On the Steady Flow of a Fluid of Variable Density Past an Obstacle

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Abstract

A model of steady airflow past a three-dimensional mountain is considered. The fluid is inviscid and incompressible. The flow is under the influence of gravity, but for most of the work the earth's rotation is neglected. Far upstream the flow is parallel and horizontal, the velocity $U(z)$ and the density $\varrho(z)$ varying with height z. By use of conservation of density and head along streamlines, and thence of the dependence of density and head on one variable z_0 only, the equations of motion are simplified, z_0 ($\mathbf{r}$) being the height far upstream of the streamline through the point with position vector $\mathbf{r}$. The flow is considered first with small and then with large density variations far upstream. In the case when these variations are small, results of the known method of simple-shear secondary flows are rederived and extended.

The chief consideration of this paper is the buoyancy effects of large variations of density in the oncoming stream. For simplicity, the shear far upstream is neglected. The primary and secondary velocities are found for large Richardson number (proportional to $gd\varrho/\varrho dz_0$). The method is time-symmetric and analogous to taking the first two terms of the Janzen-Rayleigh expansion, the inverse of the Richardson number being the analogue of the square of the Mach number. The primary velocity, for infinite buoyancy forces, is horizontal, the flow in each horizontal plane being the same as the potential flow about the section of the obstacle in that plane. In general there is shear between horizontal planes, with horizontal primary vorticity. The secondary flow is found explicitly for a circular cylinder with vertical axis and for a hemisphere resting in a horizontal plane. The secondary approximation is not uniformly valid near the horizontal plane at the height of the top (if any) of the obstacle. Near that plane there is an inviscid shear layer, and the velocity gradient cannot be neglected, however large the Richardson number.

1. Introduction

Theoretical work on airflow over mountains has been confined mainly to two types of steady inviscid flow of a parallel stream about an obstacle (cf. the review paper by CORBY in 1954). In the first type, the fluid is slightly heterogeneous, i.e. the vertical density gradient of the oncoming stream $d\varrho/dz$ is much less than the quotient ϱ/h of the density and of the height of the mountain, so that the equations may be linearised. In this way lee waves have been found by LYRA (1940) and others. Individual modes are two-dimensional, but Fourier integrals of them have been used to give the flow over a three-dimensional isolated mountain as well as a two-dimensional ridge.

The second type is two-dimensional channel flow. Such flows, both of one homogeneous fluid with a free surface and of homogeneous horizontal layers of fluids of different densities, first attracted interest last century. In 1953 LONG went on to consider the flow of a heterogeneous fluid past a two-dimensional obstacle. He found a general equation of flow and solved it in some linear cases. YIH (1959) has given further solutions.

SHEPPARD (1956) remarked that there had been no attempt to find when winds flow *over* and when *round* mountains. In work on the above two types of flow, it is assumed that flow is entirely over the mountain. For the first type of flow, it is assumed that the height of the mountain is small in order to linearise the equations and obtain the lee-wave solutions; so the flow round the mountain is neglected. For the second type, the flow is two-dimensional; so the mountain has no sides round which the wind can flow, motion being in vertical planes. However, if there are very large buoyancy forces in flow about an isolated three-dimensional mountain, the flow is constrained so that the vertical velocity is zero and motion is in horizontal planes. Thus we have two limits whose order profoundly affects the flow: (a) when an isolated mountain tends to a long ridge across the stream, its width tends to infinity; (b) the buoyancy forces due to density variation tend to infinity. This non-uniformity is important for large scale motions in the atmosphere, in which the buoyancy forces are dominant.

SHEPPARD (1956) estimated the kinetic energy of the oncoming wind necessary to supply the gravitational potential to lift a fluid particle to the top of the mountain. He supposed that the oncoming stream was in equilibrium and adiabatically cooled as it rose. With his characteristic values of wind speed and lapse rate, it appears that the flow about most mountains cannot be adequately described by either of the two extreme cases, flow entirely around or over. This is essentially equivalent to the assertion that the Richardson number, which is a characteristic ratio of the buoyancy forces due to the density variation to the inertial forces, is neither very small nor large in practice. To go further than Sheppard's rough argument it will be necessary to find the three-dimensional flow of a heterogeneous fluid, a flow which has neither a potential nor a stream function. We present a little progress toward this aim below.

To make the problem tractable we shall assume that the fluid is incompressible. This is a good approximation to adiabatic flow when the vertical motion is small (BATCHELOR 1953). Similarly, to reduce the problem to its bare essentials, we shall neglect viscosity and all heat transfer.

It is supposed that the flow far upstream is both parallel and horizontal, velocity and density varying with height alone. An equation is then deduced in section 2 by use of the conservation of density and of head along streamlines, this conservation implying that the density and head depend on only one variable, namely $z_0(\mathbf{r})$, the height far upstream of the streamline through $\mathbf{r}$. In the case of two-dimensional flow, the equation reduces to that of LONG (1953), who first used the convenient auxiliary variable z_0.

In section 3 it is assumed that the shear, Richardson number and Froude number of the oncoming stream are small. This leads to the method of simple-shear secondary flow due to SQUIRE and WINTER (1951), HAWTHORNE and MARTIN (1955), LIGHTHILL (1956) and others. Their results are deduced simply and slightly extended. The primary flow in this case is the potential flow of a uniform stream about the obstacle. The secondary flow is found by neglecting products of the differences of quantities from their primary values. This approximation requires that the height of the mountain is much less than the vertical length scale of density variation, as is so for Lyra waves.

In section 4 it is assumed that the Richardson number is large, corresponding to neglect of flow over the mountain. In this case the primary flow is in horizontal planes, between which there is shear though there is no vertical vorticity. This primary flow is irrotational only if the obstacle is a cylinder with vertical generators. Study of these two extremes, large and small Richardson number, will indicate the comparative importance of flow round and over an obstacle for the finite values of Richardson number met in practice. The velocity is expanded as a power series in the inverse of the Richardson number, and successive approximations are taken from the equations of motion. These give the primary, secondary, tertiary flows *etc.* in turn.

The secondary flows about two simple obstacles are found in this way. In section 5 a circular cylinder with vertical axis is taken as typical of a cylinder with vertical generators. The primary flow is the classical two-dimensional horizontal irrotational flow. The secondary flow is shown to be vertical, and is calculated. The second obstacle is a hemisphere resting on a horizontal plane. The primary flow is in horizontal planes with no vertical component of vorticity. The secondary flow is calculated in section 6, but found not to be a uniformly valid approximation near the level of the top of the hemisphere. In view of the limitations of ideal-flow theory for bodies that are not of streamlined shape, these cases should be regarded as illustrative examples.

In the concluding section 7, the non-uniformity of the convergence is discussed, and qualitative results are related to neglect of viscosity, to wave-motion, and to experiment.

2. Equations of flow

Consider the steady flow of a parallel horizontal stream of inviscid incompressible fluid of variable density about an obstacle. Suppose the flow is on a rotating earth and under the influence of gravity. Choose cartesian axes o (x, y, z) parallel to unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ fixed to the surface of the earth. Let the z-axis be the upward vertical and the x-axis be directed downstream; then the y-axis makes up the orthogonal right-handed triad. Let $z_0(\mathbf{r})$ be the height far upstream of the streamline through the point $\mathbf{r}$. Suppose the velocity $\mathbf{u} = U(z_0)\,\mathbf{i}$ and the density $\varrho = \varrho(z_0)$ far upstream (where $x \to -\infty$ and $z_0 \to z$) are given. By a well-known approximation in meteorology for motion of a large horizontal scale, we take the earth's angular velocity to be vertical and constant, and denote it by $\frac{1}{2} f\mathbf{k}$.

For steady flow of an incompressible fluid, density is constant on a streamline, i.e.

$$\mathbf{u} \cdot \nabla \varrho = 0, \tag{2.1}$$

or

$$\varrho(\mathbf{r}) = \varrho(z_0),$$

the density far upstream. Thus the equation of continuity becomes

$$\nabla \cdot \mathbf{u} = 0. \tag{2.2}$$

A vector form of Euler's equations of motion for an inviscid fluid under gravity referred to a frame fixed to the earth is

$$(f\mathbf{k} + \boldsymbol{\omega}) \times \mathbf{u} = -\varrho^{-1} \nabla p - \nabla \left(gz + \frac{1}{2}\mathbf{u}^2\right), \tag{2.3}$$

where the (relative) vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$. Therefore

$$(f\mathbf{k} + \boldsymbol{\omega}) \times \mathbf{u} = -\nabla H - p\varrho^{-2} \nabla \varrho, \tag{2.4}$$

where the head

$$H(\mathbf{r}) \equiv p/\varrho + gz + \frac{1}{2}\mathbf{u}^2. \tag{2.5}$$

The scalar product of $\mathbf{u}$ and equation (2.4) yields

$$\mathbf{u} \cdot \nabla H = 0. \tag{2.6}$$

This is Bernoulli's theorem that H is constant on a streamline, and therefore equals its value far upstream, i.e.

$$H = p(y_0, z_0)/\varrho(z_0) + gz_0 + \frac{1}{2} U^2(z_0). \tag{2.7}$$

But

$$p = -g\int_0^{z_0} \varrho(z)\, dz + f y_0 U(z_0)\, \varrho(z_0) + \text{constant}$$

there. We may choose the origin of pressure so that $\lim_{x \to -\infty} (p)_{y=z=0} = 0$. Then

$$H = g\varrho^{-1} \int_0^{z_0} z\, d\varrho/dz\, dz + \frac{1}{2} U^2 + f y_0 U \tag{2.8}$$

and

$$p = -g\int_0^z \varrho\, dz + g\varrho(z_0 - z) + \frac{1}{2}\varrho(U^2 - \mathbf{u}^2) + f y_0 U \varrho.$$

Now substitution of $\varrho(z_0)$ for $\varrho(\mathbf{r})$ and $H(y_0, z_0)$ for $H(\mathbf{r})$ in equation (2.4) gives

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{u} + f\mathbf{k} \times (\mathbf{u} - U\mathbf{i}) = - \\ -\left\{ -\frac{1}{\varrho}\frac{d\varrho}{dz_0}\left[g(z - z_0) - \frac{1}{2}(U^2 - \mathbf{u}^2)\right] + \right. \\ \left. + U\frac{dU}{dz_0} + f y_0 \frac{dU}{dz_0} \right\} \nabla z_0 - fU \nabla y_0. \end{aligned} \tag{2.9}$$

If $f = 0$ it follows that z_0 is constant on vortex- as well as streamlines. Therefore ϱ and H are also.

The curl of equation (2.9) is

$$\begin{aligned} \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} - f\frac{\partial}{\partial z}(\mathbf{u} - U\mathbf{i}) = \\ = \frac{d\varrho}{\varrho\, dz_0} \nabla \left(gz + \frac{1}{2}\mathbf{u}^2\right) \times \nabla z_0 + \\ + f\frac{dU}{dz_0} \nabla y_0 \times \nabla z_0. \end{aligned} \tag{2.10}$$

If $f=0$ and $d\varrho/dz_0=0$, this vorticity equation becomes the Helmholtz equation $\nabla\times(\boldsymbol{\omega}\times\mathbf{u})=0$.

Suppose the obstacle has characteristic length scale h and $U(z_0)$ has velocity scale V. We shall not distinguish between scales of height, width and length of the obstacle. For dynamical similarity it is sufficient that geometrically similar obstacles are considered for given values of the relevant dimensionless parameters. Special assumptions will be added later for a long mountain ridge (i.e. for two-dimensional flow when the width is much greater than the length) or a vertical cylinder (with infinite height).

We define the Richardson number

$$J\equiv-\frac{gh^2}{V^2}\frac{d\varrho}{\varrho dz_0},\qquad(2.11)$$

a measure of the ratio of buoyancy to shear forces. We have chosen J as a function of z_0, but shall assume later that ϱ is such that J is effectively constant, as when $\varrho(z_0)$ varies exponentially or is nearly constant. The Froude number

$$F\equiv V^2/gh.$$

In fact we shall use

$$L\equiv-\frac{h}{\varrho}\frac{d\varrho}{dz_0}=JF\qquad(2.12)$$

instead of F. It will be seen from the equation of motion that L represents the variation of inertia due to the heterogeneity. A parameter of shear,

$$M\equiv\frac{Uh}{V^2}\frac{dU}{dz_0},\qquad(2.13)$$

represents rotational inertia due to the shear far upstream (M is also a function of z_0, but is constant when U varies as $z_0^{\frac{1}{2}}$). The Rossby number,

$$Ro\equiv V/hf,\qquad(2.14)$$

measures the ratio of the inertial to the coriolis forces.

On dividing out the dimensions of all quantities in equations (2.9) and (2.10) in the usual way, we get

$$\boldsymbol{\omega}\times\mathbf{u}+(Ro)^{-1}\mathbf{k}\times(\mathbf{u}-U\mathbf{i})=-\left\{J(z-z_0)-\frac{1}{2}L(U^2-\mathbf{u}^2)+M+(Ro)^{-1}My_0/U\right\}\nabla z_0-(Ro)^{-1}U\nabla y_0\qquad(2.15)$$

and

$$\mathbf{u}\cdot\nabla\boldsymbol{\omega}-\boldsymbol{\omega}\cdot\nabla\mathbf{u}-(Ro)^{-1}\left\{\frac{\partial\mathbf{u}}{\partial z}-MU^{-1}\mathbf{i}\right\}=-\left(J\mathbf{k}+\frac{1}{2}L\nabla\mathbf{u}^2-(Ro)^{-1}MU^{-1}\nabla y_0\right)\times\nabla z_0.\qquad(2.16)$$

Equation (2.15) gives the components of $\boldsymbol{\omega}$ in the directions of ∇z_0 and $\mathbf{u}\times\nabla z_0$ from

$$\boldsymbol{\omega}\cdot\nabla z_0=(Ro)^{-1}\mathbf{k}\cdot\{\boldsymbol{\omega}\times(\mathbf{u}-U\mathbf{i}\}/\left\{J(z-z_0)-\frac{1}{2}L(U^2-\mathbf{u}^2)+M+(Ro)^{-1}My/U\right\}\qquad(2.17)$$

and

$$\boldsymbol{\omega}\cdot(\mathbf{u}\times\nabla z_0)=-(\nabla z_0)^2\left\{J(z-z_0)-\frac{1}{2}L(U^2-\mathbf{u}^2)+M+(Ro)^{-1}My_0/U\right\}-(Ro)^{-1}U\nabla y_0\cdot\nabla z_0-(Ro)^{-1}\mathbf{k}\cdot\{(\mathbf{u}-U\mathbf{i})\times\nabla z_0\}\qquad(2.18)$$

Our chief aim is to study the influence of the buoyancy forces due to the heterogeneity of the fluid. These are measured by the Richardson number, which is a maximum when the vertical scale of the motion is largest, if the other parameters do not vary. The vertical scale is a maximum when it is of the order of the height of the highest mountain, which is about the height of the troposphere and the scale height of the atmosphere's density variation, say 7 km for our crude estimate here. Then, by use of metres and seconds, $J\equiv-gh^2V^{-2}d\varrho/\varrho dz_0\lesssim10\cdot(7\times10^3)^2\cdot(30)^{-2}\cdot(7\times10^3)^{-1}\approx100$, if $V\approx30$ m sec^{-1}. Such a limit would be obtained in motion with a large horizontal scale only, such as flow round a continental mountain range. It is on such large scales that the Rossby number is small and the effects of the earth's rotation are significant. In spite of this, we shall neglect the earth's rotation hereafter in order to study the buoyancy forces more easily. Thus, putting $f=0$, we see that equation (2.15) becomes

$$\boldsymbol{\omega} \times \mathbf{u} = -\left\{J(z-z_0) - \frac{1}{2}L(U^2-\mathbf{u}^2)+M\right\} \nabla z_0, \tag{2.19}$$

which we shall solve for small and large J in succeeding sections.

The rôle of the kinetic pressure is elucidated by the transformation

$$\mathbf{u}' \equiv \varrho^{\frac{1}{2}}\mathbf{u}, \quad \boldsymbol{\omega}' \equiv \nabla \times \mathbf{u}'. \tag{2.20}$$

It leads to the dimensional equations

$$\mathbf{u}' \cdot \nabla \varrho = 0, \tag{2.1'}$$

$$\nabla \cdot \mathbf{u}' = 0, \tag{2.2'}$$

$$\boldsymbol{\omega}' \times \mathbf{u}' = -\nabla \frac{1}{2} U'^2 + g(z-z_0) \nabla \varrho \tag{2.9'}$$

if $f = 0$, where $U' \equiv \varrho^{\frac{1}{2}}U$. If we define $J'(z_0) \equiv \varrho(z_0)J/\bar{\varrho}, M'(z_0) \equiv (hU'dU'/dz_0)/\bar{\varrho}V^2$ for some constant characteristic density $\bar{\varrho}$, a dimensionless form of equation (2.9') is

$$\boldsymbol{\omega}' \times \mathbf{u}' = -\{J'(z-z_0)+M'\} \nabla z_0. \tag{2.19'}$$

Thus for each solution of equation (2.19) for given $U(z_0)$, $\varrho(z_0)$ and given obstacle there corresponds a solution of equation (2.19') and *vice versa*. Equation (2.19') is simpler, especially when $\varrho(z_0)$ or U' is constant (and therefore J' or M' respectively vanishes). This transformation gathers the acceleration of a fluid particle and the buoyancy due to the variation of density in single terms in equation (2.9'), leaving the remainder of the buoyancy and pressure forces equal to the gradient of the kinetic energy density far upstream. YIH (1959) used the above transformation for steady flow; others (cf. ECKART and FERRIS 1956) have applied it in special cases, such as wave-motions.

Before using equation (2.19) for practical problems, let us relate it to the two-dimensional case, treated by LONG in 1953. For a velocity field independent of the cross-stream co-ordinate y and perpendicular to $\mathbf{j}$, there is a stream function $\psi(z,x)$ such that $\mathbf{u} = \mathbf{j} \times \nabla\psi$. Then the vorticity $\boldsymbol{\omega} = (\nabla^2\psi)\mathbf{j}$. Both ψ and z_0 are constant on a streamline, so $\psi(\mathbf{r}) = \psi(z_0)$ with $d\psi/dz_0 = U(z_0)$. Working from the dimensional form of the vorticity equation, LONG (1953) found

$$\nabla^2 z_0 + \frac{1}{2}[(\nabla z_0)^2 - 1]\frac{d}{dz_0}(\ln U^2\varrho) = \frac{g}{U^2\varrho}\frac{d\varrho}{dz_0}(z_0 - z). \tag{2.21}$$

This is the two-dimensional form of equation (2.9) when $f = 0$. Its relative simplicity comes from the velocity's being determined by one function (ψ or z_0) in two- but not in three-dimensional flow. LONG (1953, 1958) and YIH (1959) have found some simple exact solutions of equation (2.21) by use of the transformation (2.20).

3. The method of simple-shear secondary flows

SQUIRE and WINTER (1951) considered the flow of a weakly-sheared stream (of velocity $U(z)\mathbf{i}$) of homogeneous fluid about a cascade of aerofoils (with generators parallel to $\mathbf{k}$). They used successive approximation in the Helmholtz vorticity equation. The primary flow was the two-dimensional potential flow of a uniform stream about the obstacle (the aerofoils in their example). The secondary vorticity was then found by retaining the shear dU/dz but neglecting its square.

HAWTHORNE and MARTIN (1955) extended this method in two ways, making it of meteorological interest. They took the obstacle as a hemisphere resting on a horizontal plane; so their primary flow is a three- instead of a two-dimensional potential flow. They also took a small vertical density gradient far upstream. Then they calculated the secondary vorticity.

LIGHTHILL (1956) adopted a new co-ordinate, the 'drift function', to derive the basic results of SQUIRE and WINTER (1951). It helped him spot the secondary velocity without further calculation when the primary flow is two-dimensional. Lighthill's method is also suitable for analytic estimations at infinity, for he showed that the secondary flow was smaller asymptotically than the tertiary flow far downstream. This means that the secondary approximation is not uniformly valid there. He later (LIGHTHILL 1957) found an Oseen type of approximation, and suggested that a combination of the two solutions uniformly approximates the true solution, a damped wave in the lee of the obstacle.

We shall now derive the basic formulae of simple-shear secondary flow from equation (2.19) by neglecting products of the parameters J, L, M, which are supposed to be small. For an approximation to this order we may suppose that the parameters are constant because the effect of their variation is of tertiary order. This will give the known formulae simply. A new result is the derivation of the rotational component of the secondary velocity for a three-dimensional primary flow. This enables the complete secondary velocity to be found.

If $J=L=M=0$, then equation (2.19) gives $\boldsymbol{\omega}\times\mathbf{u}=0$. Thus there is a potential solution for the primary velocity,

$$\mathbf{u}_1=\nabla\phi_1. \qquad (3.1)$$

To get the secondary velocity for small J, L, M, it is convenient to consider a small perturbation $\mathbf{u}_2$ of $U(z_0)\nabla\phi_1$ rather than of $\nabla\phi_1$. (The primary flow in both cases is the same, because $U(z_0)\nabla\phi_1=\nabla\phi_1$ when $M=0$.) Thus put

$$\mathbf{u}=U(z_0)\nabla\phi_1+\mathbf{u}_2. \qquad (3.2)$$

Therefore $\boldsymbol{\omega}=M\nabla z_0\times\nabla\phi_1+\boldsymbol{\omega}_2, \qquad (3.3)$

where $\boldsymbol{\omega}_2\equiv\nabla\times\mathbf{u}_2$, because $M/U(z_0)=M$ to first order. Next substitute $\mathbf{u}$ and $\boldsymbol{\omega}$ into equation (2.19). Then

$$\boldsymbol{\omega}_2\times\nabla\phi_1=-G(z,z_0,q_1)\nabla z_0-$$
$$-M(\nabla z_0\cdot\nabla\phi_1)\nabla\phi_1, \qquad (3.4)$$

where

$$G(z,z_0,q_1)\equiv\left\{J(z-z_0)+\left(M-\frac{1}{2}L\right)(1-q_1^2)\right\}, \qquad (3.5)$$

$$q_1\equiv|\nabla\phi_1|. \qquad (3.6)$$

The difference of z_0 and z_1 is of secondary order. Therefore we can replace z_0 by z_1 in equation (3.4) and retain the same approximation of the secondary flow. Thus we suppose that the secondary vorticity is distorted by the primary flow rather than by the exact flow, and thereby neglect tertiary effects only. So henceforth in this section we replace z_0 by z_1, the height far upstream of the streamline of the potential primary flow through $\mathbf{r}$. We can find z_0 and thence z_1 by differentiating along a streamline.

$$\mathbf{u}\cdot\nabla z=-q\partial z_0/\partial s \qquad (3.7)$$

gives

$$z-z_0=\int_{-\infty}^{s}\mathbf{u}\cdot\mathbf{k}/q\,ds \qquad (3.8)$$

and therefore

$$z-z_1=\int_{-\infty}^{\phi_1}\partial\phi_1/q_1^2\partial z_1 d\phi_1 \qquad (3.9)$$

if $J=L=M=0$.

$\nabla\phi_1$ is directed along the streamlines of the primary flow, on which z_1 is constant. Therefore $\nabla\phi_1\cdot\nabla z_1=0$. Therefore equation (3.4) becomes

$$\boldsymbol{\omega}_2\times\nabla\phi_1=-G(z,z_1,q_1)\nabla z_1. \qquad (3.10)$$

Resolution of this equation in the directions $\mathbf{z}_1\equiv\nabla z_1/|\nabla z_1|$ and $\mathbf{n}_1\equiv\mathbf{z}_1\times\nabla\phi_1/q_1$ shows that

$$\omega_{2z_1}\equiv\boldsymbol{\omega}_2\cdot\mathbf{z}_1=0, \qquad (3.11)$$

$$\omega_{2n_1}\equiv\boldsymbol{\omega}_2\cdot\mathbf{n}_1=|\nabla z_1|G/q_1. \qquad (3.12)$$

It is in fact possible to choose orthogonal curvilinear coordinates with the respective directions $\mathbf{s}_1\equiv\mathbf{u}_1/q_1$, $\mathbf{n}_1$ and $\mathbf{z}_1$, because the primary flow is irrotational. In these coordinates $\mathbf{u}_1=(q_1, 0, 0)$ and $\nabla z_1=(0, 0, |\nabla z_1|)$. The square of a line element is

$$(ds)^2=(d\phi_1)^2/q_1^2+h_2^2dn_1^2+(dz_1)^2/(\nabla z_1)^2,$$

where h_2 may be arbitrarily chosen provided that $\partial/h_2\partial n_1=(\mathbf{z}_1\times\mathbf{s}_1)\cdot\nabla$. These coordinates (known from the primary flow) can now be used to find $\boldsymbol{\omega}_2\cdot\mathbf{s}_1$ and the rotational part $\mathbf{u}_{2r}$ of the secondary velocity.

As a vector identity,

$$0=\nabla\cdot\boldsymbol{\omega}_2=\frac{q_1|\nabla z_1|}{h_2}\left\{\frac{\partial(\omega_{2s_1}h_2/|\nabla z_1|)}{\partial\phi_1}+\right.$$
$$\left.+\frac{\partial(\omega_{2n_1}/q_1|\nabla z_1|}{\partial n_1}\right\}.$$

Therefore

$$\omega_{2s_1}=-\frac{|\nabla z_1|}{h_2}\int\frac{\partial(G/q_1^2)}{\partial n_1}d\phi_1, \qquad (3.13)$$

on integrating along a streamline. The constant of integration is such that $\omega_{2s_1}\to 0$ far upstream.

The rotational part $\mathbf{u}_{2r}$ of $\mathbf{u}_2$, i.e. a vector with the same curl as $\mathbf{u}_2$, can be found from the equation

$$0=\omega_{2z_1}=\frac{q_1}{h_2}\left\{\frac{\partial(h_2u_{2n_1})}{\partial\phi_1}-\frac{\partial(u_{2s_1}/q_1)}{\partial n_1}\right\}.$$

This implies that there exists a function ϕ_2 of ϕ_1, n_1 and z_1 such that

$$\mathbf{u}_2=\mathbf{u}_{2r}+\nabla\phi_2=q_{2r}\mathbf{z}_1+\nabla\phi_2. \qquad (3.14)$$

But

$$|\nabla z_1|\,G/q_1=\omega_{2n_1}=-\,q_1|\nabla z_1|\,\frac{\partial(q_{2r}/|\nabla z_1|}{\partial\phi_1}.$$

Therefore

$$\mathbf{u}_{2r}=-(\nabla z_1)\textstyle\int G/q_1^2\,d\phi_1 \qquad (3.15)$$

Thus $\mathbf{u}_2$ has been split into an irrotational part $\nabla\phi_2$ and a rotational part $\mathbf{u}_{2r}$, which is parallel to ∇z_1. To find ϕ_2, the continuity equation $\nabla\cdot\mathbf{u}_2=0$ must be used. This gives the Poisson equation

$$\nabla^2\phi_2=\frac{q_1|\nabla z_1|}{h_2}\frac{\partial}{\partial z_1}\left\{\frac{h_2|\nabla z_1|}{q_1}\int G/q_1^2d\phi_1\right\}, \qquad (3.16)$$

with known right-hand side. The complete secondary flow can now be found such that $\mathbf{u}_2$ is tangential to the surface of the obstacle and vanishes far upstream. LIGHTHILL (1957) has formulated what is essentially this problem by use of Green's functions.

In general, $\partial/\partial n_1=h_2(\mathbf{z}_1\times\mathbf{s}_1)\cdot\nabla$ can be found from $\mathbf{u}_1$ with auxiliary coordinates (e.g. spherical polars if the obstacle is a sphere). However, n_1 can be taken as the stream function $\psi_1(x, y)$ (such that $\mathbf{u}_1=\nabla\psi_1\times\mathbf{k}$) if the obstacle is a cylinder with vertical generators. For then the primary flow is two-dimensional and $z_1=z$, so that $\nabla\phi_1$, $\nabla\psi_1$ and $\nabla z_1=\mathbf{k}$ are an orthogonal right-handed triad. This is the case considered by SQUIRE and WINTER (1951) for a homogeneous fluid. We can recover their general results for $J=0$ by putting

$$G=\left(M-\frac{1}{2}L\right)(1-q_1^2)$$

into equations (3.12) and (3.13). This gives

$$\omega_{2n_1}=\left(M-\frac{1}{2}L\right)(1-q_1^2)/q_1, \qquad (3.17)$$

$$\omega_{2s_1}=q_1\left(M-\frac{1}{2}L\right)\int\partial q_1^{-2}/\partial\psi_1 d\phi_1, \qquad (3.18)$$

because $h_2=1/q_1$ in this case.

The same substitution in equation (3.15) gives

$$\mathbf{u}_{2r}=-\left(M-\frac{1}{2}L\right)\mathbf{k}\int(q_1^{-2}-1)d\phi_1. \qquad (3.19)$$

This is the formula seen by LIGHTHILL (1956) to satisfy equations (3.17) and (3.18). He also noted that $\mathbf{u}_{2r}$ alone (with $\phi_2=0$, i.e. with a secondary velocity with zero divergence) could satisfy the boundary conditions of zero normal velocity on the cylinder. Therefore the complete secondary velocity is $\mathbf{u}_{2r}$ in this case.

These integrals for $\boldsymbol{\omega}_{2s}$ and $\mathbf{u}_{2r}$ have been evaluated by many authors for flow of a homogeneous fluid about particular obstacles. To adapt their results to a heterogeneous fluid it is only necessary to replace M by $\left(M-\frac{1}{2}L\right)$, even if J is of the same order as L, M. This is because the buoyancy effect, being the product of the small departure of the streamlines from the horizontal, of the small variation of density, and of gravity, is of tertiary order when the primary flow is two-dimensional. The similar effects of L and M in causing vorticity can be seen in making the transformation (2.20).

4. Large Richardson number

Here the contrary assumption, that J is large, is made. For simplicity, suppose that $M=0$, i.e. $U(z_0)=1$. (If $M\neq 0$ we need only replace L by $(L-2M)$, as in section 3.) Then equation (2.19) becomes

$$\boldsymbol{\omega}\times\mathbf{u}=-\left\{J(z-z_0)-\frac{1}{2}L(1-\mathbf{u}^2)\right\}\nabla z_0. \qquad (4.1)$$

We formally write

$$z_0=z_1+J^{-1}z_2+J^{-2}z_3+\ldots, \qquad (4.2)$$

$$\mathbf{u}=\mathbf{u}_1+J^{-1}\mathbf{u}_2+J^{-2}\mathbf{u}_3+\ldots \qquad (4.3)$$

(All considerations of convergence will be postponed till section 7.) Therefore

$$\boldsymbol{\omega}=\boldsymbol{\omega}_1+J^{-1}\boldsymbol{\omega}_2+\ldots,$$

where $\boldsymbol{\omega}_n\equiv\nabla\times\mathbf{u}_n\ (n=1, 2, 3,\ldots)$.

Equation of coefficients of powers of J^{-1} in the continuity equation gives

$$\nabla\cdot\mathbf{u}_n=0; \qquad (4.4)$$

and in equation (4.1)

$$0 = -(z - z_1)\nabla z_1, \tag{4.5}$$

$$\boldsymbol{\omega}_1 \times \mathbf{u}_1 = \left\{ z_2 + \frac{1}{2} L(1 - q_1^2) \right\} \nabla z_1 - (z - z_1)\nabla z_2, \tag{4.6}$$

$$\boldsymbol{\omega}_1 \times \mathbf{u}_2 + \boldsymbol{\omega}_2 \times \mathbf{u}_1 = \{z_3 - L\,\mathbf{u}_1 \cdot \mathbf{u}_2)\}\nabla z_1 + \left\{ z_2 + \frac{1}{2} L(1 - q_1^2) \right\} \nabla z_2 - (z - z_1)\nabla z_3, \quad \text{etc.} \tag{4.7}$$

Equation (4.5) shows that

$$z_1 = z, \tag{4.8}$$

i.e. that the vertical velocity of the primary flow is zero. Equation (4.6) shows further that $\boldsymbol{\omega}_1$ is horizontal. Therefore

$$\mathbf{u}_1 = (u_1, v_1, 0), \tag{4.9}$$

where $$\partial u_1/\partial y = \partial v_1/\partial x. \tag{4.10}$$

Therefore there exists $\phi_1(\mathbf{r})$ such that

$$\mathbf{u}_1 = \nabla_h \phi_1, \tag{4.11}$$

where $\nabla_h \equiv (\partial/\partial x, \partial/\partial y, 0)$ is the horizontal gradient operator. Therefore

$$\boldsymbol{\omega}_1 = (-\partial^2\phi_1/\partial z\,\partial y, \partial^2\phi_1/\partial z\,\partial x, 0). \tag{4.12}$$

Now equation (4.4) gives

$$\nabla_h^2 \phi_1 = 0; \tag{4.13}$$

and equation (4.6)

$$z_2 = -\frac{1}{2}\partial q_1^2/\partial z - \frac{1}{2} L(1 - q_1^2). \tag{4.14}$$

Thus the primary flow is in horizontal planes, the flow in each horizontal plane being the same as in the two-dimensional potential flow round a cylinder whose cross-section is that cut in the plane by the real three-dimensional obstacle. In brief, the motion in each horizontal plane is independent of motion elsewhere. The vorticity is horizontal, because the flow is irrotational in each horizontal plane, but there is shear between horizontal planes. If the obstacle is a cylinder with vertical generators, the primary flow is the two-dimensional potential flow of a uniform stream about the cylinder.

On differentiating along a streamline, we found

$$w \equiv \mathbf{u} \cdot \mathbf{k} = q\partial(z - z_0)/\partial s. \tag{4.15}$$

Therefore $w_1 = 0$, because $z_1 = z$. To secondary order

$$w_2 = -q_1 \partial z_2/\partial s_1 = -q_1^2 \partial z_2/\partial \phi_1 = \frac{1}{2} q_1^2 \frac{\partial}{\partial \phi_1}\{\partial q_1^2/\partial z + L(1 - q_1^2)\}. \tag{4.16}$$

It should be noted that partial differentiation with respect to z is for constant x, y; whereas with respect to ϕ_1 it is on a streamline, which n general will vary with z.

The complete secondary flow can now be found from equation (4.7), which becomes

$$\boldsymbol{\omega}_1 \times \mathbf{u}_2 + \boldsymbol{\omega}_2 \times \mathbf{u}_1 = (z_3 - L\mathbf{u}_1 \cdot \mathbf{u}_2)\,\mathbf{k} - \left(\frac{1}{2}\partial q_1^2/\partial z\right)\nabla z_2 \tag{4.17}$$

It is more revealing to resolve this equation in the directions of $\mathbf{s} \equiv \mathbf{u}_1/q_1$, $\mathbf{n} \equiv \mathbf{k} \times \mathbf{s}$, and $\mathbf{k}$ rather than of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The right-handed triad $\mathbf{s}$, $\mathbf{n}$, $\mathbf{k}$ corresponds to orthogonal coordinates (ϕ_1, ψ_1, z) (where $\mathbf{u}_1 = \nabla\psi_1 \times \mathbf{k}$), though the coordinates are not single-valued curvilinear ones unless the primary flow is irrotational (and therefore two-dimensional). Resolution parallel to $\mathbf{s}$ merely verifies equation (4.16). For resolution parallel to $\mathbf{n}$, we get

$$\omega_{2z} = \frac{\mathbf{u}_1 \cdot \boldsymbol{\omega}_1}{q_1^2} w_2 - \frac{\partial q_1}{\partial z} \mathbf{n} \cdot \nabla z_2. \tag{4.18}$$

Note that $\omega_{1n} = \partial q_1/\partial z$. For $\mathbf{k}$, we can get z_3.

The definition of ω_{2z} and the continuity equation give the remaining unknown components u_2, v_2 of the secondary flow in terms of the known w_2 and ω_{2z}:

$$\partial v_2/\partial x - \partial u_2/\partial y = \omega_{2z}, \tag{4.19}$$

$$\partial u_2/\partial x + \partial v_2/\partial y = -\partial w_2/\partial z. \tag{4.20}$$

Therefore

$$\nabla_h^2 u_2 = -\partial\omega_{2z}/\partial y - \partial^2 w_2/\partial x\,\partial z, \tag{4.21}$$

$$\nabla_h^2 v_2 = \partial\omega_{2z}/\partial x - \partial^2 w_2/\partial y\,\partial z. \tag{4.22}$$

These are Poisson equations to find u_2, v_2 satisfying the boundary conditions on the obstacle and at infinity.

5. Flow about a vertical cylinder

If the primary flow is the two-dimensional flow about a cylinder with vertical generators, the work of the previous section simplifies, because ϕ_1 is independent of z. It follows that

$$\mathbf{u}_1 = \nabla\phi_1, \tag{5.1}$$

$$z_2 = -\frac{1}{2}L(1-q_1^2), \tag{5.2}$$

$$w_2 = -\frac{1}{2}Lq_1^2\partial q_1^2/\partial\phi_1, \tag{5.3}$$

$$\omega_{2z} = 0, \tag{5.4}$$

and

$$z_3 = L\mathbf{u}_1\cdot\mathbf{u}_2 - q_1\omega_{2n}. \tag{5.5}$$

Equation (5.4) implies that there exists $\phi_2(\mathbf{r})$ such that $\mathbf{u}_2 = (\partial\phi_2/\partial x,\ \partial\phi_2/\partial y,\ w_2)$. Therefore $\nabla_h^2\phi_2 = \nabla\cdot\mathbf{u}_2 = 0$. The unique solution $\phi_2 = 0$ satisfies the boundary conditions of zero normal velocity on the cylinder and of zero velocity at infinity. Therefore

$$\mathbf{u}_2 = w_2\mathbf{k}, \tag{5.6}$$

$$z_3 = -q_1\omega_{2n} = q_1^2\partial w_2/\partial\phi_1 \tag{5.7}$$

These quantities can be readily found for the circular cylinder $r^2 \equiv x^2+y^2 = 1$. Then

$$\phi_1 = (r+1/r)\cos\Theta,\ \psi_1 = (r-1/r)\sin\Theta \tag{5.8}$$

where $\Theta \equiv \tan^{-1}(y/x)$, and it can be show that

$$z_2 = \frac{1}{2}Lr^{-2}(r^{-2}-2\cos 2\Theta), \tag{5.9}$$

$$w_2 = -2Lr^{-3}\{\cos 3\Theta - r^{-2}(2-r^{-2})\cos\Theta\}, \tag{5.10}$$

$$\omega_{2s} = 6Lq_1^{-1}r^{-4}\sin 2\Theta\{2\cos 2\Theta - r^{-2}(3-r^{-4})\}, \tag{5.11}$$

$$\omega_{2n} = 2Lq_1^{-1}r^{-4}\{3\cos 4\Theta - r^{-2}(9-8r^{-2}+ +3r^{-4})\cos 2\Theta - r^{-2}(4-9r^{-2}+4r^{-4})\} \tag{5.12}$$

Note that ω_2 is finite at $r=1$, $\Theta = 0$ or π, where

$$q_1 = \{1-2r^{-2}\cos 2\Theta + r^{-4}\}^{\frac{1}{2}} = 0.$$

6. Flow about a hemisphere resting on a horizontal plane

The ideas of section 4 are applied here to a three-dimensional primary flow. To simplify the calculations, suppose $L=0$. (This is essentially the Boussinesq approximation. In fact L is usually much less than J in meteorological problems, L/J being the Froude number.) Take the rigid obstacle to be the hemisphere $x^2+y^2+z^2=1$ resting on the horizontal plane $z=0$. Then, for the primary flow, the hemisphere behaves like a cylinder of radius $\delta(z) \equiv (1-z^2)^{\frac{1}{2}}$ in each horizontal plane below the top T of the hemisphere, but does not disturb the uniform stream above T. Therefore

$$\phi_1 = \begin{cases}(r+\delta^2/r)\cos\Theta\\ r\cos\Theta\end{cases},$$
$$\psi_1 = \begin{cases}(r-\delta^2/r)\sin\Theta & (0\leqslant z<1)\\ r\sin\Theta & (z>1),\end{cases} \tag{6.1}$$

where $u_{1r} = \partial\phi_1/\partial r = \partial\psi_1/r\partial\Theta$, $u_{1\Theta} = \partial\phi_1/r\partial\Theta = -\partial\psi_1/\partial r$.

Therefore

$$q_1^2 = \begin{cases}1-2\delta^2r^{-2}\cos 2\Theta+\delta^4r^{-4} & (0\leqslant z\leqslant 1)\\ 1 & (z\geqslant 1),\end{cases} \tag{6.2}$$

$$\omega_{1r} = -2zr^{-2}\sin\Theta,\ \omega_{1\Theta} = 2zr^{-2}\cos\Theta,$$
$$\omega_{1n} = 2zq_1^{-1}r^{-2}(\cos 2\Theta - \delta^2/r^2), \omega_{1s} = = -2zq_1^{-1}r^{-2}\sin 2\Theta\ (0\leqslant z<1) \tag{6.3}$$

There is an interface $z_0(\mathbf{r}) = z_0(T)$ between the two regions. To primary order, this interface is $z=1$. Above it the primary flow is uniform, and therefore z_2 and w_2 are zero. The horizontal secondary velocity must be made to satisfy a suitable boundary condition at the sloping interface, whose position should be found from the flow beneath. There

$$z_2 = -\partial\frac{1}{2}q_1^2/\partial z = -q_1\omega_{1n} = = -2zr^{-2}(\cos 2\Theta - \delta^2/r^2), \tag{6.4}$$

$$w_2 = -\mathbf{u}_1\cdot\nabla z_2 = -4zr^{-3}\{\cos 3\Theta + +(-3\delta^2/r^2+2\delta^4/r^4)\cos\Theta\} \tag{6.5}$$

$$\omega_{2z} = q_1^{-1}\{\omega_{1s}w_2 + z_2\,\mathbf{n}\cdot\nabla z_2\} = = -8z^2r^{-1}\sin\Theta(1+2\delta^2/r^2). \tag{6.6}$$

The forms of equations (4.19), (4.20) in polar coordinates are

$$\frac{\partial(ru_{2\Theta})}{\partial r} - \frac{\partial(ru_{2r})}{r\partial\Theta} = r\omega_{2z}, \tag{6.7}$$

$$\frac{\partial(ru_{2r})}{\partial r}+\frac{\partial(ru_{2\Theta})}{r\partial\Theta}=-r\partial w_2/\partial z. \quad (6.8)$$

The unique solution of these equations satisfying the boundary conditions of zero velocity and circulation at infinity and of zero normal velocity on the rigid hemisphere, i.e. the solution such that

$$\mathbf{u}_2, \int_0^{2\pi} ru_{2\Theta}\,d\Theta \to 0 \quad \text{as } r\to\infty, \quad (6.9)$$

$$\mathbf{u}_2\cdot\mathbf{r}=0 \ (r=\delta), \quad (6.10)$$

is in fact

$$u_{2r}=\frac{1}{6}r^{-4}\cos\Theta\{27-75z^2-10\delta^2/r^2+ + 54z^2\delta^2/r^2-(17+3z^2)r^2/\delta^2\} + + \frac{1}{2}r^{-2}\cos 3\Theta\{1-(1-9z^2)/r^2\}, \quad (6.11)$$

$$u_{2\Theta}=\frac{1}{6}r^{-4}\sin\Theta\{9-9z^2-2\delta^2/r^2+30z^2\delta^2/r^2- -(17+3z^2)r^2/\delta^2\}+\frac{1}{2}r^{-2}\sin 3\Theta\{3- -(1-9z^2)/r^2\}. \quad (6.12)$$

This gives the complete secondary flow for $z_0(\mathbf{r})<z_0(T)$. However $\mathbf{u}_2$ is infinite at the level of T where $\delta=0$. This shows that the secondary approximation is not uniformly valid near T, and makes the choice of condition of continuity at the interface more difficult.

7. Discussion

Convergence of the power series in J^{-1} was not considered at all in section 4. In view of the difficulty of a proof of convergence of such a solution of a non-linear partial differential equation, we shall not try to do more than make the convergence plausible. The primary flow is in accord with physical intuition. The strong variation of density inhibits all vertical motion and the flow is constrained in horizontal planes. The variation has a stabilising influence, so that a unique steady solution approximating the primary flow should exist for some range of large Richardson numbers. Our method is analogous to the Janzen-Rayleigh method to find the subsonic flow of a uniform stream of compressible fluid about an obstacle, in which method the velocity potential is expanded as a power series in the square of the Mach number. Both methods are characteristic of an elliptic partial differential equation, and give time-reversible flows.

The Janzen-Rayleigh expansion is thought to converge if the local Mach number is less than one, i.e. the flow is subsonic. Where the flow becomes sonic, a region in which the solution is of hyperbolic character may develop, if the solution of elliptic character is unstable or does not exist. Internal gravity waves, the analogue of sound waves, in an incompressible fluid of density $\varrho=\varrho_0\exp(-\beta z)$ have phase-velocity $c=\left\{g\beta/(k^2+l^2+\frac{1}{4}\beta^2)\right\}^{\frac{1}{2}}$, where the particle velocity $\mathbf{u}=\mathbf{U}_0\ Re\left\{\exp\left[\frac{1}{2}\beta z+i\,(kx+ly-kct)\right]\right\}$ (cf. LAMB 1932, p. 379). These waves are dispersive, and the Richardson number based on the length and velocity scales of the wave-motion $g\beta/\left(k^2+l^2+\frac{1}{4}\beta^2\right)c^2$ is of order one, however large g is. Thus however large the overall Richardson number is, there may exist motion for which the local Richardson number is of order one. (In the same way the overall Reynolds number of a flow may be large, while the local Reynolds number may be of order one in a boundary layer. The overall number acts a dimensionless ratio of scales; the local number is more properly a ratio of inertial to viscous forces.) This is an important difference from the Janzen-Rayleigh flow, in which the speed of sound may vary from point to point, but is independent of wave-length. Further differences in the analogy can be found by a direct study of our flow.

The uniformity of the validity of the secondary approximation can be tested for consistency by showing that the secondary is not dominated by the tertiary or higher approximations anywhere in the field of flow. Regions of doubtful validity are zeros and singularities of the primary and secondary flows.

In the examples of the cylinder and hemisphere in sections 5 and 6 respectively, we found $\phi_1=r\cos\Theta+o(1/r)$ as $r\to\infty$. Therefore $\mathbf{u}_1=\mathbf{i}+o(1/r^2)$, $z_2=o(1/r^2)$, $\mathbf{u}_2=o(1/r^4)$,

$z_3 = o(1/r^4)$, $\mathbf{u}_3 = o(1/r^5)$, etc. below the top of the obstacle. Thus at infinity below the top of the obstacle the secondary flow is greater in magnitude than higher flows. This verifies the uniformity of convergence far from a vertical cylinder, which has no top.

The singularity of the solution for the hemisphere at $z = 1$ (i.e. $\delta = 0$) needs investigation. At this height the primary vorticity is discontinuous. Also, at the top T of the hemisphere, where $\delta = 1$ and $r = 0$, the primary vorticity is infinite. The image system of the uniform stream of the primary flow includes a line of dipoles inside the obstacle. This line cuts the obstacle at its top, if any, where there results a singularity of the primary and secondary flows. A cylinder, having no top, seems to have a well-behaved secondary as well as primary flow. Modifications of the secondary approximation to flow about the hemisphere are needed, for the discontinuity of the primary vorticity near $z = 1$, and for the infinity of the primary vorticity at the top of the hemisphere.

The nature of the shear layer at the level of the top of the obstacle can be seen by use of order of magnitudes. Take u, $v \approx V$; $w \approx W$; $\partial/\partial x$, $\partial/\partial y \approx 1/h$; $\partial/\partial z \approx 1/H$; ω_x, $\omega_y \approx V/H$ *or* W/h; $\omega_z \approx V/h$ in the layer for large J. Then the thickness of the layer is of order H, and the slope of order $|\nabla_h z_0| \approx W/V$. The equation of continuity (2.2) gives $W \lesssim VH/h$. Now the vorticity equation at zero Froude number,

$$\mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \frac{g d\varrho}{\varrho d z_0} \mathbf{k} \times \nabla z_0,$$

gives $H \gtrsim hJ^{-\frac{1}{2}}$. Thus these crude arguments indicate that there is a shear layer of thickness $hJ^{-\frac{1}{2}}$ at the level of the top of the obstacle, and in that layer the vorticity gradient cannot be neglected, however large the Richardson number.

To find more rigorously the behaviour of the shear layer, the methods of singular perturbation theory with 'strained' coordinates etc. must be used. Thus the transformed vertical coordinate in the layer might be $\zeta = J^{\frac{1}{2}} z$, the layer thickness being of order $J^{-\frac{1}{2}}$, just as in viscous boundary layer theory the transverse coordinate is replaced by $\eta = R^{\frac{1}{2}} y$, the boundary layer thickness being of the order of the inverse square-root of the Reynolds number R. The nature of the flow in the shear layer would be clarified by removal of the geometry imposed by the hemisphere, and seems better considered first apart from the flow as a whole.

In this meteorological study, let us consider the steady flow as J increases from zero to infinity. When $J = 0$ there is the classical potential flow. As J increases from zero, lee-waves behind the shoulder of the mountain develop. For each value of J we suppose there is a continuously-varying velocity field. So it may be conjectured that, as J becomes large, the lee-waves are confined to a layer of decreasing thickness at the level of the top of the obstacle. In this layer there are rapid changes of shear, which enable the inertial forces to balance the buoyancy forces. (It must be admitted that the vorticity field is conceivably discontinuous for some range of values of J. This discontinuity could be associated with a characteristic surface bounding a region in which the solution is of hyperbolic nature. However there seems no evidence against our supposition that the velocity field varies continuously.)

The singularity of the secondary flow for the hemisphere could be due to some behaviour near $z = z_0(T)$ such as

$$w \sim [z - z_0(T)]^{\frac{1}{2}} \sin \{[z - z_0(T)]J\}^{-1} \text{ as } J \to \infty.$$

(This is an illustration of the kind of singular perturbation to be anticipated, and not an assertion about the actual solution.) Now

$$[z - z_0(T)]^{\frac{1}{2}} \sin \{[z - z_0(T)J\}^{-1} =$$

$$= J^{-1} [z - z_0(T)]^{-\frac{1}{2}} - \frac{1}{3!} J^{-3} [z - z_0(T)]^{-\frac{5}{2}} + \ldots$$

It is clearly impossible to approximate the series near $z = z_0(T)$ for large J by taking the first term only, for that term is infinite at $z = z_0(T)$, where the left-hand side is zero. Nonetheless, the series converges for all non-zero $J[z - z_0(T)]$, however small. This example indicates how the solutions for large J, above and below the shear layer, may be joined across it, thereby giving an approximation to the flow, uniformly valid for large J. As it stands, the solution below the shear-layer seems valid at any given point for large J.

As well as the problem of the proper solution of the equation of motion for an inviscid fluid, there is that of the correct inviscid limit of the equations of motion of a viscous fluid. The solutions in sections 5 and 6 are spatially symmetric about the obstacles and the direction of flow may be reversed. This is because the solution found in section 4 is time-reversible. There can be no drag on the mountain, and we are presented with no more than D'Alembert's paradox. Solutions in classical hydrodynamics cannot give drag due to real wakes and are not valid downstream of bluff obstacles. However we used the cylinder and sphere to investigate the nature of the flow rather than for practical purposes. A better limit for small viscosity could be got by use of a Kirchoff free-streamline or a Rankine body to model a real wake. This problem is discussed at length in the literature and is ancillary to the problem due to density variation.

The Reynolds number $R \equiv Vh/\nu$, where ν is the kinematic viscosity, is very large in the atmosphere. For if we take $V \approx 30$ m sec^{-1}, $h \approx 7 \times 10^3$ m, $\nu \approx 10^{-5}$ m^2 sec^{-1}, then $R \approx 2 \times \times 10^{10}$. To account for atmospheric turbulence, the kinematic viscosity may be replaced by an eddy viscosity K. Because of the stabilisation of the density variation and of the earth's rotation, the eddies are not too large, and $K \lesssim 10^5 \nu$. Then $R_K \equiv Vh/K \gtrsim 2 \times 10^5$ is still large. In the laboratory, scales are different, it being difficult to model J and R together.

To show that viscosity is negligible in the shear layer described above, it is necessary to show that its thickness $(hJ^{-1/2})$ is much greater than that $(hR^{-1/2})$ of a 'viscous' shear layer. Now $R_K^{1/2}/J^{1/2} = \left\{ V^3 \Big/ \left(-\frac{gKh}{\varrho} \frac{d\varrho}{dz_0} \right) \right\}^{1/2} \lesssim 50$ for the magnitudes of scales chosen above. This indicates that the viscous effects are subsidiary.

We may now describe the solutions found in sections 5 and 6. For the vertical cylinder,

$$z_0 = z + \frac{1}{2} J^{-1} L r^{-2} (r^{-2} - 2 \cos 2\Theta) + o\,(J^{-2}).$$

There is symmetry about the vertical planes $\Theta = \frac{1}{2}\pi\,(x = 0)$ and $\Theta = 0\,(y = 0)$. The secondary velocity is vertical. This gives the general pattern of the rise and fall of streamlines. Far downstream (where $x = r \cos \Theta \to -\infty$), particles rise from their initial levels z_0. For given r and z_0, there is a minimum height of the streamlines at $\Theta = \frac{1}{2}\pi$ and maxima at $\Theta = 0, \pi$. Thus streamlines rise from z_0 to a maximum height near the cylinder, fall to a minimum at $\Theta + \frac{1}{2}\pi$ as they pass round the side, rise again, and finally fall back far downstream to their original level z_0. This is due to the Bernoulli effect of conservation of head on streamlines.

For the hemisphere,

$$z = z_0 = 2J^{-1} z r^{-2} (\cos 2\Theta - \delta^2/r^2) = o\,(J^{-2}).$$

The symmetry is the same as for the cylinder, but the secondary velocity has a horizontal as well as a vertical component. In this case the secondary flow is a buoyancy, not a Bernoulli, effect.

There are few experiments to compare these results with. To accompany their calculations of the secondary vorticity for flow past a hemisphere of a stream at *small* Richardson number, HAWTHORNE and MARTIN (1955) did some experiments on a hemisphere, with a tapered fairing to prevent turbulence in its lee. Their experiments showed that increasing the Richardson number increases vorticity at the base, and, more markedly, at the shoulder in the same sense (with a component of vorticity parallel to the primary velocity). These might be explained as tendencies towards the primary flow for large J, but it is not possible to confirm a theory valid for large J with experiments at small values of J.

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