Theory of Very Long Waves in a Zonal Atmospheric Flow[1, 2]

By PIERRE WELANDER,

Swedish Research Council of the Natural Sciences and Woods Hole Oceanographic Institution

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Abstract

A derivation is given of the general wave equation for small quasi-static adiabatic perturbations in a zonal atmospheric flow. This equation is then specialized to the case of very long waves (wave-length 10,000 km or more). The proper modelling is obtained by assuming the (non-dimensional) wave number to be of order unity and by the assumptions of strong rotational constraint ($Ro \ll 1$), strong dynamic stability ($Ri \gg 1$) but weak stratification $\left(\frac{\Delta\Theta}{\Theta} \ll 1\right)$. The long waves obtained by the present model move at a speed that, to the first order, is independent of the wave number but dependent on the product $Ro \cdot Ri$. A solution is worked out in the β-plane approximation, assuming a basic wind and potential temperature that vary linearly with pressure. The existence of stationary waves in this case is demonstrated. The effect of different boundary conditions is discussed. Some consequences of the theory of interest in connection with numerical weather prediction are pointed out.

1. Introduction

On the hemispherical 500 mb maps of the atmosphere one observes clearly the existence of very long waves, with wave-lengths in the range 10,000—20,000 km. (Fig. 1.) These waves, that appear in the westerly flow of middle latitudes, remain stationary or show only a slow motion.

The first theory for long waves of this type was developed by Rossby (1939) for the case of a barotropic atmosphere (pressure depends only on density). Rossby assumed a uniform zonal flow and perturbations independent of the latitude coordinate. He further simplified the problem by introducing the so-called β-plane in which the only effect of the earth's curvature considered was the variation with latitude of the vertical component of earth's rotation. Rossby's theory gives waves moving with the speed

$$c = U - \frac{\beta}{k^2} \tag{1}$$

where U is the speed of the basic current, k is the wave-number and $\beta = \frac{2\Omega}{R}\cos\varphi$ (Ω is

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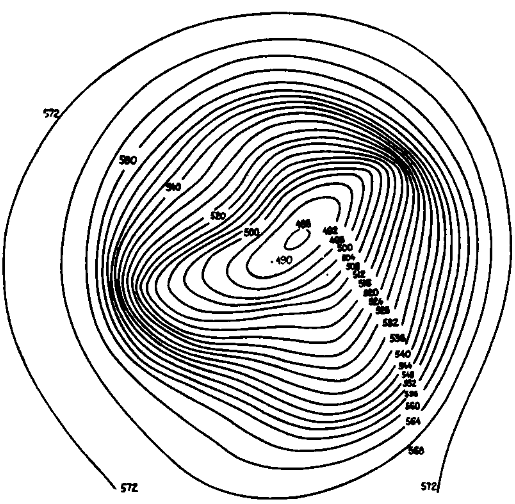


Fig. 1. Mean heigt field at 500 mb on the Northern Hemisphere. in February.

the angular velocity of earth, R the radius of earth and φ the latitude). For moderately long waves Rossby's formula is in reasonable agreement with the observations. For the very long waves that we discuss the second term in (1) will dominate, and the waves should have a strong retrograde motion. Also, the wave-speed should be very sensitive to small changes in the wave-number in this range. These results are not confirmed by the observations.

Later investigators have tried to improve Rossby's theory. HAURWITZ (1940) considered the same problem on a sphere. FJORTOFT (1950) and HOLMBOE (1953) have discussed the effects of a non-uniform basic flow. The behaviour of the waves in a physically more realistic atmosphere of baroclinic (non-barotropic) type has been studied by CHARNEY (1947), EADY (1949), FJORTOFT (1950), THOMPSON (1953) and KUO (1952). In the baroclinic theories the basic velocity is mostly assumed to vary linearly with height and the static stability is assumed constant. All changes are assumed to take place adibatically.

For very large wave-lengths these theories give generally a strong retrograde motion of the waves. In fact, many of baroclinic theories give precisely the Rossby solution in the limit of the very long waves.

A recent paper by BURGER (1959) has thrown some doubt on the simplified vorticity equation, from which most of the earlier theories start at, in the case of planetary-scale motion. For these reasons a new derivation of the wave equation that is appropriate for the very long waves seems to be needed, and an attempt in that direction is made in the present paper.

The start is made from the hydrodynamic and thermodynamic equations governing small quasi-static and adiabatic perturbations of a zonal current, retaining the spherical geometry of the problem. A single wave-equation with the associated boundary conditions is then derived for the geopotential of a pressure surface. The wave equation and the boundary conditions are now non-dimensionalized, so that all variables of the problem become of order 1. Certain non-dimensional coefficients, such as the Rossby number, the Richardson number and a stratification number will enter naturally in the equation. One has also a non-dimensional frequency and the non-dimensional wave-numbers (scaling by the period and the radius of the earth respectively). For the very long waves the non-dimensional zonal wave-number becomes of order 1. In this case it has been found possible to apply an expansion after small Rossby numbers and small inverse Richardson numbers. Finally, the study is limited to waves of small non-dimensional frequencies.

It has been attempted to justify the above expansion method in terms of the characteristic numbers that one can compute for the real atmosphere. It seems that the method will work satisfactorily for the problem we consider. It can certainly not be used for waves much shorter than 10,000 km.

The boundary conditions have been discussed in some detail. A comparison is made between the condition of vanishing vertical velocity and vanishing individual pressure change at the ground. In the upper atmosphere it is required that the geopotential remains finite when the pressure goes to zero, i.e. that the atmosphere has a limited thickness.

Numerical results have been worked out in a "β-plane" approximation for the special case where the basic zonal velocity and the potential temperature vary linearly with pressure. Diagrams will show the wave speed and the vertical structure of the waves.

A few words should be said about the coordinates used in the general derivation. The coordinates appearing are x, y, z or x, y, p. The coordinates x, y, z are not rectangular coordinates, but are quasi-coordinates that depict the spherical geometry. The rules of computation become accordingly different. For example, the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ do not commute. The coordinates x, y, p are treated the same way. The only difference is that the pressure p replaces z as vertical coordinate and z, or in our case rather the geopotential $\phi = gz$, enters as dependent variable. These coordinates go together with the hydrostatic approximation. The transformation formulas are given by ELIASSEN (1949).

2. Basic Equations

Let φ be the latitude (counted positive northward), λ the longitude (counted positive eastward) and r the distance from the earth's centre. We introduce the quasi-coordinates x and y by the non-integrable relations

$$dx = r \cos\varphi \, d\lambda$$
$$dy = r d\varphi$$

We further introduce the height $z = r - R$, where R is the radius of earth. Let (u, v, w) be the velocity in the x-y-z system, p the pressure, ϱ the density, T the temperature, q the heat per unit mass, c_v the specific heat per unit mass, (F_x, F_y, F_z) the frictional force per unit mass, g the acceleration of gravity, Ω the magnitude of earth's angular velocity, $f = 2\,\Omega \sin\varphi$ the double vertical component and $e = 2\,\Omega \cos\varphi$ the double horizontal component of the same angular velocity. The basic hydrodynamic and the thermodynamic equations are then

$$\frac{du}{dt} - \left(f + \frac{u}{r}\operatorname{tg}\varphi\right) v + \left(e + \frac{u}{r}\right) w = -\frac{1}{\varrho}\left(\frac{\partial p}{\partial x} + F_x\right) \tag{2}$$

$$\frac{dv}{dt} + \left(f + \frac{u}{r}\operatorname{tg}\varphi\right) u + \frac{vw}{r} = -\frac{1}{\varrho}\left(\frac{\partial p}{\partial y} + F_y\right) \tag{3}$$

$$\frac{dw}{dt} - eu - \frac{u^2 + v^2}{r} = -\frac{1}{\varrho}\left(\frac{\partial p}{\partial z} + F_z\right) - g \tag{4}$$

$$\frac{d\varrho}{dt} + \varrho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\operatorname{tg}\varphi}{r} v + \frac{\partial w}{\partial z} + \frac{2}{r} w\right) = 0 \tag{5}$$

$$\frac{dq}{dt} = p \frac{d}{dt}\left(\frac{1}{\varrho}\right) + c_v \frac{dT}{dt} \tag{6}$$

$$p = p(\varrho, T) \tag{7}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \cdot g$ will include the centrifugal forces due to earth's basic rotation. It is permissible to assume this apparent gravity constant and directed vertically. Consistently, the earth's surface should then be assumed spherical.

(2, 3, 4) are the equations of motion, (5) the continuity equation, (6) the first law of thermodynamics and (7) the equation of state. If q is known, or if it can be related to the other variables, we have six equations for the unknown quantities u, v, w, p, ϱ, and T. The boundary conditions are not considered at present.

3. General assumptions

The following general assumptions are made –

i) The depth (scale-depth) of the atmosphere is much smaller than the radius of earth.
ii) The vertical derivative of any of the physical variables is large compared to the horizontal derivatives.
iii) The pressure is hydrostatic.
iv) Friction and non-adiabatic heating are neglected.
v) The equation of state is the ideal gas law.

The assumptions (i) and (ii) allow us to replace r by R in the above equations. The approximation holds good also when further vertical derivatives are taken. In (1) y becomes a true coordinate, $y = R\varphi$. The assumptions further allow us to neglect the terms $\frac{uw}{R}$ and $\frac{vw}{R}$ in the equations of motion and the term $\frac{2}{R} w$ in the continuity equation.

The assumption (ii) combined with the continuity equation gives that $w \ll u, v$ and the term ew in (2) may be dropped.

The assumption (iii) states that $\frac{\partial p}{\partial z} = -g\varrho$, the assumption (iv) that $F_x = F_y = F_z = 0$ and $\frac{dq}{dt} = 0$, and the assumption (vii) that $p = R_M \varrho T$, where R_M is the gas constant per gram.

Once the hydrostatic assumption is made there is some advantage in using a new coordinate system in which the geopotential $\phi = gz$ of a given pressure surface becomes a dependent variable, while the pressure becomes an independent vertical coordinate. The transformation of the hydrodynamic and the thermodynamic equations to the p-system has been carried out by ELIASSEN (1949). In this new system the density disappears from the equations of motion as well as from the continuity equation. The adiabatic equation in combination with the equation of state gives $\frac{d}{dt}\left(\frac{1}{\varrho} p^{\frac{1}{\varkappa}}\right) = 0$ with $\varkappa = \frac{c_p}{c_v}$, and here $\frac{1}{\varrho}$ can be replaced by $-\frac{\partial \phi}{\partial p}$ according to the hydrostatic equation. The two equations of motion, the continuity equation and the transformed adiabatic equation become finally

$$\frac{du}{dt} - \left(f + \frac{u}{R} \operatorname{tg} \varphi\right) v = -\frac{\partial \phi}{\partial x} \tag{8}$$

$$\frac{dv}{dt} + \left(f + \frac{u}{R} \operatorname{tg} \varphi\right) u = -\frac{\partial \phi}{\partial y} \tag{9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\operatorname{tg} \varphi}{R} v + \frac{\partial \omega}{\partial p} = 0 \tag{10}$$

$$\frac{d}{dt}\left(p^{\frac{1}{\varkappa}} \frac{\partial \phi}{\partial p}\right) = 0 \tag{11}$$

where (u, v, ω) is the velocity in the p-system and $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$. In the above equations the x- and y-derivatives are evaluated along surfaces $p = \text{const}$. The velocity components u and v are identical with the same components in the z-system. The vertical velocity ω has, however, a different meaning in the p-system. ω stands for $\frac{dp}{dt}$ and represents thus the total rate of change of the pressure.

We next turn to the boundary conditions. Since the model is non-viscous and the changes are assumed to be adiabatic the only condition needed at the earth's surface is that the vertical velocity vanish: $w = 0$ at $z = 0$. Since $\phi = gz$ and $\frac{d\phi}{dt} = g\frac{dz}{dt} = gw$ the boundary condition in the p-system is

$$\frac{d\phi}{dt} = 0 \quad \text{at } \phi = 0 \tag{12}$$

This condition is of a different type from the one applying in the z-system, the surface $\phi = 0$ being unknown from the beginning.

What condition should be applied at the upper boundary of the atmosphere is more uncertain. In the real atmosphere frictional effects become very dominant at great heights, since the kinematic viscosity increases to the same extent as the density decreases. Thus it may be unrealistic to retain, in our ideal fluid model, waves that are very energetic at great heights and thus would be subjected to a very strong damping in reality. Formally such waves are eliminated by applying a sufficiently strong condition at the upper boundary. The condition normally used in problems of the present type is that ω should disappear at $p = 0$. This is a relatively weak condition. A more reasonable condition seems to be that the geopotential ϕ remains finite, requiring that the atmosphere has a limited extension out in space. This is a stronger condition than $\omega = 0$. We state the condition in the form

$$\phi < \infty \quad \text{when } p = 0 \tag{13}$$

In the y-direction there are no boundary conditions but one has to require regularity at certain critical points, such as the pole. However, the final wave equation that we will discuss contains y parametrically and the regularity becomes only a local problem. In this case, we can simply exclude the region close to the critical points from our analysis.

4. Wave Form Perturbation Equations

Consider a steady axially symmetric state of the atmosphere satisfying the equations (8—13). This state may be described by a geopotential $\Phi(y, p)$. The corresponding density distribution $Q(y, p)$ can be found from the hydrostatic equation, and the temperature $T(y, p)$ from the equation of state. The zonal velocity $U(y, p)$ is obtained from the second equation of motion, that in this case reduces to

$$\left(f+\frac{\operatorname{tg}\varphi}{R}U\right)U=-\frac{\partial\Phi}{\partial y} \tag{14}$$

We consider now small perturbations from this basic state. We shall assume from the beginning that these have a wave form. Thus we put

$$\begin{aligned} u &= U(y, p) + u'(y, p)\, e^{i(kx+\nu t)} \\ v &= v'(y, p)\, e^{i(kx+\nu t)} \\ \omega &= \omega'(y, p)\, e^{i(kx+\nu t)} \\ \phi &= \Phi(y, p) + \phi'(y, p)\, e^{i(kx+\nu t)} \end{aligned} \tag{15}$$

There is no restriction laid on the problem by assuming the wave form (15). In fact, this is the natural separation form of the perturbation equations. The restriction comes later by our assumption of real values of the frequency ν. This means that we only consider the neutral waves.

The appearance of x in the above form may be puzzling, since x has been defined as a quasi-coordinate. In fact, the x in (15) has a different meaning than in (1). In a spherical wave the longitudinal angle λ enters in a factor $e^{in\lambda}$, where n is an integer giving the mode. We formally identify this factor with e^{ikx} to have a notation in accordance with the β-plane analysis. x and k are then to be defined as follows —

$$\begin{aligned} x &= R\cos\varphi\cdot\lambda \\ k &= \frac{n}{R\cos\varphi} \end{aligned} \tag{16}$$

The important thing to remember here is that k, the zonal wave number, is a function of the latitude.

The differential equations for u', v', ω' and ϕ' are given by the following scheme of coefficients:

u'	v'	w'	ϕ'		
$i(\nu+kU)$	$-\left(f+\frac{\operatorname{tg}\varphi}{R}U\right)+U_y$	U_p	ik		
$f+\frac{2\operatorname{tg}\varphi}{R}U$	$i(\nu+kU)$	0	$\frac{\partial}{\partial y}$		(17)
ik	$\frac{\partial}{\partial y}-\frac{\operatorname{tg}\varphi}{R}$	$\frac{\partial}{\partial p}$	0	$=0$	
0	Φ_{yp}	Λ	$i(\nu+kU)\frac{\partial}{\partial p}$		

where

$$\Lambda = p^{-\frac{1}{\varkappa}}\left(p^{\frac{1}{\varkappa}}\Phi_p\right)_p = -\frac{1}{Q}\frac{1}{\Theta}\frac{\partial\Theta}{\partial p} \tag{18}$$

(Θ is the potential temperature $T\left(\frac{p_0}{p}\right)^{\frac{\varkappa-1}{\varkappa}}$).

5. Wave Equation for ϕ'

The next step is to derive a single wave equation and the associated boundary conditions for ϕ'. One has here the difficulty that the coefficients of the perturbations equation are functions of y and p, so that the differential operators do not normally commute. One straightforward way to derive a wave equation is, however, found: in the first, second and fourth equations the coefficients of u', v' and ω' do not contain any y- or p-derivatives. From these equations we can thus by an algebraic elimination express u', v' and ω' in terms of ϕ'. Inserting these expressions in the third equation, that is, the continuity equation, we have a single equation for ϕ'.

The first step of elimination gives us expressions of the form

$$
\begin{aligned}
u' &= \left(A\frac{\partial}{\partial y} + B\frac{\partial}{\partial p} + C\right)\phi' \\
v' &= \left(D\frac{\partial}{\partial y} + E\frac{\partial}{\partial p} + F\right)\phi' \qquad (19) \\
\omega' &= \left(G\frac{\partial}{\partial y} + H\frac{\partial}{\partial p} + I\right)\phi'
\end{aligned}
$$

and after insertion in the continuity equation one finds the following second order partial differential equation for ϕ:

$$
\begin{aligned}
&\left\{D\frac{\partial^2}{\partial y^2} + (E+G)\frac{\partial^2}{\partial y\partial p} + H\frac{\partial^2}{\partial p^2} + \right. \\
&+\left(ikA + D_y - \frac{\operatorname{tg}\varphi}{R}D + F + G_p\right)\frac{\partial}{\partial y} + \\
&+\left(ikB + E_y - \frac{\operatorname{tg}\varphi}{R}E + H_p + I\right)\frac{\partial}{\partial p} + \\
&\left. + ikC + F_y - \frac{\operatorname{tg}\varphi}{R}F + I_p\right\}\phi' = 0 \qquad (20)
\end{aligned}
$$

The coefficients A, $B-I$ are given below:

$$
\begin{aligned}
A &= -\left\{\Lambda\left(f + \frac{\operatorname{tg}\varphi}{R}U - U_y\right) + U_p\Phi_{yp}\right\}\cdot\frac{1}{N} \\
B &= -U_p(\nu + kU)^2\cdot\frac{1}{N} \\
C &= k\Lambda(\nu + kU)\cdot\frac{1}{N} \\
D &= -i\Lambda(\nu + kU)\cdot\frac{1}{N} \\
E &= -i\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)U_p(\nu + kU)\cdot\frac{1}{N} \\
F &= ik\Lambda\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)\cdot\frac{1}{N} \qquad (21) \\
G &= i\Phi_{yp}(\nu + kU)\cdot\frac{1}{N} \\
H &= -i(\nu + kU)\left[\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)\left(f + \frac{\operatorname{tg}\varphi}{R}U - U_y\right) - (\nu + kU)^2\right]\cdot\frac{1}{N} \\
I &= -ik\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)\Phi_{yp}\cdot\frac{1}{N}
\end{aligned}
$$

where

$$
\begin{aligned}
N = {} & \Lambda\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)\left(f + \frac{\operatorname{tg}\varphi}{R}U - U_y\right) + \\
&+\left(f + \frac{2\operatorname{tg}\varphi}{R}U\right)U_p\Phi_{yp} - \Lambda(\nu + kU)^2
\end{aligned}
$$

The boundary conditions (12, 13) become

$$
\begin{aligned}
&\left\{(\Phi_y D + \Phi_p G)\frac{\partial}{\partial y} + (\Phi_y E + \Phi_p H)\frac{\partial}{\partial p} + \right. \\
&\left. + i(\nu + kU) + \Phi_y F + \Phi_p I\right\}\phi' = 0 \text{ at } \Phi = 0 \quad (22) \\
&\phi' < \infty \quad \text{at } p = 0
\end{aligned}
$$

6. Non-dimensional Form

The previous equations can be brought to a non-dimensional form, in which all variables become of order 1. It is advantageous to work with such a form, since one can see directly the mathematical consequences of different physical assumptions.

At the moment we put no restriction on the horizontal and vertical scales of the basic and the perturbed states, except the one given by assumption (ii). We denote by l_0 and l the meridional wave numbers characterizing the basic state and the perturbation state, respectively. The zonal wave number k of the perturbation has already been introduced. To describe the scale in the vertical direction we introduce further the vertical wave numbers m_0 and m for the basic state and the perturbation state, respectively. We will also need the amplitude of the zonal velocity U_0, the relative variation of the potential temperature $\frac{\Delta\Theta}{\Theta_0}$ over the layer of the atmosphere considered and the standard values of pressure and density at the ground p_0 and ϱ_0.

We shall make the assumption for the basic state that the Coriolis force is at least comparable to the centrifugal force and that the vertical variation of the velocity is at least of order U_0. These assumptions, that are quite weak, allow us to normalize U by the so-called thermal wind relation.

We put

$$y^* = l_0 y,\ p^* = m_0 p \text{ and}$$

$$\frac{\partial}{\partial y} = l_0 \frac{\partial}{\partial y^*},\ \frac{\partial}{\partial p} = m_0 \frac{\partial}{\partial p^*} \text{ etc.}$$

(applying to the basic state)

$$y^{**} = ly,\ p^{**} = mp \quad \text{and } \frac{\partial}{\partial y} = l\frac{\partial}{\partial y^{**}},$$

$$\frac{\partial}{\partial p} = m\frac{\partial}{\partial p^{**}} \text{ etc.} \tag{23}$$

(applying to the perturbation state)

$$U^* = \frac{U}{U_0}$$

$$(\Phi_{yp})^* = \frac{\Phi_{yp}}{m_0 \Omega U_0}$$

The variables denoted by stars should be of order 1.

In their non-dimensional form, the wave equation and the associated boundary conditions will depend on nine independent non-dimensional coefficients. The coefficients can be chosen in different ways but those appearing most naturally are the following:

$\alpha = kR$, $\beta_0 = l_0 R$, $\beta = lR$, $\gamma_0 = m_0 p_0$, $\gamma = mp_0$ — the non-dimensional wave numbers

$\delta = \dfrac{\nu}{\Omega}$ — the non-dimensional frequency

$\varepsilon = \dfrac{U_0}{R\Omega}$ — the Rossby number (Ro)

$\eta = \dfrac{U_0^2}{\dfrac{p_0}{\varrho_0}\dfrac{\Delta\Theta}{\Theta_0}}$ — the inverse Richardson number (Ri^{-1}) [3]

$\sigma = \dfrac{\Delta\Theta}{\Theta_0}$ — the stratification number

We do not write out the non-dimensional equations explicitly but proceed directly to the expansion step.

7. Expansion of the Coefficients A, $B-I$

We now make the following important assumptions:

i) The non-dimensional wave numbers are of the order 1 ($\alpha, \beta_0, \beta, \gamma_0, \gamma \sim 1$).

ii) The Rossby number and the inverse Richardson number are small compared to 1 ($\varepsilon, \eta \ll 1$).

iii) The theory is restricted to values of the non-dimensional frequency that are small compared to 1 ($\delta \ll 1$).

The assumption (i) does not mean that the non-dimensional wave numbers have to be close to 1 but only that they are of order 1 from point of view of the expansion. The actual limits on the wave numbers is commented on further.

The assumption (iii) concerns our eigenvalue parameter and we do not, of course, dispose over this. The assumption is still legitimate but we must check our final results to be sure that we are in the range of validity of our assumptions.

Later on we will confine ourselves to cases where the stratification parameter σ takes on relatively small values. In total, we can therefore say that we concern ourselves with the *largest-scale motion* in an atmosphere of a *strong rotational constraints, strong dynamic stability* and *weak static stability*.

With these assumptions an expansion of the coefficients A, $B-I$ after δ, ε and η seems possible. In making the expansion one must

[3] Usually the Richardson number is defined as $Ri = \dfrac{\dfrac{g}{\Theta}\dfrac{\partial\Theta}{\partial z}}{\left(\dfrac{\partial U}{\partial z}\right)^2}$. With the hydrostatic assumption we can write this $\dfrac{\dfrac{1}{\varrho\Theta}\dfrac{\partial\Theta}{\partial p}}{\left(\dfrac{\partial U}{\partial p}\right)^2}$, and as a representative number for the whole atmosphere we may take $Ri \sim \dfrac{\dfrac{1}{\varrho_0\Theta_0}\dfrac{\Delta\Theta}{p_0}}{\dfrac{u_0^2}{p_0^2}} = \dfrac{\dfrac{p_0}{\varrho_0}\dfrac{\Delta\Theta}{\Theta_0}}{u_0^2}$

keep in mind that nothing has been assumed so far about the relative smallness of δ, ε and η. This is important since different ratios of these three parameters will occur. Care must be taken to keep all terms that can contribute, up to the order we want to consider, when these ratios are varied freely.

After these preliminary remarks we proceed directly to write down the expressions for the coefficients A, $B-I$, retaining only the leading terms:

$$A = -\frac{1}{2\Omega\sin\varphi}\left[1-\frac{1}{\cos\varphi}U^*\varepsilon+\dots\right]$$

$$B = -\frac{\varrho_0 U_0}{\frac{\Delta\Theta}{\Theta_0}}\frac{U^*_{p^*}}{4\sin^2\varphi\,\Lambda^*}\left[(\delta+\alpha U^*\varepsilon)^2+\dots\right]$$

$$C = \frac{k}{4\sin^2\varphi}\frac{1}{\Omega}\left[\delta+\alpha U^*\varepsilon+\dots\right]$$

$$D = \frac{-i}{4\sin^2\varphi}\frac{1}{\Omega}\left[\delta+\alpha U^*\varepsilon\dots\right]$$

$$E = \frac{-i}{2\sin\varphi}\frac{\varrho_0 U_0}{\frac{\Delta\Theta}{\Theta_0}}\left[\delta+\alpha U^*\varepsilon+\dots\right]$$

$$F = \frac{ik}{2\sin\varphi}\frac{1}{\Omega}\left[1+\left(-\frac{1}{2\cos\varphi}U^*+\frac{1}{2\sin\varphi}\beta_0 U^*_{y_*}\right)\varepsilon-\frac{U^*_{p^*}}{2\sin\varphi}\frac{1}{\Lambda^*}(\Phi_{yp})^*\,\eta+\dots\right] \tag{25}$$

$$G = \frac{i\varrho_0 U_0}{\frac{\Delta\Theta}{\Theta_0}}\frac{(\Phi_{yp})^*}{\Lambda^*}\frac{1}{4\sin^2\varphi}\left[\delta+\alpha U^*\varepsilon+\dots\right]$$

$$H = -\frac{i\varrho_0\Omega}{m_0\frac{\Delta\Theta}{\Theta_0}}\frac{1}{\Lambda^*}\left[\delta+\alpha U^*\varepsilon+\dots\right]$$

$$I = -\frac{ik\varrho_0 U_0}{\frac{\Delta\Theta}{\Theta_0}}\frac{(\Phi_{yp})^*}{2\sin\varphi\,\Lambda^*}\left[1-\frac{1}{\cos\varphi}U^*\cdot\varepsilon-\frac{1}{2\sin\varphi}\frac{U^*_{p^*}(\Phi_{yp})^*}{\Lambda^*}\cdot\eta+\dots\right]$$

8. Relation between Velocity and Geopotential

Inserting the above expressions for A, $B-I$ in the relations (19) one finds –

$$u' = -\frac{1}{2\Omega\sin\varphi}\left\{(1+\dots)\frac{\partial}{\partial y^{**}}-\left[\frac{\gamma}{2\beta\sin\varphi}\frac{U^*_{p^*}}{\Lambda^*}\frac{\eta}{\varepsilon}(\delta+\alpha U^*\varepsilon)^2+\dots\right]\frac{\partial}{\partial p^*}+o(\delta,\varepsilon)\right\}\phi'$$

$$v' = \frac{ik}{2\Omega\sin\varphi}\left\{o(\delta,\varepsilon)\frac{\partial}{\partial y^{**}}-\left(\frac{\gamma}{\alpha}\frac{U^*_{p^*}}{\Lambda}\frac{\delta\eta}{\varepsilon}+\dots\right)\frac{\partial}{\partial p^{**}}+1+\dots\right\}\phi' \tag{26}$$

$$\omega' = -\frac{i\varrho_0\Omega}{\frac{\Delta\Theta}{\Theta_0}}\left\{o(\varepsilon\delta,\delta^2)\frac{\partial}{\partial y^{**}}+\left[\frac{\gamma}{\gamma_0}\frac{1}{\Lambda^*}(\delta+\alpha U^*\varepsilon)+\dots\right]\frac{\partial}{\partial p^{**}}-\alpha\frac{U^*_{p^*}}{\Lambda^*}\varepsilon+\dots\right\}\phi'$$

In the expression for ω' use has been made of the relation

$$(\Phi_{yp})^* = -2\sin\varphi\,U^*_{p^*}+o(\varepsilon) \qquad \text{(thermal wind)} \tag{27}$$

One sees that the geostrophic approximation, in which the horizontal Coriolis force and pressure gradient balance, becomes good if $\delta\eta \ll \varepsilon$, i.e., the atmosphere should have a strong dynamic stability but not too strong rotational constraint! It becomes better for smaller frequencies and is certainly good for the stationary waves, to the present order of approximation. Since $\frac{\delta}{\varepsilon}\sim\frac{c}{U_0}$ for the longest waves, where c is the wave speed, we can also express it so that the geostrophic approximation is good when the ratio of the wave speed to the speed of the basic zonal current is small compared to the Richardson number. For the real atmosphere this condition is mostly satisfied. It may be of interest to note that in the case when the geostrophic approximation becomes invalid, the deviation becomes larger for v' than for u'.

Computing the divergence $D = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} - \frac{\operatorname{tg}\varphi}{R} v'$ and the vertical component of vorticity $Z = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} + \frac{\operatorname{tg}\varphi}{R} u'$ one verifies easily that these are of the same order, while in several earlier studies it has been assumed that $D \ll Z$. The fact that D and Z should be of the same order for the largest scales of motion in the atmosphere has already been pointed out by BURGER (1959).

9. Wave Equation

Inserting the expanded expressions $A, B - I$ in the wave equation (20) one finds

$$\Big\{ \mathrm{o}(\delta,\varepsilon)\frac{\partial^2}{\partial y^{**2}} + \mathrm{o}(\eta)\frac{\partial^2}{\partial y^{**}\partial p^{**}} + \left[\frac{\gamma}{\gamma_0^2}\frac{1}{A^*}\left(\frac{\delta\eta}{\varepsilon^2} + \alpha U^*\frac{\eta}{\varepsilon}\right) + \dots\right]\frac{\partial}{\partial p^{**2}} + \left[\frac{\beta\gamma_0}{2\sin\varphi}\left(\frac{U^*_{p^*}}{A^*}\right)_{p^*}\cdot\frac{\delta\eta}{\varepsilon} + \dots\right]\frac{\partial}{\partial y^{**}} + \left[\gamma_0\left(\frac{1}{A^*}\right)_{p^*}\left(\frac{\delta\eta}{\varepsilon^2} + \alpha U^*\frac{\eta}{\varepsilon}\right) + \left(\frac{\beta_0}{2\sin\varphi}\left(\frac{U^*_{p^*}}{A^*}\right)_{y^*} - \frac{\cos\varphi}{2\sin^2\varphi}\frac{U^*_{p^*}}{A^*}\right)\frac{\delta\eta}{\varepsilon} + \dots\right]\frac{\partial}{\partial p^{**}} + \alpha\frac{\cos\varphi}{2\sin^2\varphi} - \alpha\gamma_0\left(\frac{U^*_{p^*}}{A^*}\right)_{p^*}\frac{\eta}{\varepsilon} + \frac{\alpha\gamma_0}{2\sin\varphi}\left(\frac{U^*_{p^*}}{A^*}\right)_{p^*}\cdot\frac{\eta^2}{\varepsilon} + \dots\Big\}\phi' = 0 \tag{28}$$

All terms that can become of order 1 or more are retained. The first thing to note is that the $\frac{\partial^2}{\partial y^{**2}}$ - and $\frac{\partial^2}{\partial y^{**}\partial p^{**}}$ - terms are small. The $\frac{\partial}{\partial y^{**}}$ - term is small if $\delta\eta \ll \varepsilon$. If $\delta\eta \geq \varepsilon$ it is of order one or more but it would certainly be small compared to the $\frac{\partial^2}{\partial p^{**2}}$ - and $\frac{\partial}{\partial p^{**}}$ – terms. To the first order the y-derivatives disappear thus from the problem and y can be treated parametrically. This means, in physical terms, that the behaviour of the long waves is determined essentially by the vertical and not the horizontal structure of the atmosphere.

The goodness of the first approximation depends, of course, on the actual values of δ, ε, η and the non-dimensional wave-numbers. Inspecting (28) it seems as if the first approximation may hold equally well for the higher zonal and vertical modes while the higher meridional modes obviously bring in the $\frac{\partial}{\partial y^{**}}$-term. However, we can only judge the tendency from (24), for when the wave-numbers get large enough to change the order of the terms the whole expansion breaks down and other terms in the coefficients $A, - - I$ take over.

Once we have made the comparison between the different terms and decided which terms should enter in the first approximation we can, of course, change the vertical scaling as we like. Since our main interest is in the basic vertical mode it seems convenient to choose p_0 as vertical scale and put $p^{**} = p^* = \frac{p}{p_0}$, $\gamma = \gamma_0 = 1$. Further, we write

$$\frac{\delta}{\alpha\varepsilon} = \frac{\nu}{kU_0} = -\frac{c}{U_0} = -c' \tag{29}$$

where c' is the ratio of the wave speed c and the speed of the current U_0. The negative sign is chosen to make the wave velocity positive for eastward moving waves (these correspond to negative frequencies).

Considering only the leading terms in (28), in accordance with our previous analysis, we have then

$$(U^* - c')\frac{\partial}{\partial p^*}\left(\frac{1}{A^*}\frac{\partial\phi'}{\partial p_*}\right) + \left[\frac{1}{2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\varepsilon}{\eta} - \left(\frac{U^*_{p^*}}{A^*}\right)_{p^*}\right]\phi' = 0 \tag{30}$$

The only physical parameter entering this equation is $\frac{\varepsilon}{\eta}$, i.e., the product of the Rossby and the Richardson numbers. It is seen the zonal wave number, after introducing the wave speed, has dropped out of the equation.

10. Boundary Conditions

The expanded form of the boundary conditions is –

$$\frac{1}{Q^* A^*}(\delta + \alpha U^* \varepsilon)\frac{\partial \phi'}{\partial p^*} = \left(\frac{\alpha U^*_{p^*}}{Q^* A^*}\varepsilon - \frac{\Delta\Theta}{\Theta_0}\cdot\delta\right)\phi' \quad \text{at } \Phi = 0$$
$$\phi' < \infty \quad \text{at } p^* = 0 \qquad (31)$$

Here use has been made of the relations

$$\Phi_y = -2\Omega \sin\varphi\, U(1 + o(\varepsilon)),$$
$$\Phi_p = -\frac{1}{Q},\ Q^* = \frac{Q}{\varrho_0}$$

Now, the meridional variation of the basic pressure at the ground is given by $\frac{\partial p}{dy} = -2\Omega \sin\varphi\, \varrho_0 U_{z=0}(1 + o(\varepsilon))$. $U_{z=0}$ is normally small but in a general case we should assume it to be of order U_0. The total meridional variation of the pressure will then be of the order $R\Omega\varrho_0 U_0$ or $\frac{M^2}{\varepsilon} p_0$, where M is the Mach number or the Froude number). The relative variation becomes small and we may then replace the surface $\Phi = 0$ by the surface $p = p_0$, if $M^2 \ll \varepsilon$, i.e., if the Mach number square is small compared to the Rossby number. This condition is normally satisfied.

The stratification number $\sigma = \frac{\Delta\Theta}{\Theta_0}$ is relatively small for the real atmosphere. One may therefore feel justified in dropping the terms proportional to this parameter in (31). In this case, the boundary conditions, after introducing the wave speed c', could be written –

$$(U^* - c')\frac{\partial \phi'}{\partial p^*} = U^*_{p^*}\phi' \quad \text{at } p^* = 1$$
$$\phi' < \infty \qquad \text{at } p^* = 0 \qquad (32)$$

The first condition in (33) is equivalent to $\omega = 0$ at $p^* = 1$, which condition has been used in many earlier investigations. A more precise analysis of the error contained in this approximation will be given later.

11. Justification of the Expansion

The validity of the previous expansion of the coefficients A, $B - I$ will depend on the actual smallness of the parameters δ, ε, and η. To get an estimate of these we start out from the following basic figures:

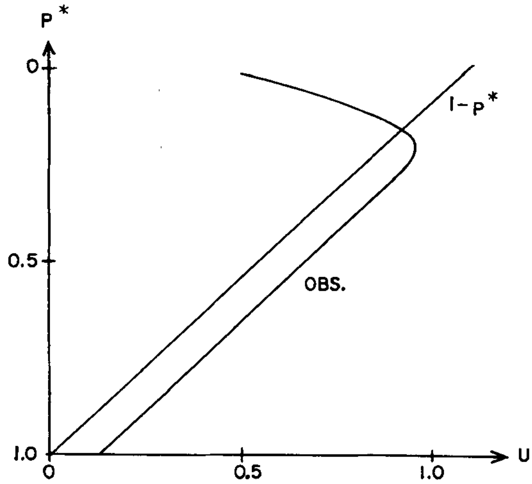
$R \cong 6.4 \cdot 10^6$ m, $\Omega \cong 7.3 \cdot 10^{-5}$ s^{-1}, $p_0 \cong 10^5$ kg m^{-1}s^{-2}, $\varrho_0 \cong 1.3$ kg m^{-3}. The vertical variation of the potential temperature over the main part of the atmosphere seems to lie in the range 30—60° A, and thus we put $\sigma \cong 0.1$—0.2. We assume the period of the waves to be at least, say, 10 days and range up to infinity for the purely stationary waves. With these values the non-dimensional parameters of interest are estimated for three different amplitudes of the zonal current, $U_0 = 5$, 15 and 25 ms^{-1}:

Table 1. Numerical values of the characteristic parameters

	δ	ε	η	σ	M	$\frac{\delta}{\varepsilon}$	$\frac{\eta}{\varepsilon}$	$\frac{\delta\eta}{\varepsilon}$	$\frac{M^2}{\varepsilon}$
$U_0 = 5$ ms^{-1} ...	0—0.1	0.01	0.002—0.004	0.1—0.2	0.02	0—10	0.2—0.4	0—0.04	0.04
$U_0 = 15$ ms^{-1} ...	0—0.1	0.04	0.02—0.04	0.1—0.2	0.06	0—3	0.5—1	0—0.01	0.1
$U_0 = 25$ ms^{-1} ...	0—0.1	0.06	0.05—0.1	0.1—0.2	0.1	0—2	1—2	0—0.2	0.2

With δ, ε and η in the above ranges the expansion is convergent and the first approximation is reasonably good, provided we limit ourselves to the basic mode in the vertical and to wave numbers less than 5—6 in the horizontal (corresponding to a wave length of 6,000—8,000 km). For the higher modes in the vertical the wave lengths must be more than 10,000 km to make the expansion valid.

The fact that $\frac{\eta}{\varepsilon}$ comes out so close to 1 indicates a balance between all the terms retained in the wave equation (30). The numbers $\frac{M^2}{\varepsilon}$



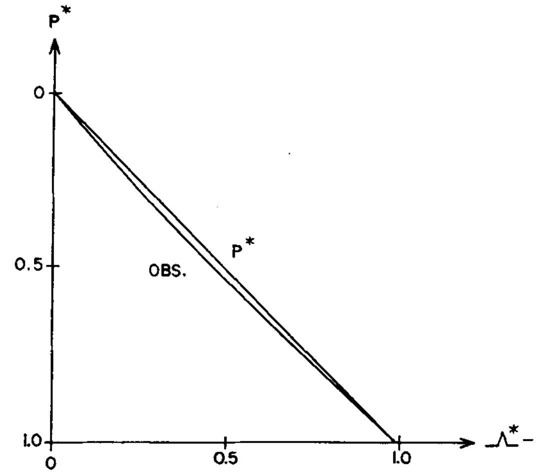


Fig. 2. Assumed and observed profiles for U* and Λ^{*-1}. Observations in winter at 50° N.

and σ seem to be small enough to give us some confidence in the simplified boundary condition (33). The smallness of $\frac{\delta\eta}{\varepsilon}$, finally, confirms the idea of a close geostrophic balance of the long waves.

12. An approximate solution for the case $U^* = 1 - p_0^*, \Lambda^* = p^{*-1}$

It has so far not been found possible to derive solutions valid over the whole sphere. One can, however, derive solutions that corresponds to the so-called β-plane solutions constructed in many earlier theories. As pointed out later the β-plane derivation yields, to the first order, the same wave equation as derived here. Equation (30) has simply to be replaced by

$$(U^* - c')\frac{\partial}{\partial p^*}\left(\frac{1}{\Lambda^*}\frac{\partial \phi'}{\partial p^*}\right) + \left[R\Omega\frac{\beta}{f^2}\frac{\varepsilon}{\eta} - \left(\frac{U^*_{p^*}}{\Lambda^*}\right)_{p^*}\right]\phi' = 0 \qquad (30\text{ a})$$

where f is the Coriolis parameter and β is the meridional variation of this parameter, respectively. In the β-plane approximation these are treated as constants.

As one simple case we make now the basic wind and potential temperature be linear functions of p. The wind is assumed to be zero at the ground, but one can of course always superimpose uniform currents (this will only correspond to a slight change in ε and η, and in the bottom boundary condition). The normalized profiles are

$$U^* = 1 - p^*$$
$$\Lambda^* = p^{*-1}$$

The variation of Λ inversely to the pressure is due to the appearance of the density in the denominator. To our degree of approximation we can put the density proportional to the pressure.

As an example, we show profiles for U^* and Λ^{*-1} computed from atmospheric data (mean winter condition at 50° N) and according to the assumed laws – (Fig. 2).

In the present case equation (30) reduces to the form

$$(1 - c' - p^*)\frac{\partial}{\partial p^*}\left(p^*\frac{\partial \phi'}{\partial p^*}\right) + \left(1 + \frac{R\Omega\beta}{f^2}\frac{\varepsilon}{\eta}\right)\phi' = 0 \qquad (34)$$

The boundary conditions become

$$c'\frac{\partial \phi'}{\partial p^*} = \phi^* \quad \text{at } p^* = 1$$
$$\phi' < \infty \quad \text{at } p^* = 0 \qquad (35)$$

(34, 35) formulates an eigen-value problem with c' as the eigen-value parameter. c' is assumed to be real (neutral waves). The assumption that the (non-dimensional) frequency is small should not be interpreted as a condition that c' is small, it merely requires that $c' \ll \varepsilon^{-1}$.

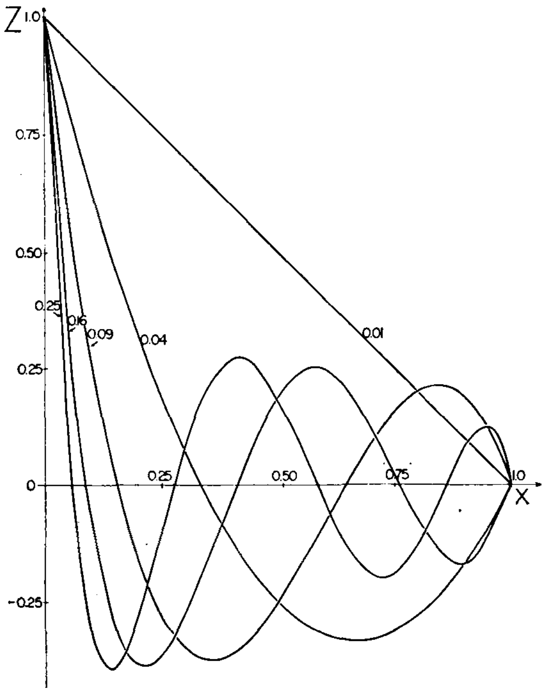


Fig. 3. Profiles of the stationary waves. The horizontal coordinate is ξ.

It has been found advantageous to transfer the eigen-value parameter to the boundary condition, by introducing the new independent variable

$$\xi = \frac{p^*}{1 - c'} \qquad (36)$$

For the sake of simplicity we introduce further

$$Z = \phi', \ \lambda = 1 + \frac{\beta}{f^2} \frac{\varepsilon}{\eta} \qquad (37)$$

Finally, with γ entering only parametrically, we may replace all partial derivatives by total derivatives. (34, 35) then take on the form

$$(1 - \xi) \frac{d}{d\xi} \left(\xi \frac{dZ}{d\xi} \right) + \lambda Z = 0 \qquad (38)$$

$$\begin{aligned} \frac{dZ}{d\xi} &= \frac{1 - c'}{c'} Z \quad \text{at } \xi = \frac{1}{1 - c'} \\ Z &< \infty \qquad \text{at } \xi = 0 \end{aligned} \qquad (39)$$

Equation (38) is of hypergeometric type. It has singularities at $\xi = 0$ and 1. Solutions that remain finite throughout the interval $0 \le \xi \le 1$ are found when $\lambda = n^2$, ($n = 0, 1, 2 \ldots$). These will be eigen-solutions for the stationary waves. They are all polynomials. The first few are listed below:

$$\begin{aligned} Z_0 &= 1 \\ Z_1 &= 1 - \xi \\ Z_2 &= 1 - 4\xi + 3\xi^2 \\ Z_3 &= 1 - 9\xi + 18\xi^2 - 10\xi^3 \\ Z_4 &= 1 - 16\xi + 60\xi^2 - 80\xi^3 + 35\xi^4 \\ Z_5 &= 1 - 25\xi + 150\xi^2 - 350\xi^3 + 350\xi^4 - 126\xi^5 \end{aligned} \qquad (40)$$

Graphs of these are seen in Fig. 3.

When λ is different from an integer square travelling waves do occur. These move very slowly compared to the Rossby waves. The wave speed and the vertical profile of the waves can be found as follows. For all values of λ the solution regular at $\xi = 0$ is given by

$$Z = 1 - \lambda\xi - \lambda \frac{1 - \lambda}{4} \xi^2 - \lambda \frac{1 - \lambda}{4} \frac{\lambda - 4}{9} \xi^3 + \ldots \qquad (41)$$

The series can be used for $\xi < 1$. By (41) Z has been computed for different values of ξ and λ and the manifold of solutions has been constructed (Fig. 4). Points have then been sought at which the curves of solution are tangent to a straight line through $(+1, 0)$. If ξ_0 is the abscissa for such a point, the value of c' is found from the relation $\xi = \frac{1}{1 - c'}$, and the eigen-solution is represented by the part of the curve from $\xi = 0$ to ξ_0 (Fig. 5). The procedure is easily understood if we note that the condition $\frac{dZ}{d\xi} = \frac{1 - c'}{c'} Z$ integrates to $Z = \text{const.}\ (\xi - 1)$ if use is made of the above relation between ξ and c'. The wave speed can be directly read as a function of λ in Fig. 6.

13. Influence of σ in the Boundary Condition

Of the approximations made the most serious one may be the neglect of terms proportional to $\sigma \left(= \frac{\Delta\Theta}{\Theta_0} \right)$ in the boundary condition.

Retaining these terms the condition replacing (35) is –

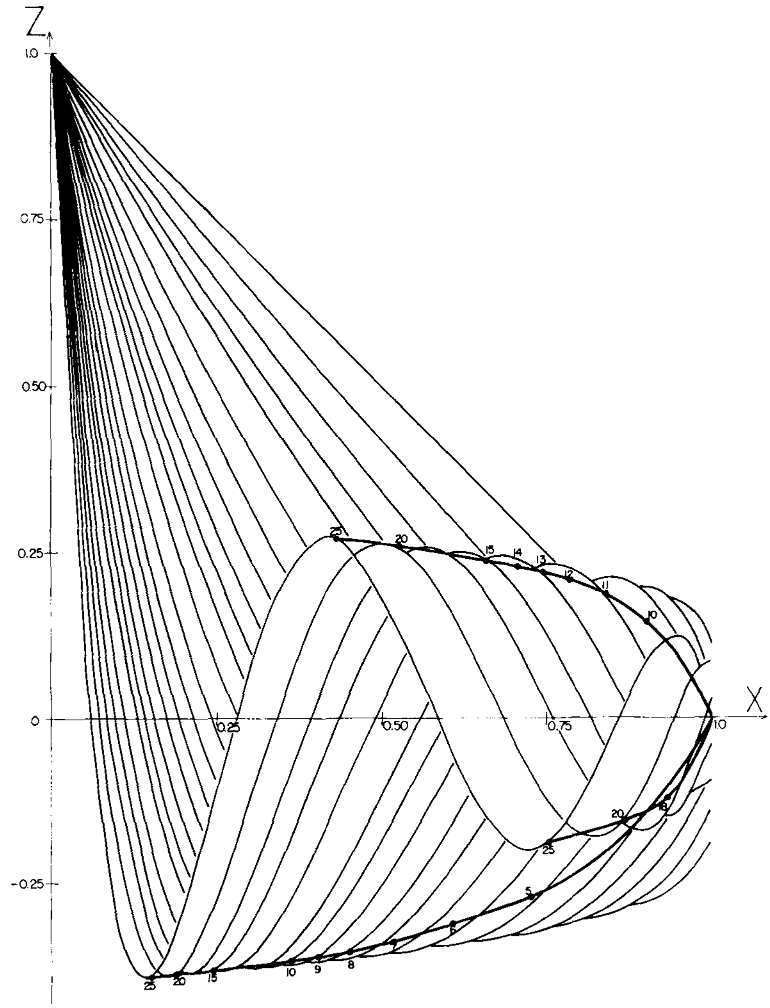


Fig. 4. The manifold $Z(\xi, \lambda)$. The horizontal coordinate is ξ.

$$c' \frac{\partial \phi'}{\partial p^*} = (1 - c'\sigma)\phi' \quad \text{at } p^* = 1$$
$$\phi' < \infty \qquad \text{at } p^* = 0 \tag{42}$$

or transformed to the new variables

$$\frac{dZ}{d\xi} = \frac{(1 - c')(1 - \sigma c')}{c'} Z \quad \text{at } \xi = \frac{1}{1 - c'}$$
$$Z < \infty \quad \text{at } \xi = 0 \tag{43}$$

In the stationary case ($c' = 0$) the condition is the same as (39). In the case of travelling waves the condition differs but there is little difficulty to extend the previous analysis to this case.

The graphical method functions again but the straight lines have now to be replaced by the family of curves $Z = \text{const.} \frac{\xi - 1}{\xi^\sigma}$. The resulting wave speeds have been computed for $\sigma = 0.2$ and the result is shown in Fig. 7. One verifies that little is changed for stationary or slowly moving waves.

14. Difference between the Spherical and β-Plane Coordinates

Using the β-plane coordinates and repeating the derivations of Section II—V a different

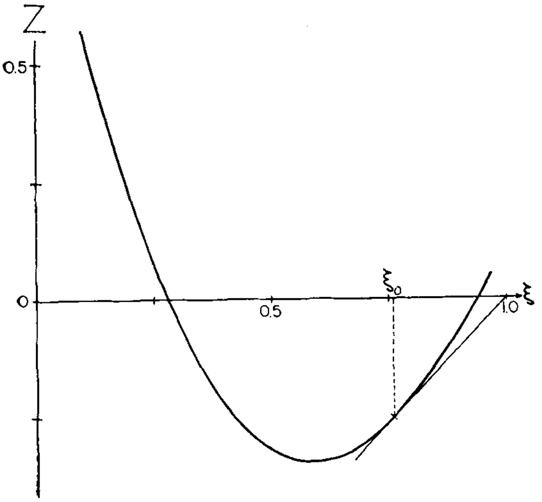


Fig. 5. Construction for the travelling waves $\left(\xi_0 = \frac{1}{1-c'}\right)$.

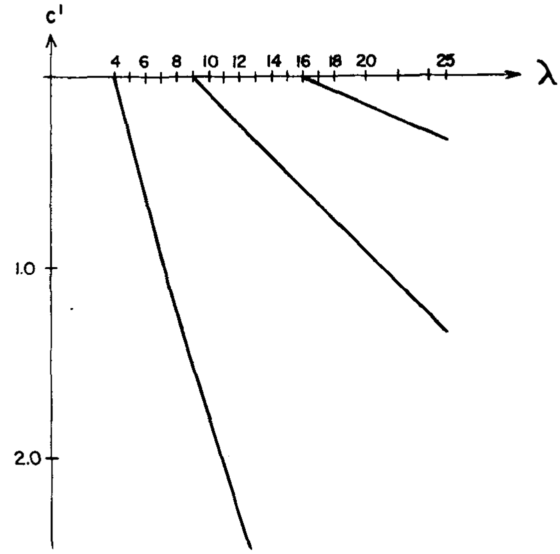


Fig. 6. c' as function of λ.

set of equations arise. However, making the same non-dimensionalization and expansion as before, the first order wave equation agrees with equation (30). Thus, the main spherical effect seems to be contained in the β-term. This result is, however, a special one, and one should not be led to the conclusion that the β-plane coordinates generally work correctly

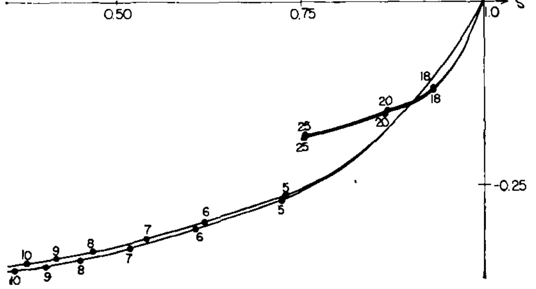


Fig. 7. Influence of σ on ξ_0. The curves represent the two cases $\sigma = 0$ and $\sigma = 0.2$.

for the largest scales of motions in the atmosphere. A closer analysis shows that the β-plane can be used only when the meridional coupling of the waves is slight, and the perturbation motion is closely geostrophic. Both these conditions are met with in our first order theory.

It may be of value to give the specific rules of computation that should be added in a β-plane analysis to make this a true spherical analysis. Subject only to our general assumptions in Section 3, the rules turn out to be the following –

i) In the equations of motion the Coriolis parameter f should be replaced by $f + \frac{\operatorname{tg}\varphi}{R} u$.

ii) In the continuity equation the term $\frac{\partial v}{\partial y}$ should be replaced by $\frac{\partial v}{\partial y} - \frac{\operatorname{tg}\varphi}{R} v$

The operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ do not commute but follow the rule

$$\frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y} + \frac{\operatorname{tg}\varphi}{R} \frac{\partial}{\partial x}$$

(When wave numbers are introduced, the meridional wave number becomes accordingly a function of y).

iii) Not only f but also β (and what higher y-derivatives of f that occur) should be considered functions of y. At the end one introduces $f=2\Omega\sin\varphi$ $\beta=\frac{2\Omega}{R}\cos\varphi$, etc.

As one simple example, we compute the divergence of the geostrophic wind by the two methods. In the β-plane as well as in the spherical case the geostrophic equations are, by definition,

$$\begin{aligned} -fv &= -\frac{\partial\phi}{\partial x} \\ fu &= -\frac{\partial\phi}{\partial y} \end{aligned} \tag{44}$$

The expressions for the divergence (D) are however, formally different. In the β-plane we have $D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ and insertion of (44) gives

$$D=-\frac{\beta}{f^2}\frac{\partial\phi}{\partial x} \tag{45}$$

In the spherical case we have

$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-\frac{\operatorname{tg}\varphi v}{R}$. Inserting (44) gives –

$$\begin{aligned} D=\frac{1}{f}\left(\frac{\partial^2}{\partial y\partial x}-\frac{\partial^2}{\partial x\partial y}\right)\phi-\frac{\beta}{f^2}\frac{\partial\phi}{\partial x}- \\ -\frac{\operatorname{tg}\varphi}{R}\frac{1}{f}\frac{\partial\phi}{\partial x}=-\frac{\beta}{f^2}\frac{\partial\phi}{\partial x} \end{aligned} \tag{46}$$

where the rule (ii) has been applied. Thus the results are identical. From point of view of the spherical analysis the success of the β-plane computation can be considered as the cancellation of two errors: the term $\frac{-\operatorname{tg}\varphi}{R}v$ is neglected in the divergence but at the same time we force a commutation of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Obviously the outcome of the two methods will not be the same when we have a general non-geostrophic motion and the one obtained from the spherical analysis has, of course, to be considered as the more correct one.

15. Some Comments on the Problem of Numerical Weather Prediction

In the attempts to make numerical weather predictions the difficulty of getting a correct movement of the very long waves have been noticed. In these computations the starting point is usually the vorticity equation. This equation reads –

$$\begin{array}{llllll} \frac{\partial Z}{\partial t} & +u\frac{\partial Z}{\partial x} & +v\frac{\partial Z}{\partial y} & +w\frac{\partial Z}{\partial p} & & +ZD \\ o(\delta) & o(\varepsilon) & o(\varepsilon) & o\left(\eta,\frac{\delta\eta}{\varepsilon}\right) & & o\left(\eta,\frac{\delta\eta}{\varepsilon}\right) \\ +fD+ & \beta v & +\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}=0 & & & \\ o(1) & o(1) & & & & \\ \multicolumn{2}{c}{\left(o(\delta,\varepsilon,\frac{\delta\eta}{\varepsilon}\right)} & -\quad o\left(\eta,\frac{\delta\eta}{\varepsilon}\right) & & & \end{array} \tag{47}$$

Inserting the expressions for u', v', ω' from (26) in the linearized form of (47) we can estimate the order of the different terms. These orders, apart from a common factor $R^{-2}\phi'$, are written under the equation (47). From the point of view of our theory, the only term that can be safely neglected is $\frac{\partial w}{\partial x}\frac{\partial v}{\partial p}$ (it does not occur at all in the linearized theory). fD and βv are both of order 1 but $fD+\beta v$ is of order $o\left(\delta,\varepsilon,\frac{\delta\eta}{\varepsilon}\right)$. Since δ, ε and η may be of the same order all the other terms have then to be retained. This is in accordance with the analysis of Burger (1958).

The fact that the divergence term and the β-term balance out, to the first order, will of course not in itself display the quasi-stationary character of the longest waves. The motion of the waves is now determined by higher order terms, but whether or not they become quasi-stationary cannot be decided merely by inspecting the vorticity equation. For one thing, the boundary conditions have to be considered.

The barotropic model states the conservation of absolute vorticity: $\frac{d}{dt}(Z+f)=0$. This case is obtained from (47) by putting $\frac{\partial}{\partial p}=0$, $D=0$. This is not a consistent approximation for the very long waves. In this case the β-term alone is of order 1 and our first order theory gives only the result $v'=0$. In fact, a more consistent approximation would be to conserve the relative vorticity: $\frac{dZ}{dt}=0$, i.e., neglect completely the effect of earth's rotation in the vorticity equation. In this case, of course, both the divergence and the β-term disappear.

In the derivation of the wave equation directly from the equations of motion, the continuity equation and the adiabatic equation the difficulty of cancelling first order does not occur. Thus, one may try to construct a scheme for numerical prediction for the largest scales of motion along a similar line. The equation used for the actual time-integration should then be the adiabatic equation. This equation would give a prediction of the potential temperature and thus through the hydrostatic equation also of the geopotential, if initial geopotential and velocity fields are known. To make a new time-step one has then to determine the new velocity field. For this, one has at one's disposal, the continuity equation and the two horizontal equations of motion.

In the first approximation, these equations of motion would be the geostrophic wind equations, but one would probably like to go one step further. One should still be able to treat the motion as balanced, i.e., merely solve for a steady state flow under the given pressure forces. The computation of the vertical velocity by help of the continuity equation is not critical in the case of the largest scales of motion, for the divergence will then be comparable to the individual terms in the continuity equation.

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