Note on the Propagation of Baroclinic Waves in a Zonal Ocean Current

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Abstract

In a zonal ocean current one sees stationary or slowly moving baroclinic waves, with a wavelength of 1,000 km or more. These waves may be equivalent to a type of very long waves studied earlier in the atmospheric case. The wave speed is independent of the wave number but depends on the speed of the basic current, the stratification etc. For the case of a zonal current with exponential vertical profile, stationary waves are found for an eastward current. The wave perturbation normally decays downward in an exponential way. The parameter that decides the nature of the waves is $\lambda = \frac{\beta}{f^2}\frac{\Delta A p_0}{U_0}$ where U_0 is the amplitude of the basic current, ΔA the vertical variation of the specific volume, p_0 the scale pressure (the pressure variation that corresponds to the scale depth of the basic current), and f the Coriolis parameter. For a westward current travelling neutral waves exist but only for special values of λ that are not normally expected in the oceans.

1. Introduction

In the open ocean, far away from the coasts, we find a system of essentially zonal ocean currents. The vertical structure of the currents is relatively uniform, and as a normal form we may take a velocity distribution with maximum velocity at the surface and an exponential-type decay downwards. Also the stratification can normally be described as exponential-type. At great depth we find only small motions and the water is practically homogeneous.

It is of interest to investigate theoretically the existence of different waves in such a current. The present investigation deals only with a special type of wave, namely low-frequency baroclinic waves having wavelengths of 1,000 km or more. These waves are the ones mainly observed in synoptic studies of the ocean circulation. Theoretically such waves have been studied in a resting two-layer ocean by VERONIS and STOMMEL (1956).

The present theory is based on the equations for very long waves in the atmosphere used earlier by the author (WELANDER, 1961). The basic equations are practically identical for the atmosphere and the ocean if the so-called p-system is used. For the ocean we may replace the law of conservation of potential temperature by the simpler law of conservation of density or specific volume. The boundary conditions are that $\omega = 0$ at the free surface ($p = 0$), and that $\phi' \to 0$ at great depths ($p \to \infty$).

The effect of the bottom is not considered here, and we allow vertical velocities to exist at infinity. For application to a real ocean we should think of the model as a boundary layer one describing the conditions in the upper stratified part of the water. It is fitted to a deep water solution in which the vertical velocity decreases linearly with depth to a zero value at the bottom. Some justification for this procedure is found in a recent study of the thermocline by H. Stommel (private communication).

In the earlier mentioned paper a simplified wave equation was obtained for low-frequency waves of planetary scale (wave-lengths many thousand kilometers), when the Rossby number (Ro) and the inverse Richardson number (Ri^{-1}) characterizing the basic state were assumed small. In the atmosphere Ro would be of the order 10^{-2} to 10^{-1}, and Ri^{-1} of the order 10^{-2}. In the open ocean we find that Ro is of the order 10^{-4} to 10^{-3} and Ri^{-1} of the order 10^{-3}. The smallness of these coefficients in the oceanic case means that the same wave equation can be applied also to shorter wave-lengths, possibly down to a wave-length of 1,000 km.

2. General equations

Let x and y be horizontal coordinates related to the latitude ϕ, the longitude λ, and to the radius of the earth R as follows:[1]

$$\begin{aligned} dx &= R\cos\phi d\lambda \\ dy &= R d\phi \end{aligned} \tag{1}$$

Further let p be the pressure, α the specific volume $\left(\alpha = \frac{1}{\varrho}\right)$, $\phi = gz$ the geopotential of a pressure surface, and let u, v, ω be the velocity components in the x-y-p system $\left(\text{thus } \omega = \frac{dp}{dt}\right)$. Assuming that the water is ideal and incompressible and that a hydrostatic balance exists in the vertical direction, and assuming further that the Rossby number is so small that we may neglect spherical acceleration terms of the type $\frac{\mathrm{tg}\phi}{R}uv$ etc., the equations governing u, v, ω, α and ϕ are

$$\frac{du}{dt} - fv = -\frac{\partial\phi}{\partial x} \tag{2}$$

$$\frac{dv}{dt} + fu = -\frac{\partial\phi}{\partial y} \tag{3}$$

$$\alpha = -\frac{\partial\phi}{\partial p} \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\mathrm{tg}\phi}{R}v + \frac{\partial\omega}{\partial p} = 0 \tag{5}$$

$$\frac{d\alpha}{dt} = 0 \tag{6}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$.

The basic current U is assumed to be uniform in the y-direction. The basic fields of the geopotential Φ and the specific volume A satisfy the relations

$$fU = -\Phi_y \tag{7}$$

$$A = -\Phi_p \tag{8}$$

(in the following derivatives of the basic quantities are denoted by a subscript).

We write

$$\left.\begin{aligned} u &= U + u' \\ v &= v' \\ \omega &= \omega' \\ \alpha &= A + \alpha' \\ \phi &= \Phi + \phi' \end{aligned}\right\} \tag{9}$$

and derive the perturbation equations. Using an expansion of the same type as in the atmospheric case, but with the Rossby number and inverse Richardson number defined as

$$\left.\begin{aligned} Ro &= \frac{U_0}{Rf} \\ Ri^{-1} &= \frac{U_0^2}{p_0\Delta A} \end{aligned}\right\} \tag{10}$$

where U_0 is the amplitude of the basic current, p_0 the scale depth, and ΔA the vertical variation of the specific volume, a correspondingly simplified wave equation is obtained. The

[1] Although a β-plane analysis will give results that are correct to the first order, the quasi-coordinates x-y-p that depict the spherical geometry correctly have been used throughout for the sake of consistency in this paper. Thus the term $\frac{\mathrm{tg}\phi}{R}v$ is kept in the continuity equation, and the permutation rule

$$\frac{\partial^2}{\partial y\partial x} = \frac{\partial^2}{\partial x\partial y} + \frac{\mathrm{tg}\phi}{R}\frac{\partial}{\partial x}$$

that is valid for the quasi-coordinates has been applied.

values of Ro and Ri^{-1} quoted earlier will be obtained by choosing $U_0 \sim 10\,\text{cm sec}^{-1}$, $p_0 \sim 10^8$ dyn cm^{-2} (scale depth 1,000 m), $\Delta A \sim 10^{-3}$ g^{-1} cm^3, $f \sim 10^{-4}$ sec^{-1}. The simplified wave equation is equivalent to the system of perturbation equations in which the two equations of motion are replaced by the geostrophic condition for the perturbation:

$$-fv' = -\frac{\partial\phi'}{\partial x} \tag{11}$$

$$fu' = -\frac{\partial\phi'}{\partial y} \tag{12}$$

$$\alpha' = -\frac{\partial\phi'}{\partial p} \tag{13}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} - \frac{\operatorname{tg}\phi}{R}v' + \frac{\partial\omega'}{\partial p} = 0 \tag{14}$$

$$\frac{\partial\alpha'}{\partial t} + U\frac{\partial\alpha'}{\partial x} + fU_p v' + A_p\omega' = 0 \quad (A_y = -\Phi_{yp} = fU_p) \tag{15}$$

Solving ω' from (15) and using (11) and (13) we get

$$\omega' = \frac{1}{A_p}\left\{\frac{\partial^2\phi'}{\partial p\partial t} + U\frac{\partial^2\phi'}{\partial x\partial p} - U_p\frac{\partial\phi'}{\partial x}\right\} \tag{16}$$

and inserting the velocity values in the continuity equation yields

$$\frac{\partial}{\partial p}\left\{\frac{1}{A_p}\left(\frac{\partial^2\phi'}{\partial p\partial t} + U\frac{\partial^2\phi'}{\partial x\partial p} - U_p\frac{\partial\phi'}{\partial x}\right)\right\} - \frac{\beta}{f^2}\frac{\partial\phi'}{\partial x} = 0. \tag{17}$$

The boundary conditions, requiring that ω' vanish at $p=0$ and that ϕ' vanish for infinite pressures, become

$$\left.\begin{aligned} &\frac{\partial^2\phi'}{\partial p\partial t} + U\frac{\partial^2\phi'}{\partial x\partial p} - U_p\frac{\partial\phi'}{\partial x} = 0 \qquad &&\text{at } p=0 \\ &\phi' = 0 &&\text{at } p=\infty \end{aligned}\right\} \tag{18}$$

3. The wave form

Introducing a wave form

$$\phi' = \hat\phi(y, p)\, e^{ik(x-ct)} \tag{19}$$

where k is the wave number and c the wave velocity (positive for eastward motion) equations (17) and (18) transform into

$$(U-c)\frac{\partial}{\partial p}\left(\frac{1}{A_p}\frac{\partial\hat\phi}{\partial p}\right) - \left\{\left(\frac{U_p}{A_p}\right)_p + \frac{\beta}{f^2}\right\}\hat\phi = 0 \tag{20}$$

$$\left.\begin{aligned} &(U-c)\frac{\partial\hat\phi}{\partial p} = U_p\hat\phi \qquad &&\text{at } p=0 \\ &\hat\phi = 0 &&\text{at } p=\infty \end{aligned}\right\} \tag{21}$$

(These equations can be directly identified with the equations (30) and (32) in the earlier paper, if we substitute A for A_p.)

The wave number k does not occur in the wave equation or the boundary conditions, hence the wave speed must be independent of k. Further we note that no y-derivatives appear, and this makes it possible to treat the y-variable parametrically.

We consider in particular the basic profiles (Fig. 1)

$$U = U_0 e^{-\frac{p}{p_0}} \tag{22}$$

$$A = A_0 + \triangle A e^{-\frac{p}{p_0}} \tag{23}$$

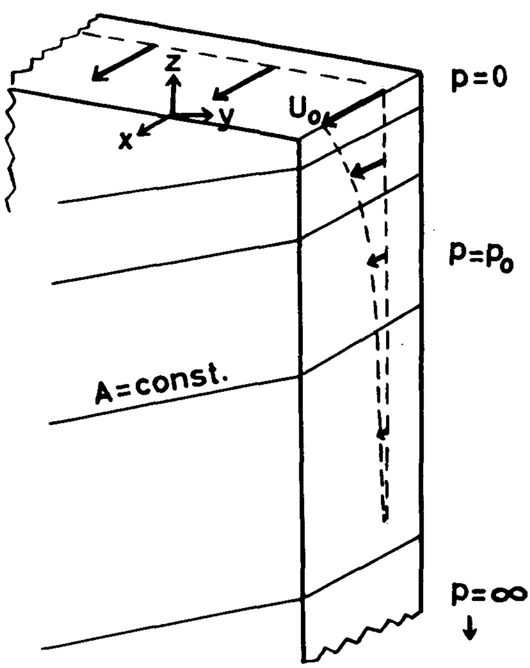

Fig. 1. Sketch of the basic current.

where p_0 is the scale pressure (this must necessarily be the same for U and A in view of the "thermal wind" equation). ΔA will show some variation in the y-direction, but we study here only a "local" solution, in which y is fixed. With the notation

$$\left.\begin{aligned} P &= \frac{p}{p_0} \\ c^\star &= \frac{c}{U_0} \\ \lambda &= \frac{\beta}{f^2}\frac{\Delta A\, p_0}{U_0} \end{aligned}\right\} \tag{23}$$

the equations (20) and (21) change to

$$(e^{-P} - c^\star)\frac{d}{dP}\left(e^P \frac{d\hat\phi}{dP}\right) + \lambda\hat\phi = 0 \tag{24}$$

$$\left.\begin{aligned} (1 - c^\star)\frac{d\hat\phi}{dP} + \hat\phi &= 0 \qquad \text{at } P = 0 \\ \hat\phi &= 0 \qquad \text{at } P = \infty \end{aligned}\right\} \tag{25}$$

4. The stationary waves

In this case $c^\star = 0$ and (24) and (25) reduce to

$$e^{-P}\frac{d}{dP}\left(e^P \frac{d\hat\phi}{dP}\right) + \lambda\hat\phi = 0 \tag{26}$$

$$\left.\begin{aligned} \frac{d\hat\phi}{dP} + \hat\phi &= 0 \qquad \text{at } P = 0 \\ \hat\phi &= 0 \qquad \text{at } P = \infty \end{aligned}\right\} \tag{27}$$

Attempting a solution of the form $e^{\alpha P}$ gives $\alpha(\alpha + 1) + \lambda = 0$, thus the general solution is of the form

$$\hat\phi = C_1 e^{\alpha_1 P} + C_2 e^{\alpha_2 P} \tag{28}$$

$$\alpha_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}, \quad \alpha_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda} \tag{29}$$

If $\lambda < 0$, as is the case when $U_0 < 0$, the second boundary condition requires that $C_1 = 0$, and the first condition gives then $\alpha_2 = -1$, i.e. $\lambda = 0$. No stationary waves occur. If $\lambda > 0$ the second condition is fulfilled and applying the first condition we find possible solutions of the form

$$\hat\phi = \text{const}\left\{(1 + \alpha_2)e^{\alpha_1 P} - (1 + \alpha_1)e^{\alpha_2 P}\right\} \tag{30}$$

For $\lambda < \frac{1}{4}$ the profiles are of exponential type, for $\lambda > \frac{1}{4}$ of oscillatory type. In the real ocean λ is generally small and we are likely to have solutions of the first type.

5. Travelling neutral waves

In the case $c \neq 0$ we introduce the new independent variable

$$\zeta = \frac{1}{c^\star} e^{-P} \tag{31}$$

This gives

$$\zeta(\zeta - 1)\frac{d^2\hat\phi}{d\zeta^2} + \lambda\hat\phi = 0 \tag{32}$$

$$\left.\begin{aligned} (1 - c^\star)\frac{d\hat\phi}{d\zeta} - c^\star\hat\phi &= 0 \quad \text{at } \zeta = \frac{1}{c^\star} \\ \hat\phi &= 0 \quad \text{at } \zeta = 0 \end{aligned}\right\} \tag{33}$$

The solution that is regular at $\zeta = 0$ is given by the series

$$\hat\phi = \zeta + \frac{1}{2}\lambda\zeta^2 + \frac{1}{2}\lambda\frac{\lambda + 1\cdot 2}{2\cdot 3}\zeta^3 + + \frac{1}{2}\lambda\frac{\lambda + 1\cdot 2}{2\cdot 3}\frac{\lambda + 2\cdot 3}{3\cdot 4}\zeta^4 + \ldots \tag{34}$$

The second boundary condition is automatically satisfied. The solution is not well behaved at $\zeta = 1$, except when $\lambda = -n(n+1)$, $n = 1, 2, 3,$ ——, and the solution becomes a polynomial[1]. In other cases, we have thus to restrict ourselves to the region $\zeta < 1$, and it must be required that $c^\star > 1$, or $c > U_0$.

The travelling neutral waves can be found

[1] The general solution to the equation (32) is $|(2\zeta-1)^2-1|\, P_\nu'(2\zeta-1)$, where P_ν is the Legendre function of order ν, and $\lambda = -\frac{1}{4}\nu(\nu+1)$. *Cf.* Kamke, Differential-Gleichungen I, eq. No. 2.231.

by the following graphical construction. We draw the family of solutions $\hat{\phi}$ (λ, ζ) given by (35) together with the curves $\hat{\phi}^{\star} = \text{const.}$ $(\zeta - 1)$ and seek for points at which the solution curves are tangent to the $\hat{\phi}^{\star}$-curves.

The abscissa of such a point represent $\frac{1}{c^{\star}}$. The construction is easily understood if we note that $\hat{\phi}^{\star}$ satisfies the relation $(\zeta - 1)\frac{d\hat{\phi}^{\star}}{d\zeta} - \hat{\phi}^{\star} = 0$ that for $\zeta = \frac{1}{c^{\star}}$ is precisely the first boundary condition (33).

Solutions are found for $\lambda \leq -2$, i.e. for a westward current. The waves travel westward with a speed of the order U_0. In the polynomial case one wave remains stationary relative to the surface current.

6. Application to the ocean

With the previously given values for U_0, p_0, and ΔA, and with f and β evaluated at middle latitudes we find a value of λ around 0.1. There may, of course, be reasons to choose other values. In the surface layer of the ocean we may have scale pressures that are one order of magnitude smaller, and λ will then be decreased correspondingly, or we may consider a strong current of 100 cm sec^{-1} that also would require a λ-value one order of magnitude smaller. Larger values of λ may be needed when the current is very weak, or when we come close to the equator (in this region the theory will, however, break down for other reasons). Under normal conditions we should expect the analysis for small λ-values to be well applicable. The general conclusions we could draw from the investigations are then: Stationary baroclinic waves can occur in an eastward current while no stationary or travelling waves are expected in a westward current of the present type.

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