On the Interactions between the Long Baroclinic Waves and the Mean Zonal Flow

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Abstract

The development of wave motions in a jet-like westerly current is examined on the basis of approximate solutions for the wave motions in a two-level, quasi-geostrophic model. Solutions of the characteristic-value equations show that two different types of amplifying waves may exist. Both types of amplifying wave motions are connected with a transformation of potential energy to kinetic energy. One of the types is found to be connected with a significant transfer of perturbation kinetic energy to mean zonal kinetic energy. Changes of the mean zonal flow caused by the eddy meridional transport of sensible heat and momentum are computed, and demonstrate the existence and the effects of the forced mean meridional circulations. Considerations of the modifications of the generation of perturbation energy connected with the modifications of the mean zonal flow indicate that the actual motion will have the character of a fluctuating state.

Introduction

One of the fundamental problems in theoretical meteorology has been the explanation of the origin and dynamics of the large-scale disturbances in the westerlies. Theoretical investigations by CHARNEY (1947), EADY (1949), FJØRTOFT (1950), KUO (1952), and PHILLIPS (1954) have shown that small-amplitude waves may amplify in consequence of the instability of the zonal baroclinic current. For the typical atmospheric conditions the waves will grow with a significant growth-rate for wavelengths of the order 3,000 to 6,000 km, and the theoretically determined kinematics of the amplifying waves seems to agree quite well with the kinematics of many actual developing disturbances. Especially the wave motion is connected with an ascent of warm air and a descent of cold air, and the corresponding decrease of potential energy must be considered as the main source for the kinetic energy of the disturbances.

A fundamental limitation in the theoretical investigations of the long baroclinic waves has been the neglect of the meridional variation of the zonal current. In the atmospheric westerlies there is normally a considerable variation of the mean zonal wind with latitude, in typical cases with a pronounced wind maximum. It is the aim of the present paper to investigate the mutual interactions between the mean zonal flow and the long baroclinic waves by considering the wave motions, which may develop in a zonal current with an idealized jet-like variation of wind, and by considering furthermore the fluctuations of the mean zonal flow, caused by the wave motions and the forced mean meridional circulations (cf. PHILLIPS, 1954). The solutions for the amplifying wave motions are obtained on the basis of rather crude approximations, considering the motion in a two-level, quasi-geostrophic model, and describing the meridional variation of the waves by a finite sine series. In a recent theoretical study by CHARNEY (1959) of the general circulation a similar approach has been used for obtaining solutions for the wave motions in a broad band of westerlies, resulting from a simple meridional variation of heating.

1. Basic equations

In investigating the development of long waves in a zonal flow with both vertical and horizontal shear, we shall assume that the motion is adequately described by the adiabatic, frictionless, and quasi-geostrophic equations for a two-level model (CHARNEY and PHILLIPS, 1952). With some slight additional simplifications, these may be written

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\nabla\right)(\zeta_1+f)-\frac{f}{\Delta p}\,\omega_2=0 \qquad (1)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_3\cdot\nabla\right)(\zeta_3+f)+\frac{f}{\Delta p}\,\omega_2=0 \qquad (2)$$

$$\left(\frac{\partial}{\partial t}+\frac{\mathbf{v}_1+\mathbf{v}_3}{2}\cdot\nabla\right)(\phi_1-\phi_3)-\frac{f^2}{\Delta p\varkappa^2}\,\omega_2=0 \qquad (3)$$

The subscripts 1, 2, and 3 refer to quantities at the constant pressure levels p_1, p_2, and p_3, respectively, with $p_3-p_2=p_2-p_1=\frac{1}{2}\Delta p$. The motion is described in a cartesian coordinate system (x to the east, y to the north), ∇ is the horizontal gradient operator and $\mathbf{v}$ the geostrophic velocity vector. ω denotes the individual time rate of change of pressure, and ϕ is the geopotential. The relative vorticity is determined by its geostrophic value, $\zeta=\frac{1}{f}\nabla^2\phi$. The Coriolis parameter, f, is considered to be a constant except in the term $\mathbf{v}\cdot\nabla f$, where we assume that $\frac{\partial f}{\partial y}=\beta=$constant. With θ denoting the potential temperature, the quantity $\varkappa^2$ is given by the formula

$$\varkappa^2=\frac{f^2}{\phi_1-\phi_3}\,\frac{\theta_2}{\theta_1-\theta_3}$$

and is considered as a constant, representing typical atmospheric conditions. Assuming that the values of ω become relatively small in a layer a little above the tropopause, we shall interpret the levels p_1, p_2, and p_3 as the 400-, 600- and 800-mb levels, respectively.

In the case of a zonal basic current without meridional shear the two-level approximation leads to very simple solutions of the perturbation equations for the wave motion, a fact which has been utilized by PHILLIPS (1954) in a detailed study of the properties of the growing waves, especially regarding their effects upon the mean zonal current. In the following we shall apply equations (1), (2), and (3) to perturbations of a basic zonal flow with the velocities U_1 and U_3 being functions of y. Indicating perturbation quantities by primed letters, and neglecting terms of second order in the perturbation quantities, equations (1), (2), and (3) may be combined into the following two equations

$$\frac{\partial}{\partial t}(\nabla^2\phi_1'-\varkappa^2\phi_1')+U_1\frac{\partial}{\partial x}\nabla^2\phi_1'-\frac{d^2U_1}{\partial y^2}\frac{\partial\phi_1'}{\partial x}+ \\ +\beta\frac{\partial\phi_1'}{\partial x}-\varkappa^2U_3\frac{\partial\phi_1'}{\partial x}+\varkappa^2\left(\frac{\partial\phi_3'}{\partial t}+U_1\frac{\partial\phi_3'}{\partial x}\right)=0 \qquad (4)$$

$$\frac{\partial}{\partial t}(\nabla^2\phi_3'-\varkappa^2\phi_3')+U_3\frac{\partial}{\partial x}\nabla^2\phi_3'-\frac{d^2U_3}{dy^2}\frac{\partial\phi_3'}{\partial x}+ \\ +\beta\frac{\partial\phi_3'}{\partial x}-\varkappa^2U_1\frac{\partial\phi_3'}{\partial x}+\varkappa^2\left(\frac{\partial\phi_1'}{\partial t}+U_3\frac{\partial\phi_1'}{\partial x}\right)=0 \qquad (5)$$

These equations are satisfied by wave solutions of the form

$$\phi_j'=\alpha_j(y)\,e^{i\mu(x-ct)}, \qquad j=1.3 \qquad (6)$$

with the understanding that the real parts of the solutions represent the physical quantities. In general α_1 and α_3 are complex functions of y. The quantity μ is connected with the wavelength, L, by the relation $\mu=\frac{2\pi}{L}$. For a fixed value of μ the possible values of c are determined as solutions of the characteristic-value problem. If c is complex, $c=c_r+ic_i$, the exponential factor in (6) becomes

$$e^{\mu c_i t}\,e^{i\mu(x-c_r t)}$$

The second factor represents a wave travelling in the x-direction with the constant velocity c_r. The first factor means an exponential increase or decrease of the amplitude, according as c_i is positive or negative. Inserting (6) into the equations (4) and (5), we obtain

$$-(U_1-c)\frac{d^2\alpha_1}{dy^2}+\left\{\mu^2(U_1-c)+\frac{d^2U_1}{dy^2}-\beta+\varkappa^2(U_3-c)\right\}\alpha_1-\varkappa^2(U_1-c)\alpha_3=0 \qquad (7)$$

$$-(U_3-c)\frac{d^2\alpha_3}{dy^2}+\left\{\mu^2(U_3-c)+\frac{d^2U_3}{dy^2}-\beta+\varkappa^2(U_1-c)\right\}\alpha_3-\varkappa^2(U_3-c)\alpha_1=0 \qquad (8)$$

Geometrically we shall represent the zone of the extra-tropical disturbances by the conditions

$$0 \leqq y \leqq D$$

where D is of the order 6,000 km. For the mean zonal flow it is then appropriate to consider the following simple jet-like structure

$$U_j = B_j(1-\cos 2\lambda y), \qquad j=1.3 \qquad (9)$$

where B_1 and B_3 are constants, and $\lambda=\dfrac{\pi}{D}$. As boundary conditions for ϕ' we shall require that

$$\phi_1' = \phi_3' = 0 \quad \text{at } y=0 \text{ and } y=D \qquad (10)$$

which exclude any geostrophic inflow or outflow through the southern and northern boundaries. In view of (10), the general solutions of the system of equations (7) and (8) may be written as the series

$$\alpha_j = \sum_n \alpha_j^{(n)} \sin(n\lambda y), \qquad n=1,2,3,--- \qquad (11)$$

Inserting the expressions (11) together with (9) into (7) and (8), we obtain the following system of relations

$$\left\{\frac{1}{2}(3\mu^2-\lambda^2)B_1+\frac{3}{2}\varkappa^2B_3-\beta-(\mu^2+\lambda^2+\varkappa^2)c\right\}\alpha_1^{(1)}-\varkappa^2\left(\frac{3}{2}B_1-c\right)\alpha_3^{(1)}-\frac{1}{2}\{(\mu^2+5\lambda^2)B_1+\varkappa^2B_3\}\alpha_1^{(3)}+\frac{1}{2}\varkappa^2B_1\alpha_3^{(3)}=0$$

$$-\varkappa^2\left(\frac{3}{2}B_3-c\right)\alpha_1^{(1)}+\left\{\frac{1}{2}(3\mu^2-\lambda^2)B_3+\frac{3}{2}\varkappa^2B_1-\beta-(\mu^2+\lambda^2+\varkappa^2)c\right\}\alpha_3^{(1)}+\frac{1}{2}\varkappa^2B_3\alpha_1^{(3)}-\frac{1}{2}\{(\mu^2+5\lambda^2)B_3+\varkappa^2B_1\}\alpha_3^{(3)}=0$$

$$\{(\mu^2+4\lambda^2)B_1+\varkappa^2B_3-\beta-(\mu^2+4\lambda^2+\varkappa^2)c\}\alpha_1^{(2)}-\varkappa^2(B_1-c)\alpha_3^{(2)}-\frac{1}{2}\left\{(\mu^2+12\lambda^2)B_1+\varkappa^2B_3\right\}\alpha_1^{(4)}+\frac{1}{2}\varkappa^2B_1\alpha_3^{(4)}=0$$

$$-\varkappa^2(B_3-c)\alpha_1^{(2)}+\{(\mu^2+4\lambda^2)B_3+\varkappa^2B_1-\beta-(\mu^2+4\lambda^2+\varkappa^2)c\}\alpha_3^{(2)}+\frac{1}{2}\varkappa^2B_3\alpha_1^{(4)}-\frac{1}{2}\{(\mu^2+12\lambda^2)B_3+\varkappa^2B_1\}\alpha_3^{(4)}=0$$

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$$-\frac{1}{2}[\{\mu^2+(n^2-4n)\lambda^2\}B_1+\varkappa^2B_3]\alpha_1^{(n-2)}+\frac{1}{2}\varkappa^2B_1\alpha_3^{(n-2)}+\{(\mu^2+n^2\lambda^2)B_1+\varkappa^2B_3-\beta-(\mu^2+n^2\lambda^2+\varkappa^2)c\}\alpha_1^{(n)}-\varkappa^2(B_1-c)\alpha_3^{(n)}-\frac{1}{2}[\{\mu^2+(n^2+4n)\lambda^2\}B_1+\varkappa^2B_3]\alpha_1^{(n+2)}+\frac{1}{2}\varkappa^2B_1\alpha_3^{(n+2)}=0$$

$$\frac{1}{2}\varkappa^2B_3\alpha_1^{(n-2)}-\frac{1}{2}[\{\mu^2+(n^2-4n)\lambda^2\}B_3+\varkappa^2B_1]\alpha_3^{(n-2)}-\varkappa^2(B_3-c)\alpha_1^{(n)}+\{(\mu^2+n^2\lambda^2)B_3+\varkappa^2B_1-\beta-(\mu^2+n^2\lambda^2+\varkappa^2)c\}\alpha_3^{(n)}+\frac{1}{2}\varkappa^2B_3\alpha_1^{(n+2)}-\frac{1}{2}[\{\mu^2+(n^2+4n)\lambda^2\}B_3+\varkappa^2B_1]\alpha_3^{(n+2)}=0$$

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(12)

It is seen that this system of equations may be separated into two independent parts, the one containing only $\alpha_1^{(n)}$ and $\alpha_3^{(n)}$ with n odd, the other only $\alpha_1^{(n)}$ and $\alpha_3^{(n)}$ with n even. Consequently we may treat separately those waves for which ϕ_j' is either symmetrical or antisymmetrical in y with respect to $y=\frac{1}{2}D$. In the following we shall consider only the symmetrical waves, as it is in this case we find the solutions, which may represent the growing disturbances.

Breaking off the series (11) at $n = N$ (N odd), we obtain from (12) a system of $N+1$ linear equations, which determine, for a given basic flow and wavelength, the possible values of c and, apart from an arbitrary factor, the corresponding values of the $N+1$ unknowns, $\alpha_1^{(1)}$, $\alpha_3^{(1)}$, $\alpha_1^{(3)}$, $\alpha_3^{(3)}$, - - - . The exclusion of all terms with n larger than N from the series (11) involves the actual assumption that the characteristics of the long growing waves can be studied to some extent, even if all components with wavelengths in the y-direction smaller than $\frac{2D}{N}$ are neglected. In so far this assumption seems to be consistent with the use of the quasi-geostrophic, two-level model, as this model is likely to be a usable approximation only for motions having rather large horizontal scales. A certain justification of the assumption may be obtained from considerations of the initial-value problem, if we take into account only initial states of the perturbations having almost all the kinetic energy in the components with the longest wavelengths in the y-direction. The development of the perturbations according to the equations (4) and (5) may then be described with a fair approximation up to a certain time on the basis of the total set of characteristic-value solutions, obtained from (12) with a finite number of terms in the series (11). From this point of view the problem about the development of disturbances in a barotropic flow was treated in a previous paper (ELIASEN, 1954).

2. Numerical solutions for the wave motion

For the numerical treatment of the equations (12) it seems appropriate to introduce the following non-dimensional quantities

$$q = \frac{B_1}{B_3}$$

$$\sigma = \frac{\varkappa^2}{\lambda^2}$$

$$\gamma = \frac{\beta}{\lambda^2 B_3}$$

$$k = \frac{\mu}{\lambda} = \frac{2D}{L}$$

$$\tau = \frac{c^{\bullet}}{B_3}$$

Thus the part of (12) containing $\alpha_1^{(n)}$ and $\alpha_3^{(n)}$ with n odd may be written in the case $N = 5$

$$\left\{\frac{1}{2}(3k^2 - 1)q + \frac{3}{2}\sigma - \gamma - (k^2 + 1 + \sigma)\tau\right\}\alpha_1^{(1)} - \sigma\left(\frac{3}{2}q - \tau\right)\alpha_3^{(1)} - \frac{1}{2}\{(k^2 + 5)q + \sigma\}\alpha_1^{(3)} + \frac{1}{2}\sigma q\alpha_3^{(3)} = 0$$

$$-\sigma\left(\frac{3}{2} - \tau\right)\alpha_1^{(1)} + \left\{\frac{1}{2}(3k^2 - 1) + \frac{3}{2}\sigma q - \gamma - (k^2 + 1 + \sigma)\tau\right\}\alpha_3^{(1)} + \frac{1}{2}\sigma\alpha_1^{(3)} - \frac{1}{2}\{k^2 + 5 + \sigma q\}\alpha_3^{(3)} = 0$$

$$-\frac{1}{2}\{(k^2 - 3)q + \sigma\}\alpha_1^{(1)} + \frac{1}{2}\sigma q\alpha_3^{(1)} + \{(k^2 + 9)q + \sigma - \gamma - (k^2 + 9 + \sigma)\tau\}\alpha_1^{(3)} - \sigma(q - \tau)\alpha_3^{(3)} - \frac{1}{2}\{(k^2 + 21)q + \sigma\}\alpha_1^{(5)} + \frac{1}{2}\sigma q\alpha_3^{(5)} = 0$$

$$\frac{1}{2}\sigma\alpha_1^{(1)} - \frac{1}{2}\{k^2 - 3 + \sigma q\}\alpha_3^{(1)} - \sigma(1 - \tau)\alpha_1^{(3)} + \{k^2 + 9 + \sigma q - \gamma - (k^2 + 9 + \sigma)\tau\}\alpha_3^{(3)} + \frac{1}{2}\sigma\alpha_1^{(5)} - \frac{1}{2}\{k^2 + 21 + \sigma q\}\alpha_3^{(5)} = 0$$

$$-\frac{1}{2}\{(k^2 + 5)q + \sigma\}\alpha_1^{(3)} + \frac{1}{2}\sigma q\alpha_3^{(3)} + \{(k^2 + 25)q + \sigma - \gamma - (k^2 + 25 + \sigma)\tau\}\alpha_1^{(5)} - \sigma(q - \tau)\alpha_3^{(5)} = 0$$

$$\frac{1}{2}\sigma\alpha_1^{(3)} - \frac{1}{2}\{k^2+5+\sigma q\}\alpha_3^{(3)} - \\ -\sigma(1-\tau)\alpha_1^{(5)} + \\ +\{k^2+25+\sigma q-\gamma-(k^2+25+\sigma)\tau\}\alpha_3^{(5)}=0 \quad (13)$$

With U_1 and U_3 fixed from the mean zonal velocities about the 400-mb and 800-mb levels, respectively, we shall assume that $q=3$. Furthermore the basic state shall be characterized by the value $\varkappa^2=3\times 10^{-12}$ m^{-2}, corresponding to quite normal atmospheric conditions. Thus, with $D=6{,}000$ km we have $\sigma=10.8$. Taking $\beta=16\times 10^{-12}$ sec^{-1} m^{-1}, corresponding to a mean latitude of 45°, the value of γ becomes equal to the value of σ, when $B_3=5^1/_3$ msec^{-1}, which from observed zonal velocities seems to be a quite reasonable value of B_3. With the stated values

$$q=3,\ \sigma=\gamma=10.8$$

the characteristic values of τ have been determined for different values of k, i.e. for different wavelengths, in the cases $N=1$, $N=3$, and $N=5$. For $k=3\,(L=4{,}000$ km) we obtain

$N=1$: $\tau=2.026\pm 0.736i$

$N=3$: $\tau=2.628\pm 0.861i$, $0.962\pm 0.227i$

$N=5$: $\tau=2.912\pm 0.686i$, $1.901\pm 0.352i$, 0.369, 0.674

and for $k=2.4$ ($L=5{,}000$ km) we obtain

$N=1$: $\tau=1.644\pm 0.579i$

$N=3$: $\tau=2.335\pm 0.854i$, $0.795\pm 0.288i$

$N=5$: $\tau=2.783\pm 0.732i$, $1.624\pm 0.508i$, 0.363, 0.527.

The solutions in the case $N=3$ involve a neglect of the components of the disturbances with wavelengths in the y-direction equal to and smaller than 2,400 km, whereas $N=5$ means a neglect of components with wavelengths in the y-direction equal to and smaller than 1,700 km.

According to (6), the growth-rate of the waves may be expressed by the quantity μc_i. Fig. 1 shows for wavelengths smaller than 6,000 km the values of μc_i in the cases $N=3$ and $N=5$, computed from $\mu c=k\lambda B_3\tau$, still with $D=6{,}000$ km and $B_3=5^1/_3$ msec^{-1}. It is seen that for wavelengths larger than about 3,500 km, we have two groups of solutions representing amplifying waves (cf. Charney, 1959). The maximum amplification corresponds in the case $N=3$ to a time of 26 hours, and in the case $N=5$ to a time of 33 hours for a doubling of the amplitude. The velocity of wave propagation, c_r is determined as the real part of τ multiplied by B_3. For each group of solutions corresponding to amplifying waves, c_r is decreasing slightly with increasing wavelength. The values of c_r occurring for wavelengths from 3,000 to 6,000 km may therefore be represented quite well by the numerical solutions for τ given above.

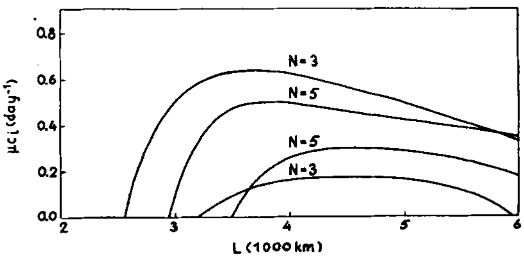

Fig. 1. Growth-rate of the waves as function of wavelength for the solutions obtained with $N = 3$ and $N = 5$.

A likely physical interpretation of the instability conditions may be obtained by a comparison with the simple case of a zonal flow without horizontal shear (cf. Phillips, 1954). From this it may be concluded that the absence of instability for the smaller wavelengths is due to the stabilizing effect of the static stability, an effect which increases with decreasing scale of the disturbances. For the larger scales, on the other hand, the stabilizing effect connected with the variation of the Coriolis parameter becomes dominating. An increase of the thermal vind, U_1—U_3, results in a general increase of the instability. From these physical effects we may infer immediately about the influence of the different parameters entering in the above determination of the amplification of the waves.

In so far it seems most likely that for each wavelength the wave solution with the largest growth-rate will dominate statistically the development of a disturbance. Depending on the initial formation, however, the development of a disturbance may also be influenced essentially by the wave solution with the smaller amplification. Especially this may be the case for the longer wavelengths, for which the difference in growth-rate for the two groups of solutions becomes relatively small. In the following, therefore, we shall examine the characteristics of the growing waves by considering both groups of solutions.

For fixed values of the parameters q, σ, γ, k, and a corresponding characteristic value of τ, we may obtain from the equations (13) the solutions for $\alpha_1^{(n)}$ and $\alpha_3^{(n)}$. In accordance with (6) and (11) we may write the resulting solutions for ϕ_1' and ϕ_3' in the form

$$\phi_j' = A_0 e^{\mu c_i t}[\{a_j^{(1)} \sin \lambda y + a_j^{(3)} \sin 3\lambda y + a_j^{(5)} \sin 5\lambda y\} \cos\{\mu(x - c_r t)\} + \{b_j^{(1)} \sin \lambda y + b_j^{(3)} \sin 3\lambda y + b_j^{(5)} \sin 5\lambda y\} \sin\{\mu(x - c_r t)\}] \quad (14)$$

As a representative value for the wavelengths of the long baroclinic waves we shall use $L = 5{,}000$ km, corresponding to $k = 2.4$. For the solutions obtained with $N = 3$ the values of $a_j^{(n)}$ and $b_j^{(n)}$ are given in Table 1a, and for the solutions obtained with $N = 5$ in Table 1b.

Table 1a. Values of $a_j^{(n)}$ and $b_j^{(n)}$ for $N = 3$.

	$\tau = 2.335 + 0.854i$		$\tau = 0.795 + 0.288i$	
n	1	3	1	3
$a_1^{(n)}$	0.601	−0.420	2.411	0.710
$b_1^{(n)}$	−1.856	0.862	−1.453	−0.428
$a_3^{(n)}$	1.000	−0.418	1.000	0.599
$b_3^{(n)}$	0.000	−0.122	0.000	0.292

The values are given in the way, that A_0 becomes the amplitude of the component with $n = 1$ at the lower level and at the time $t = 0$. From the values we may realize the relations between the different components of each solution. In order to represent more directly the resulting structure of the waves, the expression (14) may be written

$$\phi_j' = A_0 e^{\mu c_i t} a_j(y) \cos[\mu\{x - \delta_j(y)L - c_r t\}] \quad (15)$$

For the solutions given in Tables 1a and 1b, the values of $a_j(y)$ and $\delta_j(y)$ are listed in Tables 2a and 2b for $y = \frac{1}{2}D$, $y = \frac{1}{3}D$, $\frac{2}{3}D$, and $y = \frac{1}{6}D$, $\frac{5}{6}D$. In each case the zero of x is chosen in the way that $\delta_1 = 0$ at $y = \frac{1}{2}D$. The values in the tables illustrate clearly the vertical tilt of the waves and the differences between the two groups of solutions, concerning the meridional variations of amplitude and phase.

3. Energy transformations

The energy equations adapted to the quasi-geostrophic two-level model has been formulated by PHILLIPS (1954, 1956). As a measure of the kinetic energy we may define

$$K = \frac{1}{LD}\int_0^D\int_0^L \frac{1}{2}(u_1^2 + v_1^2 + u_3^2 + v_3^2)\,dx\,dy \quad (16)$$

where u denotes the eastward and v the northward component of the geostrophic wind. As representing the potential energy we may consider

$$P = \frac{1}{LD}\int_0^D\int_0^L \frac{\varkappa^2}{2f^2}(\phi_1 - \phi_3)^2\,dx\,dy \quad (17)$$

The kinetic energy and the potential energy may each be divided up into a part representing the energy of the mean zonal current and a part representing the energy of the perturbation, i.e.

$$K = \bar{K} + K' \text{ and } P = \bar{P} + P'$$

Table 1b. Values of $a_j^{(n)}$ and $b_j^{(n)}$ for $N = 5$.

	$\tau = 2.783 + 0.732i$			$\tau = 1.624 + 0.508i$		
n	1	3	5	1	3	5
$a_1^{(n)}$	0.100	−0.680	0.505	1.599	−0.224	−0.302
$b_1^{(n)}$	−1.405	0.834	−0.306	−2.159	0.224	0.356
$a_3^{(n)}$	1.000	−0.756	0.271	1.000	0.302	−0.388
$b_3^{(n)}$	0.000	−0.134	0.092	0.000	−0.071	−0.092

Table 2a. Meridional variation of amplitude and phase. $N=3$.

	$\tau=2.335+0.854i$			$\tau=0.795+0.288i$		
y	$\frac{1}{2}D$	$\frac{1}{3}D$, $\frac{2}{3}D$	$\frac{1}{6}D$, $\frac{5}{6}D$	$\frac{1}{2}D$	$\frac{1}{3}D$, $\frac{2}{3}D$	$\frac{1}{6}D$, $\frac{5}{6}D$
a_1	2.90	1.69	0.14	1.99	2.44	2.24
δ_1	0.00	−0.01	−0.23	0.00	0.00	0.00
a_3	1.66	0.87	0.15	0.50	0.87	1.14
δ_3	0.21	0.19	0.04	−0.01	0.09	0.13

Table 2b. Meridional variation of amplitude and phase. $N=5$.

	$\tau=2.783+0.732i$			$\tau=1.624+0.508i$		
y	$\frac{1}{2}D$	$\frac{1}{3}D$, $\frac{2}{3}D$	$\frac{1}{6}D$, $\frac{5}{6}D$	$\frac{1}{2}D$	$\frac{1}{3}D$, $\frac{2}{3}D$	$\frac{1}{6}D$, $\frac{5}{6}D$
a_1	2.85	1.02	0.38	2.53	2.73	0.80
δ_1	0.00	−0.13	−0.32	0.00	0.00	−0.01
a_3	2.04	0.64	0.15	0.31	1.20	0.62
δ_3	0.19	0.16	−0.22	0.14	0.16	0.12

with

$$\bar{K}=\frac{1}{D}\int_0^D \frac{1}{2}(\bar{u}_1^2+\bar{u}_3^2)\,dy \tag{18}$$

$$K'=\frac{1}{D}\int_0^D \frac{1}{2}(\overline{u_1'^2}+\overline{v_1'^2}+\overline{u_3'^2}+\overline{v_3'^2})\,dy \tag{19}$$

$$\bar{P}=\frac{1}{D}\int_0^D \frac{\varkappa^2}{2f^2}(\bar{\phi}_1-\bar{\phi}_3)^2\,dy \tag{20}$$

$$P'=\frac{1}{D}\int_0^D \frac{\varkappa^2}{2f^2}\overline{(\phi_1'-\phi_3')^2}\,dy \tag{21}$$

where the bar denotes mean values with respect to x.

The time rate of change of the different energy forms may be obtained from equations (1), (2), and (3). Leaving out of consideration any mean meridional circulation, we get

$$\frac{d\bar{K}}{dt}=-\frac{1}{D}\int_0^D\left\{\bar{u}_1\frac{\partial(\overline{u_1'v_1'})}{\partial y}+\bar{u}_3\frac{\partial(\overline{u_3'v_3'})}{\partial y}\right\}dy \tag{22}$$

$$\frac{dK'}{dt}=-\frac{1}{D\Delta p}\int_0^D\overline{\omega_2'(\phi_1'-\phi_3')}\,dy+$$

$$+\frac{1}{D}\int_0^D\left\{\bar{u}_1\frac{\partial(\overline{u_1'v_1'})}{\partial y}+\bar{u}_3\frac{\partial(\overline{u_3'v_3'})}{\partial y}\right\}dy \tag{23}$$

$$\frac{d\bar{P}}{dt}=\frac{1}{D}\frac{\varkappa^2}{f^2}\int_0^D\overline{v_3'(\phi_1'-\phi_3')}\,\frac{\partial(\bar{\phi}_1-\bar{\phi}_3)}{\partial y}\,dy \tag{24}$$

$$\frac{dP'}{dt}=\frac{1}{D\Delta p}\int_0^D\overline{\omega_2'(\phi_1'-\phi_3')}\,dy-$$

$$-\frac{1}{D}\frac{\varkappa^2}{f^2}\int_0^D\overline{v_3'(\phi_1'-\phi_3')}\,\frac{\partial(\bar{\phi}_1-\bar{\phi}_3)}{\partial y}\,dy \tag{25}$$

By adding these equations, we obtain the joint energy equation

$$\frac{d}{dt}(\bar{K}+K'+\bar{P}+P')=0 \tag{26}$$

expressing the conservation of the total kinetic and potential energy. In equations (22) and (23) the expression

$$\frac{1}{D}\int_0^D\left\{\bar{u}_1\frac{\partial(\overline{u_1'v_1'})}{\partial y}+\bar{u}_3\frac{\partial(\overline{u_3'v_3'})}{\partial y}\right\}dy$$

represents a transformation of mean kinetic energy into perturbation kinetic energy, in equations (23) and (25) the expression

$$-\frac{1}{D\Delta p}\int_0^D \overline{\omega_2'(\phi_1'-\phi_3')}\,dy$$

represents a transformation of perturbation potential energy into perturbation kinetic energy, and finally in equations (24) and (25) the expression

$$-\frac{1}{D}\frac{\varkappa^2}{f^2}\int_0^D \overline{v_3'(\phi_1'-\phi_3')}\;\frac{\partial(\bar\phi_1-\bar\phi_3)}{\partial y}\,dy$$

represents a transformation of mean potential energy into perturbation potential energy. From these expressions we may study the energy transformations connected with the amplification of the waves considered in the foregoing. From the solutions for these wave motions, however, we may also compute the time rate of change of the perturbation kinetic and potential energies directly as

$$\frac{dK'}{dt}=2\mu c_i K' \qquad (27)$$

$$\frac{dP'}{dt}=2\mu c_i P' \qquad (28)$$

Inserting the expressions (14) for ϕ_1' and ϕ_3' into (19) and (21), we get

$$K'=\frac{\lambda^2}{f^2}A^2\frac{1}{8}\left[(k^2+1)\{a_1^{(1)^2}+b_1^{(1)^2}+a_3^{(1)^2}+b_3^{(1)^2}\}+\right.$$
$$+(k^2+9)\{a_1^{(3)^2}+b_1^{(3)^2}+a_3^{(3)^2}+b_3^{(3)^2}\}+$$
$$\left.+(k^2+25)\{a_1^{(5)^2}+b_1^{(5)^2}+a_3^{(5)^2}+b_3^{(5)^2}\}\right] \qquad (29)$$

and

$$P'=\frac{\lambda^2}{f^2}A^2\frac{\sigma}{8}\left[\{a_1^{(1)}-a_3^{(1)}\}^2+\{b_1^{(1)}-b_3^{(1)}\}^2+\right.$$
$$+\{a_1^{(3)}-a_3^{(3)}\}^2+\{b_1^{(3)}-b_3^{(3)}\}^2+$$
$$\left.+\{a_1^{(5)}-a_3^{(5)}\}^2+\{b_1^{(5)}-b_3^{(5)}\}\right] \qquad (30)$$

where $A=A_0e^{\mu c_i t}$. For the solutions given in Tables 1a and 1b the distribution of kinetic and potential energy on the different wavelengths in the y-direction, indicated by the upper index, is listed in Table 3. In each case the values are given with the total perturbation kinetic energy as unit.

According to (27) and (28), $\frac{dK'}{dt}$ and $\frac{dP'}{dt}$ may be computed from the values of K' and P'. With ϕ_j' given by (14), the expressions (22) and (24) for $\frac{d\bar K}{dt}$ and $\frac{d\bar P}{dt}$, respectively, become

$$\frac{d\bar K}{dt}=\frac{\mu\lambda^2B_3}{f^2}A^2\left[q\{a_1^{(3)}b_1^{(1)}-a_1^{(1)}b_1^{(3)}+\right.$$
$$+2(a_1^{(5)}b_1^{(3)}-a_1^{(3)}b_1^{(5)})\}+a_3^{(3)}b_3^{(1)}-a_3^{(1)}b_3^{(3)}+$$
$$\left.+2(a_3^{(5)}b_3^{(3)}-a_3^{(3)}b_3^{(5)})\right] \qquad (31)$$

and

$$\frac{d\bar P}{dt}=\frac{\mu\lambda^2B_3}{f^2}A^2\frac{(q-1)\sigma}{8}\left[3\{a_3^{(1)}b_1^{(1)}-a_1^{(1)}b_3^{(1)}\}+\right.$$
$$+2\{a_3^{(3)}b_1^{(3)}-a_1^{(3)}b_3^{(3)}+a_3^{(5)}b_1^{(5)}-a_1^{(5)}b_3^{(5)}\}+$$
$$+a_1^{(3)}b_3^{(1)}-a_3^{(1)}b_1^{(3)}+a_1^{(1)}b_3^{(3)}-a_3^{(3)}b_1^{(1)}+$$
$$\left.+a_1^{(5)}b_3^{(3)}-a_3^{(3)}b_1^{(5)}+a_1^{(3)}b_3^{(5)}-a_3^{(5)}b_1^{(3)}\right] \qquad (32)$$

For the wave motions given in Tables 1a and 1b the time rate of change of the four energy forms is tabulated in Table 4. The values are given in units of total perturbation kinetic energy per day. It is seen that in each case the

Table 3. Distribution of kinetic and potential energy on the components with different wavelength in the y-direction.

		$K'^{(1)}$	$K'^{(3)}$	$K'^{(5)}$	K'	$P'^{(1)}$	$P'^{(3)}$	$P'^{(5)}$	P'
$N=3$	$\tau=2.335+0.854i$	0.66	0.34	—	1.00	0.80	0.21	—	1.01
	$\tau=0.795+0.288i$	0.78	0.22	—	1.00	0.58	0.07	—	0.65
$N=5$	$\tau=2.783+0.732i$	0.34	0.44	0.22	1.00	0.51	0.17	0.04	0.72
	$\tau=1.624+0.508i$	0.79	0.04	0.17	1.00	0.77	0.06	0.03	0.86

Table 4. Time rate of change of the four energy forms in units of total perturbation kinetic energy per day.

		$\frac{d\bar{P}}{dt}$	$\frac{dP'}{dt}$	$\frac{dK'}{dt}$	$\frac{d\bar{K}}{dt}$
$N=3$	$\tau=2.335+0.854i$	-2.07	1.00	0.99	0.09
	$\tau=0.795+0.288i$	-0.53	0.22	0.33	-0.02
$N=5$	$\tau=2.783+0.732i$	-1.78	0.61	0.85	0.32
	$\tau=1.624+0.508i$	-1.14	0.51	0.59	0.04

increase of the perturbation kinetic energy is connected with a decrease of the total potential energy. A part of the loss of zonal potential energy is restored as perturbation potential energy. According to (23) and (25) the actual generation of perturbation kinetic energy from perturbation potential energy is achieved by a positive correlation between $-\omega_2'$ and $\phi_1'-\phi_3'$, i.e. by a rising of warm air and a sinking of cold air. For the wave motion with the largest amplification a considerable part of the generated kinetic energy is organized into the mean zonal current, especially when we consider the solution, in which the component with wavelength $\frac{2}{5}D$ in the y-direction is taken into account.

4. Eddy transport of sensible heat and momentum

According to (24) and (25) the transfer of zonal potential energy into perturbation potential energy is connected with a meridional eddy transport of sensible heat. In view of the geostrophic approximation we may consider.

$$\frac{1}{2}\overline{(v_1'+v_3')(\phi_1'-\phi_3')}=\overline{v_3\phi_1}=-\overline{v_1'\phi_3'} \quad (33)$$

as a measure of this transport. With ϕ_j' given by (14), we get

$$\begin{aligned}\overline{v_3'\phi'} &= \frac{\mu}{f}A^2\frac{1}{4}\,[\{a_1^{(1)}b_3^{(1)}-a_3^{(1)}b_1^{(1)}+\\
&+a_1^{(3)}b_3^{(3)}-a_3^{(3)}b_1^{(3)}+a_1^{(5)}b_3^{(5)}-a_3^{(5)}b_1^{(5)}\}+\\
&+\{a_3^{(1)}b_1^{(1)}-a_1^{(1)}b_3^{(1)}+a_1^{(3)}b_3^{(1)}-a_3^{(1)}b_1^{(3)}+\\
&+a_1^{(1)}b_3^{(3)}-a_3^{(3)}b_1^{(1)}+a_1^{(3)}b_3^{(5)}-a_3^{(5)}b_1^{(3)}+\\
&+a_1^{(5)}b_3^{(3)}-a_3^{(3)}b_1^{(5)}\}\cos 2\lambda y+\{a_3^{(1)}b_1^{(3)}-a_1^{(3)}b_3^{(1)}+\\
&+a_3^{(3)}b_1^{(1)}-a_1^{(1)}b_3^{(3)}+a_1^{(1)}b_3^{(5)}-a_3^{(5)}b_1^{(1)}+\\
&+a_1^{(5)}b_3^{(1)}-a_3^{(1)}b_1^{(5)}\}\cos 4\lambda y+\{a_3^{(1)}b_1^{(5)}-a_1^{(5)}b_3^{(1)}+\\
&+a_3^{(5)}b_1^{(1)}-a_1^{(1)}b_3^{(5)}+a_3^{(3)}b_1^{(3)}-a_1^{(3)}b_3^{(3)}\}\cos 6\lambda y+\\
&+\{a_3^{(3)}b_1^{(5)}-a_1^{(5)}b_3^{(3)}+a_3^{(5)}b_1^{(3)}-a_1^{(3(}b_3^{(5)}\}\cos 8\lambda y+\\
&+\{a_3^{(5)}b_1^{(5)}-a_1^{(5)}b_3^{(5)}\}\cos 10\lambda y] \quad (34)\end{aligned}$$

For the wave motions given in Table 1b we obtain in the case $\tau=2.783+0.732i$

$$\begin{aligned}\overline{v_3'\phi_1'} &= \frac{\mu}{f}A^2(0.564-0.976\cos 2\lambda y+\\
&+0.651\cos 4\lambda y-0.354\cos 6\lambda y+\\
&+0.147\cos 8\lambda y-0.032\cos 10\lambda y) \quad (35)\end{aligned}$$

and in the case $\tau=1.624+0.508i$

$$\begin{aligned}\overline{v_3'\phi_1'} &= \frac{\mu}{f}A^2(0.568-0.456\cos 2\lambda y-\\
&-0.414\cos 4\lambda y+0.348\cos 6\lambda y-\\
&-0.005\cos 8\lambda y-0.041\cos 10\lambda y) \quad (36)\end{aligned}$$

Fig. 2 contains a picture of the two types of transport of heat. Put together they represent a cooling of the southern half and a warming up of the northern half of the region.

The transfer of perturbation kinetic energy into mean zonal kinetic energy given by (22) is depending on the mean northward transport of momentum $\overline{u_j'v_j'}$. The correlation between u_j' and v_j' is visualized by the meridional tilt of the waves in the way that a SW—NE orientation of the axes involves a positive value, whereas a SE—NW orientation involves a negative value of $\overline{u_j'v_j'}$. In this way the values

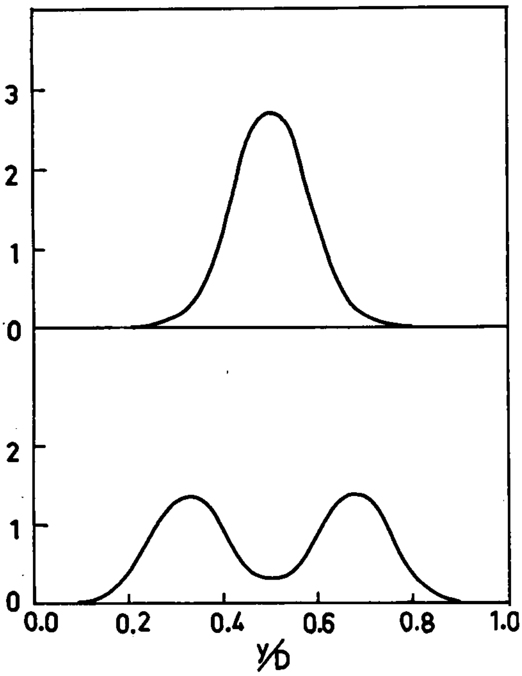

Fig. 2. Northward eddy transport of sensible heat. Values of $\left(\frac{\mu}{f}A^2\right)^{-1}\overline{v_3'\phi_1'}$, in the upper part for the first type of wave motion, in the lower part for the second type.

of a_j and δ_j in Tables 2a and 2b may give an impression of the corresponding values of $\overline{u_j'v_j'}$. With ϕ_j' given by (14), we obtain for $\overline{u_j'v_j'}$ the expression

$$\begin{aligned}\overline{u_j'v_j'} = \frac{\mu\lambda}{f^2}A^2\,[\{a_j^{(3)}b_j^{(1)} - a_j^{(1)}b_j^{(3)} &+ \\ + 2\,(a_j^{(5)}b_j^{(3)} - a_j^{(3)}b_j^{(5)}\}\sin 2\lambda y &+ \\ + \left\{\frac{1}{2}\,(a_j^{(1)}b_j^{(3)} - a_j^{(3)}b_j^{(1)}) \right. &+ \\ \left. + \frac{3}{2}\,(a_j^{(5)}b_j^{(1)} - a_j^{(1)}b_j^{(5)})\right\}\sin 4\lambda y &+ \\ + \{a_j^{(1(}b_j^{(5)} - a_j^{(5)}b_j^{(1)}\}\sin 6\lambda y &+ \\ + \frac{1}{2}\{a_j^{(3)}b_j^{(5)} - a_j^{(5)}b_j^{(3)}\}\sin 8\lambda y] & \qquad (37)\end{aligned}$$

For the values of $a_j^{(n)}$ and $b_j^{(n)}$ given in Table 1b we get in the case $\tau = 2.783 + 0.732i$

$$\left.\begin{aligned}\overline{u_1'v_1'} &= \frac{\mu\lambda}{f^2}A^2(1.298\sin 2\lambda y - 1.454\sin 4\lambda y + \\ &\quad + 0.679\sin 6\lambda y - 0.107\sin 8\lambda y) \\ \overline{u_3'v_3'} &= \frac{\mu\lambda}{f^2}A^2(0.200\sin 2\lambda y - 0.205\sin 4\lambda y + \\ &\quad + 0.092\sin 6\lambda y - 0.017\sin 8\lambda y)\end{aligned}\right\} \qquad (38)$$

and in the case $\tau = 1.624 + 0.508i$

$$\left.\begin{aligned}\overline{u_1'v_1'} &= \frac{\mu\lambda}{f^2}A^2(0.150\sin 2\lambda y + 0.062\sin 4\lambda y - \\ &\quad - 0.083\sin 6\lambda y - 0.006\sin 8\lambda y) \\ \overline{u_3'v_3'} &= \frac{\mu\lambda}{f^2}A^2(0.181\sin 2\lambda y + 0.102\sin 4\lambda y - \\ &\quad - 0.092\sin 6\lambda y - 0.028\sin 8\lambda y)\end{aligned}\right\} \qquad (39)$$

From these values it is seen that the first type of wave motion ($\tau = 2.783 + 0.732i$) is producing an eddy transport of momentum towards the north in the southern half and towards the south in the northern half of the region. The transport at the upper level is about 7 times as large as the transport at the lower level. For the wave motion of the other type ($\tau = 1.624 + 0.508i$) the meridional variation of the momentum transport is somewhat more complicated, and the values are essentially smaller than for the first type of wave motion.

In accordance with equations (1) and (2) in their x-averaged form, the eddy transport of momentum will contribute to a change of the mean zonal flow with a time rate of change equal to $-\frac{\partial(\overline{u_j'v_j'})}{\partial y}$. The dashed curve in Fig. 3 shows this change of the mean zonal flow at the upper level for the wave motion of the first type ($\tau = 2.783 + 0.732i$). As the corresponding values at the lower level are markedly smaller, the curve may represent, approximately, also the change of the vertical shear of the mean zonal wind determined from the eddy transport of momentum. The eddy transport of sensible heat, on the other hand, involves, in accordance with equation (3) in its averaged form, a change of the vertical shear of the mean zonal wind with a time rate of change equal to $\frac{1}{f}\frac{\partial^2(\overline{v_3'\phi_1'})}{\partial y^2}$. The values of the change of the mean vertical wind shear determined from the eddy transport of heat given by (35) and (36), respectively, are shown by the dashed curves in Fig. 4. It is seen that for both types of wave motion, the change of the vertical shear of the mean zonal flow determined from the eddy transport of heat is quite different from the change determined from the eddy transport of momentum at the two levels. From this fact we may conclude about the existence of a mean meridional circulation represented in the averaged forms of the equations (1), (2), and (3) by the terms containing $\bar{\omega}_2$. The properties of such meridional circulations connected with the large-scale baroclinic disturbances have been studied by PHILLIPS (1954, 1956). Also in connection with the development of frontal waves in a simple two-layer model the induced meridional circulation is of fundamental importance (ELIASEN, 1960).

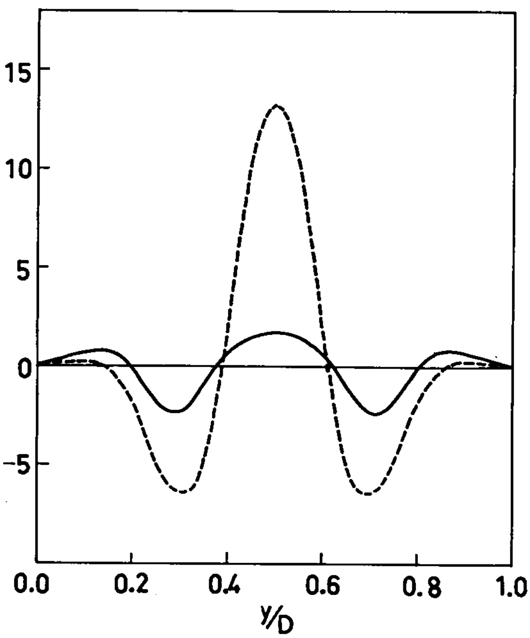

Fig. 3. Changes of the mean zonal flow at the upper level for the first type of wave motion. Values of $\left(\frac{\mu\lambda^2}{f^2}A^2\right)^{-1}\frac{\partial\bar{u}_1}{\partial t}$ (full curve) and $-\left(\frac{\mu\lambda^2}{f^2}A^2\right)^{-1}\frac{\partial(\overline{u_1'v_1'})}{\partial y}$ (dashed curve).

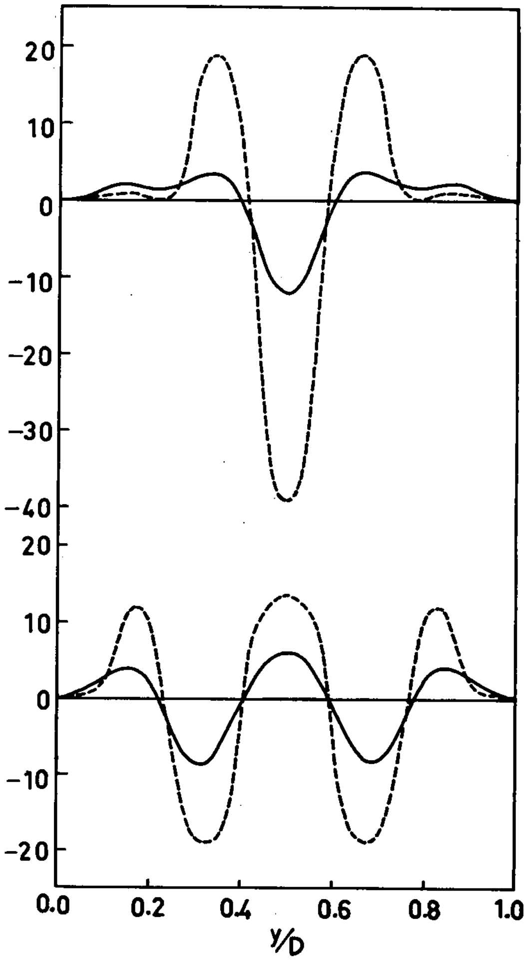

Fig. 4. Changes of the vertical shear of the mean zonal wind. Values of $\left(\frac{\mu\lambda^2}{f^2}A^2\right)^{-1}\frac{\partial(\bar{u}_1-\bar{u}_3)}{\partial t}$ (full curves) and $\left(\frac{\mu\lambda^2}{f^2}A^2\right)^{-1}\frac{\partial^2(\overline{v_3'\phi_1'})}{\partial y^2}$ (dashed curves), in the upper part for the first type of wave motion, in the lower part for the second type.

5. Changes of the mean zonal flow

Equations (1), (2), and (3) averaged with respect to x, may be combined into the following two equations

$$\left.\begin{aligned}\frac{\partial}{\partial t}\left\{\frac{\partial^2\bar{\phi}_1}{\partial y^2}-\varkappa^2(\bar{\phi}_1-\bar{\phi}_3)\right\}&=\\=-f\,\overline{\mathbf{v}_1\cdot\nabla(\zeta_1+f)}&+\varkappa^2\,\overline{\mathbf{v}_3\cdot\nabla\phi_1}\\\frac{\partial}{\partial t}\left\{\frac{\partial^2\bar{\phi}_3}{\partial y^2}+\varkappa^2(\bar{\phi}_1-\bar{\phi}_3)\right\}&=\\=-f\,\overline{\mathbf{v}_3\cdot\nabla(\zeta_3+f)}&-\varkappa^2\,\overline{\mathbf{v}_3\cdot\nabla\phi_1}\end{aligned}\right\}\quad(40)$$

In consequence of the quasi-geostrophic approximation these equations may be written

$$\left.\begin{aligned}\frac{\partial}{\partial t}\left\{\frac{\partial^2\bar{\phi}_1}{\partial y^2}-\varkappa^2(\bar{\phi}_1-\bar{\phi}_3)\right\}&=\\ &=f\frac{\partial^2(\overline{u_1'v_1'})}{\partial y^2}+\varkappa^2\frac{\partial(\overline{v_3'\phi_1'})}{\partial y}\\ \frac{\partial}{\partial t}\left\{\frac{\partial^2\bar{\phi}_3}{\partial y^2}+\varkappa^2(\bar{\phi}_1-\bar{\phi}_3)\right\}&=\\ &=f\frac{\partial^2(\overline{u_3'v_3'})}{\partial y^2}-\varkappa^2\frac{\partial(\overline{v_3'\phi_1'})}{\partial y}\end{aligned}\right\}$$

Forming further the sum and difference of these equations, we get

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\bar{u}_1+\bar{u}_3)=-\frac{\partial}{\partial y}(\overline{u_1'v_1'}+\overline{u_3'v_3'})\\ &\frac{\partial}{\partial t}\left\{\frac{\partial^2}{\partial y^2}(\bar{u}_1-\bar{u}_3)-2\varkappa^2(\bar{u}_1-\bar{u}_3)\right\}=\\ &=\frac{\partial}{\partial y}\left\{\frac{\partial^2}{\partial y^2}(\overline{u_1'v_1'}-\overline{u_3'v_3'})+2\varkappa^2\frac{\partial}{\partial y}(\overline{v_3'\phi_1'})\right\}\end{aligned}\right\}\quad(41)$$

The solution of equations (41) requires boundary conditions at $y=0$ and $y=D$. From the first equation of motion in its averaged form, we find that the kinematic boundary condition, $v_j=0$ at $y=0$ and $y=D$, implies the further condition that

$$\frac{\partial\bar{u}_j}{\partial t}=0\qquad\text{at } y=0\text{ and } y=D\qquad(42)$$

From the values of $\overline{v_3'\phi_1'}$ given by (35) and the values of $\overline{u_j'v_j'}$ given by (38), we get as solutions of equations (41) and (42) the following expressions for $\frac{\partial\bar{u}_j}{dt}$

$$\left.\begin{aligned}\frac{\partial\bar{u}_1}{\partial t}&=\frac{\mu\lambda^2}{f^2}A^2\Bigg[-0.02\cos2\lambda y+1.39\cos4\lambda y-1.02\cos6\lambda y-0.42\cos8\lambda y+\\ &\qquad+0.28\cos10\lambda y-0.20\,\frac{\cosh\left\{\pi\sqrt{2\sigma}\left(\frac{y}{D}-\frac{1}{2}\right)\right\}}{\cosh\left\{\frac{\pi}{2}\sqrt{2\sigma}\right\}}\Bigg]\\ \frac{\partial\bar{u}_3}{\partial t}&=\frac{\mu\lambda^2}{f^2}A^2\Bigg[-2.98\cos2\lambda y+5.25\cos4\lambda y-3.60\cos6\lambda y+1.42\cos8\lambda y-\\ &\qquad-0.28\cos10\lambda y+0.20\,\frac{\cosh\left\{\pi\sqrt{2\sigma}\left(\frac{y}{D}-\frac{1}{2}\right)\right\}}{\cosh\left\{\frac{\pi}{2}\sqrt{2\sigma}\right\}}\Bigg]\end{aligned}\right\}\quad(43)$$

For the other type of wave motion with the transport terms given by (36) and (39), we get

$$\left.\begin{aligned}\frac{\partial\bar{u}_1}{\partial t}&=\frac{\mu\lambda^2}{f^2}A^2\Bigg[0.44\cos2\lambda y+1.61\cos4\lambda y-1.84\cos6\lambda y+0.11\cos8\lambda y+\\ &\qquad+0.37\cos10\lambda y-0.69\,\frac{\cosh\left\{\pi\sqrt{2\sigma}\left(\frac{y}{D}-\frac{1}{2}\right)\right\}}{\cosh\left\{\frac{\pi}{2}\sqrt{2\sigma}\right\}}\Bigg]\\ \frac{\partial\bar{u}_3}{\partial t}&=\frac{\mu\lambda^2}{f^2}A^2\Bigg[-1.10\cos2\lambda y-2.27\cos4\lambda y+2.90\cos6\lambda y+0.17\cos8\lambda y-\\ &\qquad-0.37\cos10\lambda y+0.69\,\frac{\cosh\left\{\pi\sqrt{2\sigma}\left(\frac{y}{D}-\frac{1}{2}\right)\right\}}{\cosh\left\{\frac{\pi}{2}\sqrt{2\sigma}\right\}}\Bigg]\end{aligned}\right\}\quad(44)$$

The values of $\frac{\partial \bar{u}_1}{\partial t}$ determined by (43) are shown by the full curve in Fig. 3. In accordance with equations (1) and (2) in their averaged form, the difference between $\frac{\partial \bar{u}_j}{\partial t}$ and $-\frac{\partial (\overline{u_j' v_j'})}{\partial y}$ is due to the contribution from a mean meridional circulation. From the values in Fig. 3 it is seen that the effect of the mean meridional circulation is to reduce strongly the effect of the eddy transport, quite in agreement with the result obtained by PHILLIPS (1956). At the lower level the contribution to $\frac{\partial \bar{u}_3}{\partial t}$ from the eddy transport given by (38) is relatively unimportant, and the change of $\bar{u}_3$ is caused mainly by the mean meridional circulation. It follows from (43) that $\frac{\partial \bar{u}_3}{\partial t}$ has a pronounced maximum at $y = \frac{1}{2} D$.

The fundamental difference concerning the change of the mean zonal flow produced by the two different types of wave motion, appears clearly from Fig. 4, showing the values of the time rate of change of the mean zonal wind shear with height, $\frac{\partial (\bar{u}_1 - \bar{u}_3)}{\partial t}$. From the figure it is seen that the change of the vertical shear of the mean zonal wind, which would result from the eddy transport of sensible heat (represented by the dashed curves) is reduced considerably. Also this effect of the mean meridional circulation was found by PHILLIPS (1956).

The magnitude of the change of the mean zonal flow produced by the wave motion and the associated mean meridional circulation is seen to be proportional to $\frac{\mu \lambda^2}{f^2} A^2$. With $D = 6{,}000$ km, $L = 5{,}000$ km, and $f = 10^{-4}$ sec^{-1} we have

$$\frac{\mu \lambda^2}{f^2} = 3.0 \times 10^{-6}\, \mathrm{m}^{-3} \sec^3 \mathrm{day}^{-1}$$

From the values of a_1 given in Table 2b it is seen that for both solutions the maximum amplitude of the perturbation of the height of the pressure surface p_1 is about $2.8\, g^{-1} A$, where g is the acceleration of gravity. Harmonic analyses of the height of the 500-mb surface at 40—50° N (cf. ELIASEN, 1958) indicate that $A = 250$ m^2 sec^{-2} may be taken as a reasonable mean value for the waves with wavelengths of the order 5,000 km, corresponding to wavenumber 5 or 6. With this value of A we have

$$\frac{\mu \lambda^2}{f^2} A^2 \approx 0.2\ \mathrm{m\,sec}^{-1}\, \mathrm{day}^{-1}$$

as unit for the numerical values shown in Fig. 3 and Fig. 4.

In the two-level model the equation of continuity in its x-averaged form may be written

$$\frac{\partial \bar{v}_1}{\partial y} = -\frac{\partial \bar{v}_3}{\partial y} = -\frac{\bar{\omega}_2}{\Delta p} \tag{45}$$

Combining these equations with the equations (1) and (2) in their x-averaged form, we get by integrating with respect to y

$$\frac{\partial \bar{u}_j}{\partial t} = -\frac{\partial (\overline{u_j' v_j'})}{\partial y} + f \bar{v}_j \tag{46}$$

From the values of $\frac{\partial \bar{u}_j}{\partial t}$ given by (43) and the values of $\frac{\partial (\overline{u_j' v_j'})}{\partial y}$ determined from (38), we get

$$\bar{v}_1 = -\bar{v}_3 = \frac{\mu \lambda^2}{f^3} A^2 \Bigg[2.58 \cos 2\lambda y - 4.43 \cos 4\lambda y + 3.05 \cos 6\lambda y - 1.28 \cos 8\lambda y + \\ + 0.28 \cos 10\lambda y - 0.20 \frac{\cosh \left\{ \pi \sqrt{2\sigma} \left(\frac{y}{D} - \frac{1}{2} \right) \right\}}{\cosh \frac{\pi}{2} \sqrt{2\sigma}} \Bigg] \tag{47}$$

and from the values of $\frac{\partial u_j}{\partial t}$ given by (44) and the values of $\frac{\partial(\overline{u_j' v_j'})}{\partial y}$ determined by (34), we get

$$\bar{v}_1 = -\bar{v}_3 = \frac{\mu\lambda^2}{f^3} A^2 \left[0.74 \cos 2\lambda y + 1.86 \cos 4\lambda y - 2.34 \cos 6\lambda y + 0.06 \cos 8\lambda y + \right.$$
$$\left. + 0.37 \cos 10\lambda y - 0.69 \frac{\cosh\left\{\pi\sqrt{2\sigma}\left(\frac{y}{D} - \frac{1}{2}\right)\right\}}{\cosh \frac{\pi}{2}\sqrt{2\sigma}} \right] \tag{48}$$

These values of $\bar{v}_1$ as function of y are shown in Fig. 5. From the figure it follows that the first type of wave motion ($\tau = 2.783 + 0.732i$) is connected with a mean meridional circulation with an indirect cell in the middle area and two weaker direct cells to the south and north. As shown by PHILLIPS (1954, 1956) a similar meridional circulation is connected with the amplifying waves which may develope in a zonal current without horizontal shear, and also with the large-scale disturbances, developing in his numerical experiment. From the lower part of Fig. 5, however, it follows that the other type of wave motion ($\tau = 1.624 + 0.508i$) is connected with a quite different mean meridional circulation, consisting of five cells, a direct cell in the middle area, an indirect cell on each side, and further two direct cells farthest to the south and north.

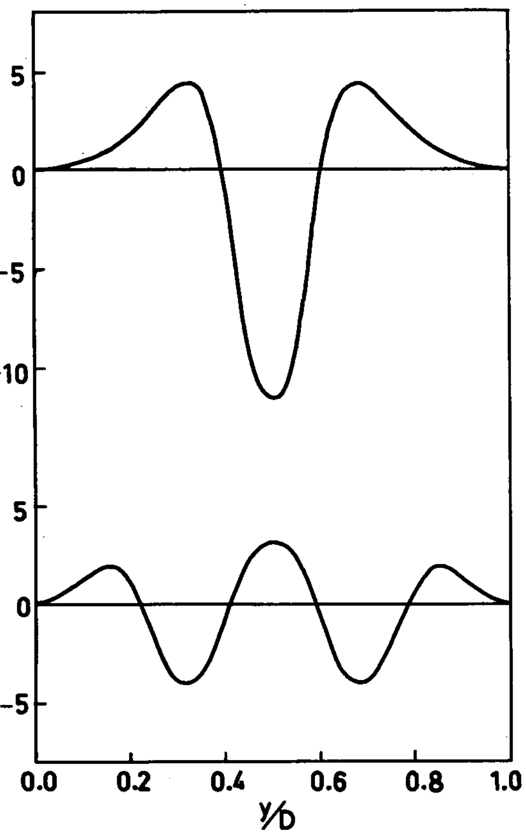

Fig. 5. Mean meridional circulations connected with the wave motions. Values of $\left(\frac{\mu\lambda^2}{f^3} A^2\right)^{-1} \bar{v}_1$, in the upper part for the first type of wave motion, in the lower part for the second type.

In order to estimate the strength of the meridional circulations, we note that with the numerical values used previously

$$\frac{\mu\lambda^2}{f^2} = 0.35 \times 10^{-6} \text{ m}^{-3} \text{ sec}^3$$

For $A = 250$ m^2 sec^{-2} it then follows from the values in Fig. 5, that the mean meridional velocities are of the order of magnitude 0.1 m sec^{-1}.

6. Combined fluctuations of the wave motion and the mean zonal flow

The generation of perturbation potential and kinetic energy was considered in section 3 for the wave motions in the original mean zonal flow, U_j, given by (9). In the following we shall consider for each type of amplifying wave motion the modifications of the energy transformations connected with the changes of the mean zonal flow caused by the wave motions. For the change of the mean zonal flow produced during the time from t_0 to t, we may write

$$\Delta \bar{u}_j = \frac{\int_{t_0}^{t} A^2 dt}{A^2} \frac{\partial \bar{u}_j}{\partial t} \tag{49}$$

where $\frac{\partial \bar{u}_j}{\partial t}$ is determined as solution of equations (41) and (42). From the formulae (23) and (25) we may compute the time rate of change of the total perturbation energy for the wave motions in the changed mean zonal flow, $\bar{u}_j = U_j + \Delta\bar{u}_j$. At first we consider the changes of the mean zonal flow caused by the first type of wave motion ($\tau = 2.783 + 0.732i$), i.e. the changes given by (43). In this case we get for the wave motion itself the following rate of conversion of mean zonal potential energy to perturbation potential energy

$$\{\bar{P}\cdot P'\} = \frac{\mu\lambda^2}{f^2} A^2(22.7 B_3 - 35.7 Q) \quad (50)$$

where $Q = \frac{\mu\lambda^2}{f^2} \int_{t_0}^{t} A^2\,dt$. For the rate of conversion of perturbation kinetic energy to mean zonal kinetic energy we get

$$\{K'\cdot\bar{K}\} = \frac{\mu\lambda^2}{f^2} A^2(4.1 B_3 + 9.8 Q) \quad (51)$$

Consequently we get for the rate of change of total perturbation energy

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2} A^2(18.6 B_3 - 45.5 Q) \quad (52)$$

From (52) it follows that the wave motion of the first type will cause a change of the mean zonal flow, which will diminish the generation of perturbation energy for this type of wave motion. As long as we may consider the solution of the form (14) as a valid approximation for the wave motion, the amplitude will grow exponentially with time. In this case we may write

$$Q \approx \frac{\mu\lambda^2}{f^2}\,\frac{A^2}{2\mu c_i} = \frac{\lambda^2}{2\tau_i f^2}\,\frac{A^2}{B_3} \quad (53)$$

With the numerical values used in the foregoing, equation (52) then becomes

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2} A^2 B_3(18.6 - 30.5 \times 10^{-6} A^2) \quad (54)$$

where A is measured in m^2 sec^{-2}. For $A \approx 250$ m^2 sec^{-2} the decrease of the generation of perturbation energy is seen to be about 10 per cent. Quite formally we obtain from (54) that the generation of perturbation energy will stop, when $A \approx 800$ m^2 sec^{-2}, corresponding to a maximum amplitude for the perturbation of the height of the pressure level p_1 equal to about 220 m, a value which is, after all, of the same order of magnitude as the observed upper limit for the amplitude of the actual waves in the atmosphere. From the results of the foregoing section it also follows that values for A of the order of magnitude 800 m^2 sec^{-2} are just the values for which the changes of the mean zonal flow, produced during a time as short as 1 day, are of the same order of magnitude as the values assumed for the original mean zonal flow.

Still considering the change of the mean zonal flow caused by the first type of wave motion, we get for the second type of wave motion ($\tau = 1.624 + 0.508i$) the following energy transformations

$$\{\bar{P}\cdot P'\} = \frac{\mu\lambda^2}{f^2} A^2(17.2 B_3 + 6.1 Q) \quad (55)$$

$$\{K'\cdot\bar{K}\} = \frac{\mu\lambda^2}{f^2} A^2(0.6 B_3 - 1.8 Q) \quad (56)$$

i. e.

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2} A^2(16.6 B_3 + 7.9 Q) \quad (57)$$

From this expression it is seen that the wave motion of the first type involves a change of the mean zonal flow, which will increase the generation of perturbation energy connected with the second type of wave motion.

For the change of the mean zonal flow caused by the second type of wave motion, we obtain quite similar results. The energy transformations connected with the first type of wave motion become

$$\{\bar{P}\cdot P'\} = \frac{\mu\lambda^2}{f^2} A^2(22.7 B_3 + 14.4 Q) \quad (58)$$

$$\{K'\cdot\bar{K}\} = \frac{\mu\lambda^2}{f^2} A^2(4.1 B_3 + 6.4 Q) \quad (59)$$

i. e.

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2} A^2(18.6 B_3 + 8.0 Q) \quad (60)$$

which means an increase of the generation of perturbation energy. The energy transformations connected with the second type of wave motion become

$$\{\bar{P}\cdot P'\} = \frac{\mu\lambda^2}{f^2}A^2(17.2B_3 - 21.5Q) \quad (61)$$

$$\{K'\cdot\bar{K}\} = \frac{\mu\lambda^2}{f^2}A^2(0.6B_3 + 0.8Q) \quad (62)$$

i.e.

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2}A^2(16.6B_3 - 22.3Q) \quad (63)$$

which means a decrease of the generation of perturbation energy. Using (53), the expression (63) becomes

$$\frac{d}{dt}(P' + K') = \frac{\mu\lambda^2}{f^2}A^2B_3(16.6 - 21.4\times 10^{-6}A^2) \quad (64)$$

where again A is measured in $m^2\ sec^{-2}$. The formal determination of the upper limit for the amplitude of the wave motion based upon the expression (64) leads to a value slightly larger than that determined from (54).

As a general result of the above computations it may be concluded that each type of wave motion, represented by the two amplifying solutions of the characteristic-value equations, will be connected with changes of the mean zonal flow, which will brake the growth of that type of wave motion, and simultaneously favour the growth of the other type of wave motion. So it seems (cf. also CHARNEY, 1959) that the actual wave motion will have the character of fluctuations between two extreme forms, associated with two extreme forms of the mean zonal flow.

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