Weather Prediction", p. 19, by Arnt Eliassen, University of California, Dept. of Met., 1955.

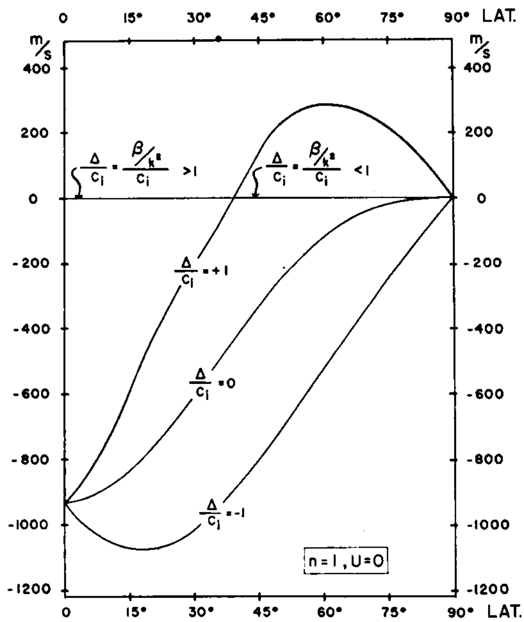

Fig. 4. Shows the curves c_R, and c_1, c_2, c_3 corresponding to $H=0$ (see Fig. 2) labelled in terms of the number Δ/C_i. Note that $\Delta/C_i > 1$ for c_2, $0 \le \Delta/C_i \le 1$ for c_3, and $\Delta/C_i \le -1$ for c_1.

(see also Fig. 2). It follows from Fig. 4, and what was said in conjunction with Fig. 2 in the previous section, that *for* c_1, $\Delta/C_i \le -1$; *for* c_2, $\Delta/C_i \ge 1$ *and for* c_3, $0 \le \Delta/C_i \le 1$. Recalling that c_3 did not give realistic displacements of the planetary waves below a certain latitude as compared with the atmosphere, see p. 163, *we conclude that in order to forecast properly the movement of very long waves at low latitudes we may have to allow values of the horizontal divergence that are in excess of the relative vorticity.* Observed behavior of the planetary waves in the atmosphere certainly implies that horizontal divergence has an important role in controlling their movement. A corollary to the above statement is that the windfield that is associated with a planetary pressure wave is both solenoidal and potential and *may therefore not be properly represented by a stream function.* A related question is whether the term $\frac{d}{dt}$ div $\mathbf{v}$ in the divergence equation is negligible compared with the term $f\zeta$ in the same equation, see eq. (2.2) and eq. (2) in the appendix. If not, a stream function derived by means of the balance equation would not be the proper stream function. Since the operator $\frac{d}{dt}$ corresponds to $-ik(c-U)$ in our case it follows from (5,11) that

$$\frac{\frac{d}{dt}(\operatorname{div} \mathbf{v})}{f\zeta} = \frac{c-U}{C_i}\frac{\Delta}{C_i} = \pm R_0 \qquad (5.12)$$

The above ratio varies greatly depending on which of the three solutions c_1, c_2, c_3 we are dealing with. For $c=c_3$ it is mostly much smaller than unity, except for low latitudes, and decreases with increasing wave-number n. For $c=c_2$ it is small for the combination low latitudes and small H (or small ε), but exceeds unity with increasing latitude and increasing H (or increasing ε). It is numerically greater than unity for $c=c_1$. It may be added that the ratios (5.11) and (5.12) occur frequently in the meteorological literature as useful concepts in dynamical analysis. CHARNEY (1955), for instance, distinguishes ocean waves from ocean currents by ascribing $\frac{\text{div}}{\zeta} \ll 1$ to currents and $\frac{\text{div}}{\zeta} \gg 1$ to waves. BURGER (1958) finds from scale considerations that the ratio (5.11) may exceed unity for the planetary waves, even if the Rossby number, R_0, is less than unity. We point out that the ratio (5.12) is essentially a Rossby number, although not expressed in its most conventional form.

It has been shown on the previous pages that none of the three solutions to the cubic frequency equation (3.9) agrees fully with observed speeds of disturbances for all wave lengths and all latitudes. Apparently the root c_1 can be discarded as not reflecting displacements typical of the atmosphere; but a combination of c_2 and c_3 would seem capable of realistic displacement at all latitudes and all wave lengths including the planetary waves.

6. Possible application to weather prediction

In this section we shall discuss the possibility of applying the stratified model to weather prediction on the assumption that the approximate solution to the frequency eq. (3.9), see also eq. (4.11),

$$c_a = U - \frac{\beta}{k^2+k_1^2}\frac{g'H}{g'H+C_i^2} \qquad (6.1)$$

represents dynamics that is meteorologically meaningful. For the most part c_a coincides with one of the three roots (c_3) of (3.9), but at very low latitudes these two may differ. In this case the adoption of (6.1) implies a solution to the forecasting problem which includes all three roots of (3.9) in accordance with the principle of superposition, and whose divergence and vorticity are identical with those described implicitly by eq. (6.1).[7]

By means of (6.1) we derive the following expressions:

$$c - U = -\frac{\beta}{k^2 + k_1^2}\frac{g'H}{g'H + C_i^2} \tag{6.2}$$

$$\Delta = \frac{\beta}{k^2 + k_1^2}\frac{C_i^2}{g'H + C_i^2} \tag{6.3}$$

Utilizing (6.2) and (6.3), equations (5.11), (5.12) and (5.8) may be written as:

$$\left|\frac{\operatorname{div}\mathbf{v}}{\zeta}\right| = \frac{\Delta}{C_1} = \frac{C_i\dfrac{\beta}{k^2+k_1^2}}{g'H + C_i^2} \tag{6.4}$$

$$R_0 = \left|\frac{c-U}{C_i}\frac{\Delta}{C_i}\right| = \frac{g'H}{g'H + C_i^2}\frac{\left(\dfrac{\beta}{k^2+k_1^2}\right)^2}{g'H + C_i^2}. \tag{6.5}$$

$$\operatorname{div}\mathbf{v} = \frac{c-U}{g'H}\frac{\partial\phi}{\partial x} = -\frac{\beta}{k^2+k_1^2}\frac{\dfrac{\partial\phi}{\partial x}}{g'H + C_i^2} \tag{6.6}$$

where C_i is defined by (4.10) and g' by (3.10); R_0 is the Rossby number. We have omitted the subscript "a" from c and Δ as superfluous from here on. An alternative form for (6.6) is

$$\operatorname{div}\mathbf{v} = \frac{C_i^2}{C_i^2 + g'H}(\operatorname{div}\mathbf{v})_g \tag{6.7}$$

where $(\operatorname{div}\mathbf{v})_g$ is the divergence of the geostrophic wind. Thus, the horizontal wind divergence compatible with (6.1) *is proportional to the geostrophic wind divergence.* The proportionality factor depends on: a) scale of the disturbance, b) latitude, c) static stability, d) the thickness of the layer penetrated by the perturbation.

The equations (6.1) to (6.7) above represent the main dynamic characteristics of the stratified model as it now stands. These may be summarized as follows:

1. The westward displacement of a disturbance, due to the gradient of the earth's vorticity, f, is always less than in the non-divergent, barotropic model and for $U > 0$ is also less than in the divergent barotropic model.
2. There is always wind convergence east of a trough $\left(\dfrac{\partial\phi}{\partial x} > 0\right)$ and wind divergence west of it $\left(\dfrac{\partial\phi}{\partial x} < 0\right)$; *this is independent of the direction in which the wave moves.*
3. The ratio $\dfrac{\operatorname{div}\mathbf{v}}{\zeta}$ is mostly $\ll 1$; this is particularly true for short waves occurring at high and middle latitudes. At low latitudes, the horizontal divergence associated with the planetary waves becomes an appreciable portion of the relative vorticity; consequently both stream function and velocity potential may be needed for adequate representation of the wind.
4. The Rossby number is mostly smaller than the ratio div $\mathbf{v}/\zeta$ and therefore generally $\ll 1$. As a result one may use the balance equation, i.e., the divergence equation without the $\dfrac{d}{dt}$ div $\mathbf{v}$ term, for deriving a stream function.[8]

To the extent that we can observe the dynamics of large-scale disturbances in the lower half of the troposphere, they show characteristics similar to those listed above. This is best illustrated by the model's ω-distribution along the vertical which greatly resembles that of the atmosphere as compared with the trivial ω-distribution of the autobarotropic models. On the other hand, the temperature wave of the stratified model is more out of

[7] Construction of a general solution compatible with the three roots of the cubic frequency equation (3.9) and satisfying given initial conditions is described by HINKELMANN (1951).

[8] We note that (6.1) is the only solution to the systems of perturbations (1) to (5) in the appendix if the divergence equation is replaced by the geostrophic wind law.

phase with the pressure wave than we generally observe in the atmosphere and for this reason the vertical structure of the pressure wave is somewhat unrealistic. We infer that the stratified model can be useful in predicting displacements of atmospheric pressure systems *only insofar as the phase between pressure and temperature waves is not crucial for their movements.* Linear theory of baroclinic models capable of development shows that this phase is crucial for the development of waves but not for their displacements.

According to the description of waves in the Easterlies by RIEHL (1954, pp. 210—234), these disturbances seem to resemble the perturbations of the stratified model. The following characteristics are attributed to the waves in the Easterlies by Riehl:

a) An average westward speed slightly in excess of the basic flow.
b) The trough line is preceded by a wind divergence; and convergence, associated with precipitation, occurs to the rear of the trough.
c) The wave penetrates the layer 1,000—300 mb and has a maximum amplitude between 700 and 500 mb.
d) Near the ground, the east component of the wind is strong in the ridges but it weakens progressively toward a line a short distance east of the trough line (fig. 9.3 in Riehl's book).

These features are qualitatively in agreement with those of the perturbations of the model. For an easterly basic current (negative U), equation (6.1) gives a wave speed toward west in excess of U and b) and d) are in agreement with point 2 on page 166, and with the expression (5.6) for u when it is combined with (6.2). The vertical structure of the Easterly waves, as described in c), deviates, however, from that of a single wave of the model. A curious observation reported by PALMER (1952) is that equatorial waves appear to reverse their phase somewhere between 5,000 and 25,000 feet. Although the structure and dynamics of waves occurring at low levels in the Tropics are incompletely known, it is suggested that their displacements might be predicted by the simple model described in this paper.

We shall now deal with the possibility of incorporating the results of the linear theory of the stratified model to non-linear forecasting for levels below and up to 500 mb. For this purpose we write equation (5.8) as follows:

$$(\text{div }\mathbf{v})_{k,m} = \frac{c_{k,m} - U}{g'H}\frac{\partial \phi_{k,m}}{\partial x} \qquad (6.8)$$

where k denotes the wave numbers k and k_1 and m denotes the number of half waves along the vertical. The quantity g' depends on the static stability of the basic flow but not on k, and H will be considered a parameter at our disposal. Replacing $c_{k,m}\frac{\partial \phi_{k,m}}{\partial x}$ by $-\frac{\partial \phi_{k,m}}{\partial t}$ and integrating over all k's, we can replace (6.8) by

$$(\text{div }\mathbf{v})_m = -\frac{1}{g'H}\left(\frac{\partial \phi'_m}{\partial t} + U\frac{\partial \phi'_m}{\partial x}\right) \qquad (6.9)$$

where a prime denotes deviation from a zonal mean. In order to obtain the total divergence, div $\mathbf{v}$, we have to sum over all m:s. This we can do only if the observed pressure field were decomposed according to m. In practice this is not the case; at the best we know the heights of a number of pressure surfaces. For practical purposes we shall therefore replace (6.9) by

$$\text{div }\mathbf{v} = -\frac{1}{g'H}\left(\frac{\partial \phi'}{\partial t} + U\frac{\partial \phi'}{\partial x}\right) \qquad (6.10)$$

where we interpret $g'H$ as a parameter to be determined empirically; it may be taken as a constant over the whole map or possibly as a function of latitude. In approximating the expression (6.10) we may write:

$$\frac{\partial \phi'}{\partial t} = \frac{\partial \phi}{\partial t} - \frac{\partial \bar{\phi}}{\partial t} = \frac{\partial \phi}{\partial t} \qquad (6.11)$$

and

$$U\frac{\partial \phi'}{\partial x} = U\frac{\partial \phi}{\partial x} = \overline{\mathbf{v}} \cdot \nabla \phi \qquad (6.12)$$

where

$$\overline{\mathbf{v}} = U\mathbf{i} = -\frac{\partial \overline{\psi}}{\partial y}\mathbf{i} \qquad (6.13)$$

A bar denotes a zonal mean, ψ is the stream function, and $\mathbf{i}$ is a unit vector directed toward the East.

Substitution into (6.10) by means of (6.11), (6.12), and (6.13) results in

$$\operatorname{div} \mathbf{v} = -\frac{1}{g'H}\left(\frac{\partial \phi}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \phi\right) \quad (6.14)$$

We introduce the expression (6.14) for div **v** into the vorticity equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \eta + \eta \operatorname{div} \mathbf{v} = 0 \quad (6.15)$$

and obtain

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \eta - \frac{\eta}{g'H}\left(\frac{\partial \phi}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \phi\right) = 0 \quad (6.16)$$

where $\zeta = \nabla^2 \psi$ and $\eta = \zeta + f$.

Equation (6.16) has two dependent variables, ψ and ϕ. In order to make an iterative forecast by means of it we need an additional equation relating ψ and ϕ. Consistent with the assumption leading to (6.1) (see footnote page 166), we shall use the balance equation

$$f \nabla^2 \psi + 2\left[\psi_{xx}\psi_{yy} - \psi_{xy}{}^2\right] + \nabla \psi \cdot \nabla f - \nabla^2 \phi = 0 \quad (6.17)$$

as the second predictive equation. For a complete prediction scheme we need the additional equations

$$\mathbf{v} = \mathbf{k} \times \nabla \psi + \nabla \chi \quad (6.18)$$

and

$$\nabla^2 \chi = \operatorname{div} \mathbf{v} = -\frac{1}{g'H}\left(\frac{\partial \phi}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \phi\right) \quad (6.19)$$

The main steps in making a forecast by means of the above system are: a) simultaneous relaxation of equations (6.16) and (6.17); b) derivation of the wind **v** from (6.18) and (6.19).

Considering the simplicity of the proposed model, the labor and time involved in using the predictive scheme we have just outlined may be somewhat out of proportion. A simpler scheme is the following:

a) replace the balance equation (6.17) by the geostrophic relation

$$\nabla \phi = f \nabla \psi \quad (6.20)$$

b) replace (6.18) by

$$\mathbf{v} = \mathbf{k} \times \nabla \psi \quad (6.21)$$

Substitution into (6.16) then gives:

$$\left(\nabla^2 - \frac{f\eta}{g'H}\right)\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \eta - \frac{f\eta}{g'H}\overline{\mathbf{v}} \cdot \Delta \psi = 0 \quad (6.22)$$

which has ψ as the only dependent variable. Initially the balance equation (6.17) will be used for deriving the stream function ψ from ϕ but is not required in successive time steps. Conversion of ψ to ϕ, when required, is accomplished by solving (6 17) for ϕ; the geostrophic relation (6.20) is used *exclusively* for approximating the expression (6, 14) for div **v** and does *not* otherwise replace the balance equation.[9]

The vorticity equation (6.22) is of the same form as given by CRESSMAN (1958) and WIIN-NIELSEN (1959) except for the last term which is peculiar to the stratified model. In applying (6.22) to the 500 mb stream function for the purpose of prediction, its last term may be important in that it provides a better control over the displacement of planetary waves. This is quite apparent in comparing (6.1) with (4.4). In the latter the wave speed will always have the same sign as the speed, c_R, of the Rossby wave, independent of the choice of gH. On the other hand, (6.1) provides for possibility of a wave speed of a sign opposite to c_R, depending on U and the choice of $g'H$; we have here the advantage of suppressing the β-term without at the same time suppressing the basic current. This feature may be quite important in extending the usefulness of 500 mb forecasts to lower latitudes. Presently, 500 mb forecasts appear to have little skill south of approximately 35° N.

In applying eq. (6.22), we are, of course, not limited to the 500 mb surface. Notwithstanding the simplicity of the stratified model, its theory does not imply that it would, per se, apply better to one level than another. It appears, however, that owing to the model's inadequate perturbation structure as compared with that of the atmosphere, its usefulness is limited to the layer 500—1,000 mb. In application we face the problem of how to

[9] In deriving **v** in (6.22) from a stream function alone and not including the irrotational component, one introduces a spurious vorticity source which may be of some importance, cf. WIIN-NIELSEN (1959) and 'ARNASON and CARSTENSEN (1959).

interpret the concept of the basic current U. In the atmosphere the zonal wind will vary from one level to another, and in using (6.22) as a prediction equation to a certain pressure surface, we may not wish to use $\overline{\mathbf{v}}$ from that particular surface. For low-level forecasting, it would seem preferable to use an average along the vertical for two or more levels, thereby implicitly introducing the concept of a *steering current*, familiar from synoptic meteorology. Moreover, it may be advisable to interpret the basic current as a flow obtained by horizontal averaging (space smoothing) rather than a zonal flow. Quite possibly the choice of g' and H might differ from one level to another.

Acknowledgement

The writer wishes to thank Professor Lawrence Gates of the Department of Meteorology, UCLA, and Professor George J. Haltiner of the Department of Meteorology and Oceanography, U.S. Naval Postgraduate School, Monterey, for reading the manuscript and offering helpful suggestions.

Thanks are also extended to Mr. Ronald McDonald for drafting the figures and computing the three roots of the frequency equation (4, 1), and to Mrs. Octavia Mitchell for typing the manuscript.

Appendix

Application to the system (2.1)—(2.5) of the method of small perturbations results in the following system of equations given in the same order:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial v}{\partial x}+\beta v+f\frac{\partial u}{\partial x}=0 \tag{1}$$

$$\frac{\partial v}{\partial x}-\beta u-\frac{\partial^2\phi}{\partial x^2}-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x}=0 \tag{2}$$

$$\frac{\partial \omega}{\partial p}+\frac{\partial u}{\partial x}=0 \tag{3}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\vartheta+\omega\frac{\partial\bar{\vartheta}}{\partial p}=0 \tag{4}$$

$$\frac{\partial\phi}{\partial p}+F\cdot\vartheta=0 \tag{5}$$

where

$$F=\frac{R}{p}\left(\frac{p}{p_0}\right)^{R/c_p} \tag{6}$$

ϑ is the potential temperature, R the gas constant, c_p the specific heat for constant pressure, and $\beta=\frac{df}{dy}$. Both f and β will be treated as constants. A bar over a quantity (e.g., $\bar{\vartheta}$) refers to the basic flow.

The perturbation quantities u, v, ω, ϕ, and ϑ, assumed to be independent of y, are expressed in terms of x, p, and t as follows:

$$\begin{aligned} u&=A(p)e^{ik(x-ct)} & v&=B(p)e^{ik(x-ct)}\\ \omega&=D(p)e^{ik(x-ct)} & \phi&=C(p)e^{ik(x-ct)}\\ \vartheta&=E(p)e^{ik(x-ct)} \end{aligned} \tag{7}$$

where c is the speed (positive toward the East) of a harmonic wave, k the wave number, $\frac{2\pi}{L}$, where L is the wave length, and i the imaginary unit. The amplitudes A, B, C, D, and E may be complex numbers.

It follows from (4) and (5) that

$$\frac{dD}{dp}+ikA=0 \tag{8}$$

and

$$F\cdot E+\frac{dC}{dp}=0 \tag{9}$$

Substitution for u, v, ω, ϕ, and ϑ from (7) into equations (1), (2), and (3) leads to:

$$(c-U+\beta/k^2)B+\frac{if}{k}A=0 \tag{10}$$

$$(c-U+\beta/k^2)A-\frac{if}{k}B-C=0 \tag{11}$$

$$\frac{c-U}{F}\frac{dC}{dp}-\frac{i}{k}\frac{d\bar{\vartheta}}{dp}D=0 \tag{12}$$

where E has been eliminated by means of (9). Elimination of B between (10) and (11) gives:

$$\frac{C_i^2-\Delta^2}{\Delta}A+C=0 \tag{13}$$

where

$$\Delta=c-U+\beta/k^2 \tag{14}$$

and $C_i=f/k$, the speed of inertia waves.

We continue the process of elimination and arrive at the following second order differential equation for D:

$$\frac{dD^2}{dp^2} + \frac{\Delta F \dfrac{d\bar{\vartheta}}{dp}}{(c-U)(C_i^2-\Delta^2)} D = 0 \qquad (15)$$

Both F and $\bar{\vartheta}$ are known functions of p. One finds by using the definition for potential temperature and the hydrostatic equation that

$$F\frac{d\bar{\vartheta}}{dp} = \frac{R}{p^2}\left[\frac{d\bar{T}}{d\ln p} - \frac{R\bar{T}}{c_p}\right] \qquad (16)$$

We introduce $Z = \ln\left(\frac{p_0}{p}\right)$ as a new independent variable; this transforms (15) into

$$D'' + D' - \frac{\Delta R}{(c-U)(C_i^2-\Delta^2)}\left(\frac{R\bar{T}}{c_p} + \frac{d\bar{T}}{dZ}\right) D = 0 \qquad (17)$$

where a prime means differentiation with respect to Z, and p_0 is a reference pressure (1,000 mb). Equation (17) has constant coefficients for an isothermal atmosphere. When T is a linear function of Z, a proper coordinate transformation will reduce (17) to a Bessel equation, KAMKE (1943) p. 440; see also BJERKNES, et al., (1933) p. 370. Here we shall confine ourselves to an approximate solution of (17) in that we write

$$\frac{R\bar{T}}{c_p} + \frac{d\bar{T}}{dZ} = \frac{R\bar{T}}{g}\left[\frac{g}{c_p} + \frac{g}{R\bar{T}}\frac{d\bar{T}}{dZ}\right] \approx$$

$$\approx H_m\left[\frac{g}{c_p} + \frac{\partial\bar{T}}{\partial z}\right] \qquad (18)$$

where $z = \frac{R\bar{T}_m}{g} Z$, $\bar{T}_m$ is the mean vertical temperature for the layer considered, and $H_m = \frac{R\bar{T}_m}{g}$. Thus we replace (17) by

$$D'' + D' - \frac{\Delta R H_m \cdot s}{(c-U)(C_i^2-\Delta^2)} D = 0 \qquad (19)$$

where

$$s = \frac{g}{c_p} + \frac{\partial\bar{T}}{\partial z} = \text{const.} \qquad (20)$$

REFERENCES

ARNASON, G. and CARSTENSEN, L. P., 1959: The effects of vertical vorticity advection and turning of the vortex tubes in hemispheric forecasts with a two-level model. *Monthly Weather Review*, Vol. **87**, pp. 119—128.

BJERKNES, V. et al., 1933: *Physikalische Hydrodynamik.* Julius Springer 1933, pp. 277—420.

BOLIN, B., 1953: The adjustment of non-balanced velocity field toward geostrophic equilibrium in a stratified fluid. *Tellus*, **5**, pp. 373—385.

— 1955: Numerical forecasting with the barotropic model. *Tellus*, **7**, pp. 27—49.

— 1956: An improved baroclinic model and some aspects of using the balance equation for three-dimensional flow. *Tellus*, **8**, pp. 61—75.

BURGER, A. P., 1958: Scale considerations of planetary motions. *Tellus*, **10**, pp. 195—205.

CHARNEY, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Met., Vol.* **4**, pp. 135—162.

CHARNEY, J. G., 1955: The generation of ocean currents by wind. *Journal of Marine Research*, **14**, pp. 477—498.

CRESSMAN, G. P., 1958: Barotropic divergence and very long atmospheric waves. *Monthly Weather Review*, **86**, pp. 293—298.

ELIASSEN, A., 1955: "*Lectures on physical weather prediction.*" University of Calif., Dept. of Meteorology, 109 pp.

FJØRTOFT, R., 1950: Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. *Geofysiske Publikasjoner*, **17**, pp. 1—52.

HINKELMANN, K., 1951: Der Mechanismus des meteorologischen Lärmes. *Tellus*, **3**, pp. 285—296.

KAMKE, E., 1943: Differentialgleichungen: *Lösungsmethoden und Lösungen*, Vol. I. Akademische Verlagsgesellschaft, Leipzig, pp. 1—642.

MARTIN, D. E., 1958: An investigation of systematic errors in the barotropic forecasts. *Tellus*, **10**, pp. 451—465.

PALMER, C. E., 1952: Tropical meteorology. *Q.J.R.M. Soc.*, **78**, pp. 126—163.

RIEHL, H., 1954: *Tropical Meteorology*, New York McGraw Hill, pp. 1—392.

ROSSBY, C.-G., 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacement of semi-permanent centers of action. *Journal of Marine Research*, **2**, pp. 38—55.

THOMPSON, P. D., 1956: A theory of large scale disturbances in non-geostrophic flow. *J. Met.*, **13**, pp. 251—261.

WIIN-NIELSEN, A., 1959: On barotropic and baroclinic models, with special emphasis on ultra-long waves. *Monthly Weather Review*, **87**, pp. 171—184.

— 1959: On certain integral constraint for the time-integration of baroclinic models. *Tellus*, **11**, pp. 45—59.

WOLFF, P. M., 1958: The error in numerical forecasts due to retrogression of ultra-long waves. *Monthly Weather Review*, **86**, pp. 245—250.