SVENSKA GEOFYSISKA FÖRENINGEN

VOLUME 12, NUMBER 4 Tellus NOVEMBER 1960

A QUARTERLY JOURNAL OF GEOPHYSICS

The Effects of Thermal Stratification on Turbulent Diffusion from a Continuous Fixed Source

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(Manuscript received May 4, 1960)

Abstract

The similarity principle is employed to determine the maximum velocity of the vertical propagation of a diffusing smoke flowing from a continuous fixed source in a stratified fluid. With the use of the velocity profile (KAO, 1959) for a stratified fluid the profile of the boundaries of the smoke plume is evaluated numerically for various values of the stability parameter $s_0 = kgz_0 Q (c_p \bar{\varrho} \bar{T})^{-1} u_{*\mu}^{-3}$. It is shown that the spreading of a smoke plume generally increases with increasing heat flux from the boundary surface.

1. Introduction

Since the publication of Taylor's pioneering paper, "Diffusion by continuous movements" (TAYLOR, 1921), in 1921, the statistical characteristics of turbulent diffusion of particles released serially from a fixed point in a steady, homogenous and stratified fluid have been studied by many investigators, notably by RICHARDSON (1926), SUTTON (1932), BATCHELOR (1949, 1950, 1952), BRIER (1950), EDINGER (1953), FRENKIEL (1953), PANOFSKY and MCCORMICK (1954), CORRSIN (1956), ELLISON (1957), GIFFORD (1957, 1959), TOWNSEND (1958), BARAD (1959), INOUE (1959), and STEWART (1959). Advances in both experiments and theories have been made.

In an analysis of the effects of stratification on the turbulent transfers in the steady atmospheric flow over an infinite uniform plane KAO (1959) has shown that for a constant free stream velocity the mean velocity in the surface boundary layer may be expressed in terms of the dimensionless height, $\eta = z/z_0$, and the dimensionless parameters s_0 and f defined respectively by

$$s_0 = \frac{kgz_0 Q}{c_p \bar{\varrho} \bar{T} u_{*\mu}^3} \tag{1}$$

and

$$f = \frac{1}{2}\left\{1 + \left[1 + \frac{4 B s_0}{\ln \frac{H_s}{z_0}}\left(\frac{H_s}{z_0} - 1\right)\right]^{\frac{1}{2}}\right\} \tag{2}$$

where k is von Karman constant, g is the gravity acceleration, Q is the vertical heat flux, z_0 is the roughness height, c_p is the specific heat of the air at constant pressure, $u_{*\mu}$ is the friction velocity in neutral stratification, B is a constant, H_s is the height of the free stream velocity, $\bar{\varrho}$ and $\bar{T}$ are respectively the mean density and temperature of the air. Applying the similarity principle to the study of the turbulent flow in the surface layer of a stratified fluid MONIN (1959) has analysed the characteristics of turbulent diffusion and obtained some empirical results. The objective of this paper is to give a theoretical estimate of the maximum

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velocity of the vertical propagation of the diffusing smoke and the profile of the boundaries of the smoke plume making use of the above mentioned velocity profile for a stratified fluid.

2. Energy Balance Equation and the Maximum Velocity of the Vertical Propagation of the Diffusing Smoke

The Navier-Stokes equations of motion and the equation of continuity for a viscous incompressible fluid may respectively be written

$$\frac{\partial}{\partial t}(\bar{u}_i + u_i') + (\bar{u}_l + u_l')\frac{\partial}{\partial x_l}(\bar{u}_i + u_i') = \\ = \frac{g_i}{\bar{\bar{T}}}T' - \frac{1}{\bar{\varrho}}\frac{\partial(\bar{p} - p')}{\partial x_i} + \nu\nabla^2(\bar{u}_i + u_i') \quad (3)$$

$$\frac{\partial(\bar{u}_l + u_l')}{\partial x_l} = 0 \quad (4)$$

where repeated suffixes imply summation over three values of the repeated suffix, p is the pressure, u_i and $-g_i$ are respectively the i-components of the velocity and the acceleration of the gravitational field. The barred and primed quantities in (3) and (4) represent respectively the time-averaged and turbulent quantities defined as

$$\varphi(x, y, z, t; \tau) = \frac{1}{2\tau}\int_{t-\tau}^{t+\tau}\varphi(x, y, z, t)\,dt \quad (5)$$

$$\varphi' = \varphi - \bar{\varphi} \quad (6)$$

Multiplying (3) by u_i' then taking time-average of each member of the equation, we obtain the equation for the turbulent kinetic energy per unit mass

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{q'^2}\right) + \frac{\partial}{\partial x_l}\left(\frac{1}{2}\overline{q'^2 u'_l} + \frac{1}{\bar{\varrho}}\overline{p' u_i'}\right) + \\ + \bar{u}_l\frac{\partial}{\partial x_l}\left(\frac{1}{2}\overline{q'^2}\right) + \overline{u_i' u_l'}\frac{\partial \bar{u}_i}{\partial x_l} = \\ = \frac{g_i}{\bar{\bar{T}}}\overline{u_i' T'} + \gamma\overline{u_i'\nabla^2 u_i'} \quad (7)$$

where $\frac{1}{2}\overline{q'^2} = \frac{1}{2}\left(\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}\right)$ is the kinetic energy of the turbulent flow. On the left hand side of (7) the first term is the local rate of change of kinetic energy of the turbulent flow, the second term represents the transport of kinetic energy by diffuse motion, the third term represents the transport of kinetic energy through advection by the mean motion. The remaining terms in (7) represent energy production by transfer from the mean flow and energy loss through work done against buoyancy forces and through viscous dissipation.

Let the x-axis be directed along the mean velocity and the z-axis vertical. Taking average of (7) over the xy-plane, defined by

$$\tilde{\bar{\varphi}}(z, t; \tau) = \\ = \lim_{X, Y \to \infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\bar{\varphi}(x, y, z, t; \tau)\,dx\,dy \quad (8)$$

we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\tilde{\overline{q'^2}}\right) + \tilde{\overline{u'w'}}\frac{\partial \bar{u}}{\partial z} = \\ = \frac{g}{\bar{\bar{T}}}\tilde{\overline{w'T'}} + \gamma\tilde{\overline{u_i'\nabla^2 u_i'}} \quad (9)$$

Here equation of continuity is employed, $\tilde{\bar{u}} = \bar{u}$ and $\tilde{\bar{T}} = \bar{T}$ are assumed.

It is well-known that for high Reynolds numbers the rates of dissipation are very largely determined by the typical length and velocity of the turbulence and not by molecular quantities such as viscosity. In accordance with the similarity principle the rate of dissipation of turbulent energy may be expressed by $(\overline{w'^2})^{3/2}l^{-1}$ where l is the scale of turbulence. Introducing the concepts of eddy viscosity K_m and eddy conductivity K_h defined respectively by the following momentum- and heat-transfer equation

$$K_m\frac{d\bar{u}}{dz} = -\tilde{\overline{u'w'}}$$

and

$$K_h\frac{d\bar{\theta}}{dz} = -\tilde{\overline{w'T'}}$$

in (9), we obtain for the stationary case the energy balance equation

$$\frac{w_*^3}{l} = K_m \left(\frac{d\bar{u}}{dz}\right)^2 - K_h \frac{d\bar{\theta}}{dz} \frac{g}{\bar{T}} \tag{10}$$

where $w_* = (\overline{w'^2})^{\frac{1}{2}}$. Since $K_m \alpha w_* l$, (10) may be written

$$\frac{w_*}{u_*} \alpha (1 - R_f)^{\frac{1}{4}} \tag{11}$$

where

$$R_f = \frac{K_h}{K_m} \frac{\frac{g}{\bar{T}} \frac{d\bar{\theta}}{dz}}{\left(\frac{d\bar{u}}{dz}\right)^2} \tag{12}$$

is the flux Richardson number.

According to the similarity principle the maximum velocity of the vertical propagation of the diffusing smoke may be written

$$w_* = \beta u_* I(s_0\eta) \tag{13}$$

where $I(s_0\eta)$ is a universal function, and is subjected to the condition that $I \to 1$ as $|s_0| \to 0$. β is therefore equal to w_*/u_* under neutral stratification. Monin (1959) obtained from experiment that $\beta \sim 0.75$.

It has been shown (Kao, 1959) that

$$R_f = -\frac{\frac{s_0}{f^3}\eta}{\frac{1}{2}\left\{1 + \left(1 + \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}}\right\}} \tag{14}$$

Substitution of (13) and (14) in (11) gives

$$I(s_0\eta) = \left\{1 + \frac{\frac{s_0}{f^3}\eta}{\frac{1}{2}\left[1 + \left(1 + \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}}\right]}\right\}^{\frac{1}{4}} \tag{15}$$

For small values of $s_0\eta$, (15) may be approximated by

$$I(s_0\eta) \approx 1 + \frac{1}{2}\frac{s_0}{f^3}\eta \tag{16}$$

The ratio of the maximum velocity of the vertical propagation of the smoke and the friction velocity increases with $s_0\eta$.

3. The Shape of the Boundaries of the Smoke Plume

We consider a smoke plume of neutral stratification flowing out of a stationary point source at the height z_s in the surface layer of the stratified fluid. The equations for smoke particles at the boundaries of the plume may be written

$$\frac{dx}{dt} = \bar{u}, \qquad \frac{dz}{dt} = \pm w_* \tag{17}$$

where the positive and negative signs refer respectively to the upper and lower boundaries of the smoke plume. In view of (13) and (15) the equations for the boundaries of the smoke plume take the form

$$\frac{dx}{dz} = \pm \frac{G({}_0\eta)}{I(s_0\eta)} \tag{18}$$

where $G(s_0\eta)$ has been shown (Kao, 1959) to be

$$G(s_0\eta) = \frac{k\bar{u}}{u_*} = \begin{cases} \frac{1}{2}\ln\eta + \left(1 - \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}} - \tanh^{-1}\left(1 - \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}} - \left(1 - \frac{4Bs_0}{f^3}\right)^{\frac{1}{2}} + \tanh^{-1}\left(1 - \frac{4Bs_0}{f^3}\right)^{\frac{1}{2}}, \\ \qquad \text{for } Q > 0,\ \frac{s_0}{f^3} \leqslant \frac{s_0}{f^3}\eta \leqslant \frac{1}{4B} \\ \ln\eta, \quad \text{for } Q = 0, \quad \eta \geqslant 1 \\ \frac{1}{2}\ln\eta + \left(1 - \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}} - \coth^{-1}\left(1 - \frac{4Bs_0}{f^3}\eta\right)^{\frac{1}{2}} - \left(1 - \frac{4Bs_0}{f^3}\right)^{\frac{1}{2}} + \coth^{-1}\left(1 - \frac{4Bs_0}{f^3}\right)^{\frac{1}{2}}, \\ \qquad \text{for } Q < 0, \quad \eta \geqslant 1 \end{cases} \tag{19}$$

Table 1. Values of $\xi(s_0\eta)$

$s_0\,10^4$ \ η	10	100	200	300	400	500	600	700	800	900	1000
5	14.007	358.499	847.386	1380.547	1939.017	2513.432	3097.852	3687.705	4278.283	4865.532	5458.419
4	14.009	358.861	848.991	1384.404	1946.263	2525.368	3116.036	3714.174	4316.344	4918.823	5517.163
3	14.012	359.319	851.017	1389.256	1955.327	2540.187	3138.362	3746.075	4360.444	4979.043	5599.531
2	14.016	359.895	853.556	1395.310	1966.577	2558.450	3165.613	3784.492	4412.480	5047.558	5688.080
1	14.020	360.614	856.721	1402.823	1980.465	2580.850	3198.775	3830.787	4474.404	5127.776	5789.443
0	14.026	361.517	860.663	1412.134	1997.584	2608.300	3236.153	3886.750	4548.680	5223.143	5908.740
−1	14.033	362.642	865.556	1423.639	2018.645	2641.925	3288.395	3954.701	4638.461	5337.899	6051.637
−2	14.042	364.039	871.620	1437.847	2044.574	2683.212	3348.723	4037.796	4748.088	5477.865	6225.792
−3	14.052	365.784	879.153	1455.452	2076.643	2734.234	3423.282	4140.581	4883.905	5651.650	6442.629
−4	14.067	367.966	888.539	1477.353	2116.559	2797.868	3516.578	4269.768	5055.547	5872.703	6720.501
−5	14.085	370.704	900.296	1504.792	2166.755	2878.364	3635.524	4436.066	5279.036	6164.377	7092.789

For small values of $s_0\eta$, the diffusion angle may be approximated by

$$\tan^{-1}\frac{dz}{dx} \approx \tan^{-1}\left\{\beta k\,\frac{1+\dfrac{s_0}{2f^3}\eta}{\ln\eta-\dfrac{Bs_0}{f^3}(\eta-1)}\right\} \tag{20}$$

Integration of (18) from the dimensionless source-height, $\eta_s = z_s/z_0$ to an arbitrary height η gives the profile of the boundaries of the smoke plume as a function of dimensionless height η and stability parameter s_0

$$\left.\begin{aligned}\xi(s_0\eta;\eta_s) &= \beta k\frac{x}{z_0} = \pm\int_{\eta_s}^{\eta}\frac{G(s_0\zeta)}{I(s_0\zeta)}\,d\zeta = \\ &= \pm\left\{\int_1^{\eta}\frac{G(s_0\zeta)}{I(s_0\zeta)}\,d\zeta - \right. \\ &\qquad \left. - \int_1^{\eta_s}\frac{G(s_0\zeta)}{I(s_0\zeta)}\,d\zeta\right\} = \\ &= \pm\{\xi(s_0\eta)-\xi(s_0\eta_s)\}\end{aligned}\right\} \tag{21}$$

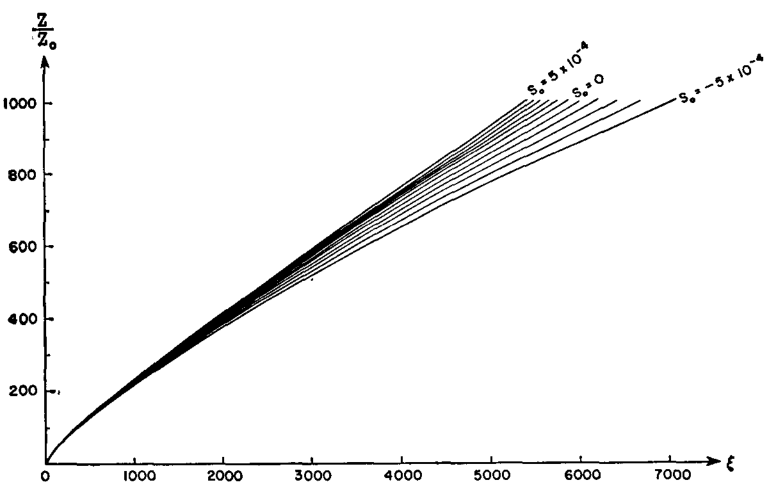

Fig. 1. The variation of ξ as a function of η, for various values of the stability parameter s_0.

where

$$\xi(s_0\eta) = \int_1^{\eta} \frac{G(s_0\zeta)}{I(s_0\zeta)}\, d\zeta \qquad (22)$$

Integral (22) has been evaluated numerically for various values of s_0 and η, and are shown in Table 1 in which $B=1$ and $H_s/z_0 = 3 \cdot 10^3$ are chosen. It is seen from Table 1 that for a fixed value of η the value of $\xi(s_0\eta)$ decreases with increasing value of s_0. This indicates that the spreading of a smoke plume increases with increasing value of s_0. It may be noted that at $\eta = 10^3$ the difference in the values of ξ for $s_0 = -5 \cdot 10^{-4}$ and the neutral case is more than 17 % of the value for the latter. The $\xi(s_0\eta)$-curves for various values of s_0 and η are shown in Fig. 1.

4. Conclusion

The effects of thermal stratification on turbulent diffusion from a continuous fixed source are examined. The similarity principle is employed to determine the maximum velocity of propagation of a diffusing smoke in a stratified fluid. The profile of the boundaries of the smoke plume is evaluated numerically for various values of the stability parameter $s_0 = kgz_0Q(c_p\bar{\varrho}\bar{T})^{-1}u_{*\mu}^{-3}$. It is shown that the spreading of a smoke plume generally increases with increasing heat flux from the boundary surface.

Acknowledgements

The writer wishes to thank Mr. Phillip Emig for programming and processing the numerical integration of equation (22) for 709 computor at the Western Data Processing Center.

This work was conducted under U.S. Weather Bureau Contract CWB-9666.

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