The Numerical Calculation of Wind Effect on Sea Level Elevations

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Abstract

An iteration method has been developed which uses finite differences for calculating water level elevation and water transport. To this end, the equations of motion were integrated over the depth and transformed into an implicit difference equation system. The equations were solved by the aid of an electronic computer. Two main cases were calculated, and in one instance the water levels were checked with the real ones.

1. Introduction

When the hydrodynamic equations of motion of liquids cannot be solved by analytical means, a numerical method can be applied for the calculation of specific results. Most scientists have avoided this method because of the rather burdensome numerical work involved. Nevertheless, such calculations can often be carried out with the aid of electronic computers. Numerical calculations relating to a sea-level problem in the North Sea were made by HANSEN (1956) and FISCHER (1959). They took the vertically integrated equations of motion, and transferred these two-dimension equations into difference equations. These equations were then used directly for calculations of the transport and water level differences by a step-by-step integration in time. Instead of using such explicit equations, the implicit ones have been adopted; this has entailed gaining some advantage at the boundary of our region, the Gulf of Bothnia, which is somewhat complicated in shape.

2. Differential Equations

Hydrodynamical equations can be written in the form:

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \zeta}{\partial x} - \nu\frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \zeta}{\partial y} - \nu\frac{\partial^2 v}{\partial z^2} = 0,$$

were x, y, z are the Cartesian coordinates, z being calculated as positive from the undisturbed surface of the sea upwards.

ζ is the elevation of the sea surface
t is the time
u, v, w are the velocity components
ν is the vertical eddy viscosity coefficient
$f = 2\,\omega \sin \varphi$, the Coriolis parameter.

It is assumed that the basin is not too complicated in shape, and that the water is incompressible and homogeneous.

In these equations, the term of horizontal viscosity and the non-linear terms are omitted; it appears that their effect is relatively minor.

The above equations can be integrated over the depth from the bottom at $-h$ to the surface at ζ, thus

$$\frac{\partial U}{\partial t} = -g(h+\zeta)\frac{\partial \zeta}{\partial x} + fV + \nu\int_{-h}^{\zeta}\frac{\partial^2 u}{\partial z^2}\partial z$$

$$\frac{\partial V}{\partial t} = -g(h+\zeta)\frac{\partial \zeta}{\partial y} - fU + \nu\int_{-h}^{\zeta}\frac{\partial^2 v}{\partial z^2}dz,$$

where $U = \int_{-h}^{\zeta} u\, dz$, and

$V = \int_{-h}^{\zeta} v\, dz$ are the components of horizontal water transport.

The stress components at bottom $\nu\left(\frac{\partial u}{\partial z}\right)_{-h}$ and $\nu\left(\frac{\partial v}{\partial z}\right)_{-h}$ are replaced by rU and rV respectively, where $r = \frac{\nu\pi}{4h^2}$, assuming the validity of steady state of EKMAN (1905). On using Ekman's friction terms it is assumed that the depth is small.

As is usual, the wind shear components $\left(\frac{\partial u}{\partial z}\right)_{\zeta}$ and $\left(\frac{\partial v}{\partial z}\right)_{\zeta}$ have been taken as $\tau_x = \gamma u_a w_a$ and $\tau_y = \gamma v_a w_a$. The wind components are u_a and v_a, w_a the total wind speed, and γ a constant.

Following substitution of the shear values, the final equations are obtained, and these, together with the continuity equations integrated over the depth, constitute the complete set of equations necessary for the calculations. In addition, we have that $\zeta \ll h$; thus we can write h instead of $h + \zeta$, except in derivatives. The equations are:

$$\frac{\partial U}{\partial t} = -gh\frac{\partial \zeta}{\partial x} + fV - rU + \tau_y$$

$$\frac{\partial V}{\partial t} = -gh\frac{\partial \zeta}{\partial y} - fU - rV + \tau_y$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}.$$

It is our aim to transform these equations, in order to make them suitable for calculations in a quadratic network. It is not advisable to take the available difference equations which are the simplest possible: it is easier to introduce the boundary conditions if they are made as symmetrical as possible. Thus we take:

$$U^{n+1}_{j,k} = U^{u}_{j,k} - \frac{\Delta t}{2}\Big[r\left(U^{n}_{j,k} + U^{n+1}_{j,k}\right) - f\left(V^{n}_{j,k} + V^{n+1}_{j,k}\right) + \frac{gh}{2\Delta s}\left(\zeta^{n}_{j+1,k} - \zeta^{n}_{j-1,k} + \zeta^{n+1}_{j+1,k} - \zeta^{n+1}_{j-1,k}\right)\Big] + \tau_x \Delta t$$

$$V^{n+1}_{j,k} = V^{n}_{j,k} - \frac{\Delta t}{2}\Big[f\left(U^{n}_{j,k} + U^{n+1}_{j,k}\right) + r\left(V^{n}_{j,k} + V^{n+1}_{j,k}\right) + \frac{gh}{2\Delta s}\left(\zeta^{n}_{j,k+1} - \zeta^{n}_{j,k-1} + \zeta^{n+1}_{j,k+1} - \zeta^{n+1}_{j,k-1}\right)\Big] + \tau_y \Delta t$$

$$\zeta^{n+1}_{j,k} = \zeta^{n}_{j,k} - \frac{\Delta t}{4\Delta s}\left(U^{n}_{j+1,k} - U^{n}_{j-1,k} + U^{n+1}_{j+1,k} - U^{n+1}_{j-1,k} + V^{n}_{j,k+1} - V^{n}_{j,k-1} + V^{n+1}_{j,k+1} - V^{n+1}_{j,k-1}\right),$$

where Δt is the length of the time step used,
- Δs is the mesh width,
- n gives the number of the time step calculated,
- j, k give the grid points which correspond to the coordinates x and y.

These equations are implicit in the unknowns $U^{n+1}_{j,k}$, $V^{n+1}_{j,k}$ and $\zeta^{n+1}_{j,k}$, and thus must be calculated by an iteration method. When the calculations are being started for a given time step, say $n+1$, the unknowns on the right side of the equations will have as substitutes other values which are suitable, here the known values for the time step n. The values thus obtained from the equations are used in the following step of iteration, and so on until the desired accuracy is attained.

The first step in the calculations is that of feeding the initial values of U, V and ζ into

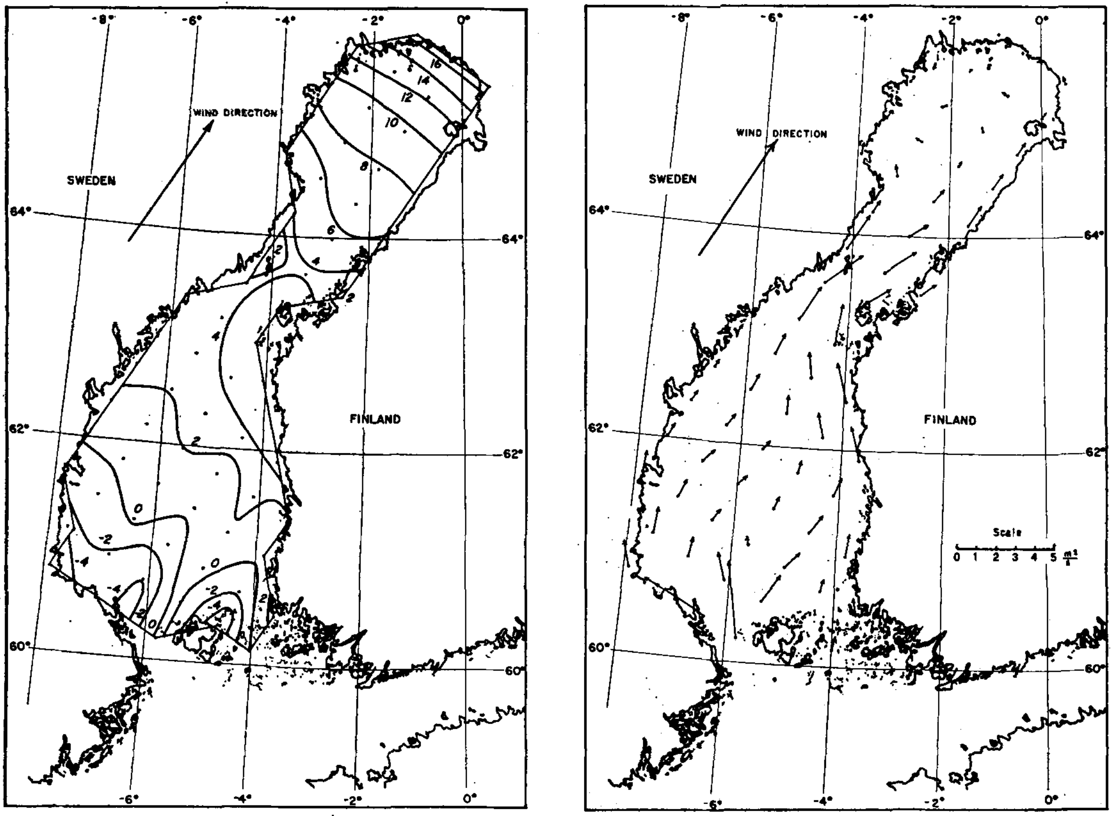


Fig. 1 a, b. Water elevations in cm after 6^h wind of 10 m/s from a southwesterly direction and the corresponding water transports.

the machine. The boundary conditions must be taken into account with each step of iteration. They are as follows:

(*i*) Near the coast, the current must be parallel to the coastal line.

(*ii*) At the open ends, the water elevations must be given.

3. The Stability of the Model

When computations are being made by electronic calculators, an examination must be made of the stability or instability of the difference equations. If the method of LAX and RICHTMYER (1956) is used, it is easy to demonstrate that our equations, without Coriolis and frictions terms, are stable. Nevertheless, practical calculations with the machine showed that the bottom roughness, the boundary conditions and the Coriolis force were causing instability and the friction decreasing the instability. To make the equations stable, it was necessary to add a smoothing term to the right sides. For the first equation, the smoothing term was:

$$\alpha\left(\frac{U^n_{j-1,k} + U^n_{j,k-1} + U^n_{j,k+1} + U^n_{j+1,k}}{4} - U^n_{j,k}\right).$$

Similar expressions were used for the other two equations. The added Laplacian does not influence the difference equations tending to the differential equations, when Δs and Δt tend to zero. In order to eliminate the instability occasioned by bottom roughness, α had to be chosen at as high value as 0.25.

4. Numerical Results

To clarify the phenomena of current and water level in the Gulf of Bothnia, knowledge of which is still rather inadequate, two principal cases were examined: that of a constant

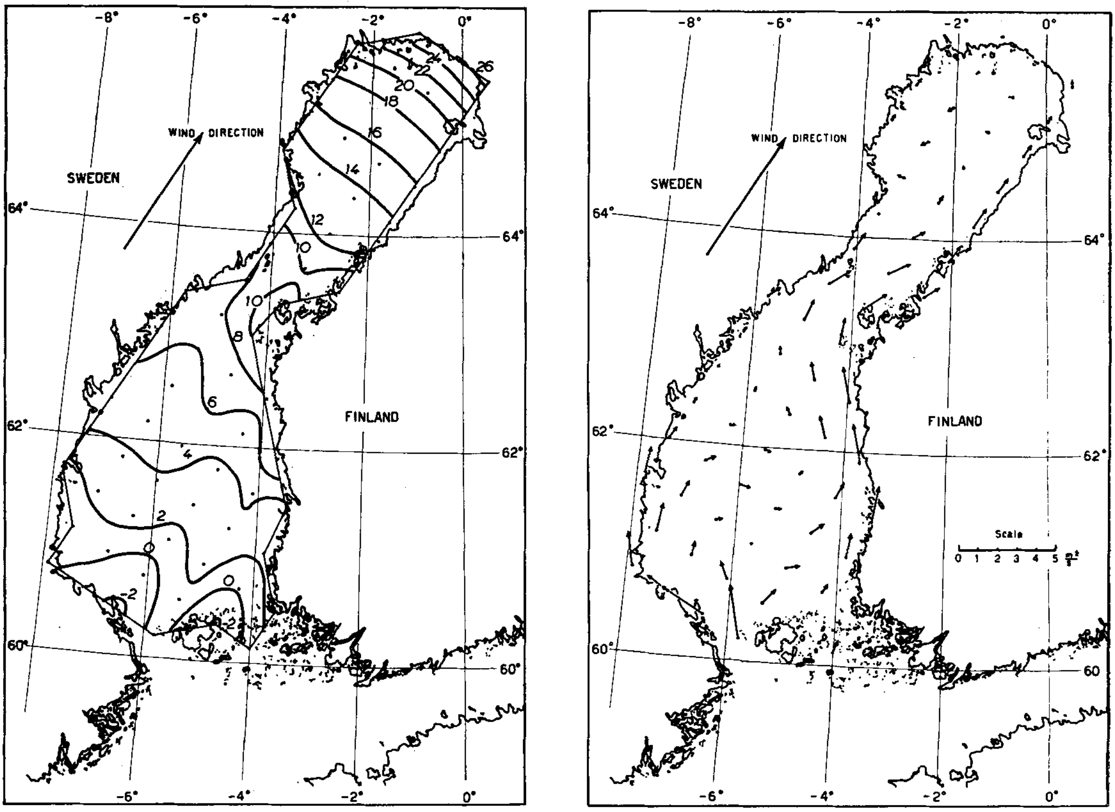


Fig. 1 c, d. Water elevations in cm after 12^h wind of 10 m/s from a southwesterly direction and the corresponding water transports.

wind blowing lengthwise over the bay, and that of a similar wind blowing transversely across it. For the calculations, the mesh width Δs of 44 km was chosen, and the time step Δt as 10 minutes. With these values and the depth $h = 200$ m, the Courant-Friedrich-Lewy criterion

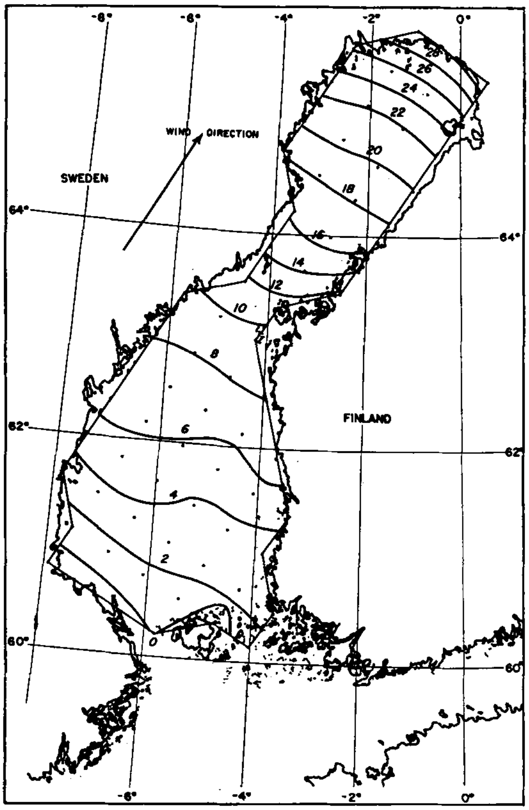
$$\sqrt{gh}\,\frac{\Delta t}{\Delta s} < \sqrt{2}$$

for the stability of a simple wave equation is fulfilled. The wind-stress constant was chosen as $\gamma = 1.9 \cdot 10^{-6}$ (cf. FISCHER, 1959).

I

In the first case, a constant wind was blowing from the southwest in the direction of the axis of the Gulf of Bothnia. As regards the boundary condition at the open end, the sea level was assumed to be normal (i.e. zero). The initial conditions were such that sea level elevation and current velocities vanished. Thus the wind started to blow simultaneously over the whole of the region. The water transport became strong in narrow sounds, and was on the whole more marked on the eastern coast and weaker on the western coast, as a result of the Coriolis force. (See Fig. 1 d, e, f.)

At the same time, the water in most of the southwestern part displayed a slight subsidence, and then rose steadily in the northeast except in the Quark, where there was at first also a subsidence in water elevation. On the eastern coast, there was a noticeable piling up of the water masses due to the Coriolis force, as there was in the Quark, where the tilting of the water surface was very marked. The water transport to the northeast was first stopped in the north, and in the Bay of Bothnia after a period of six hours from the start of the wind. (See Fig. 1.) At the same time, the sea level had



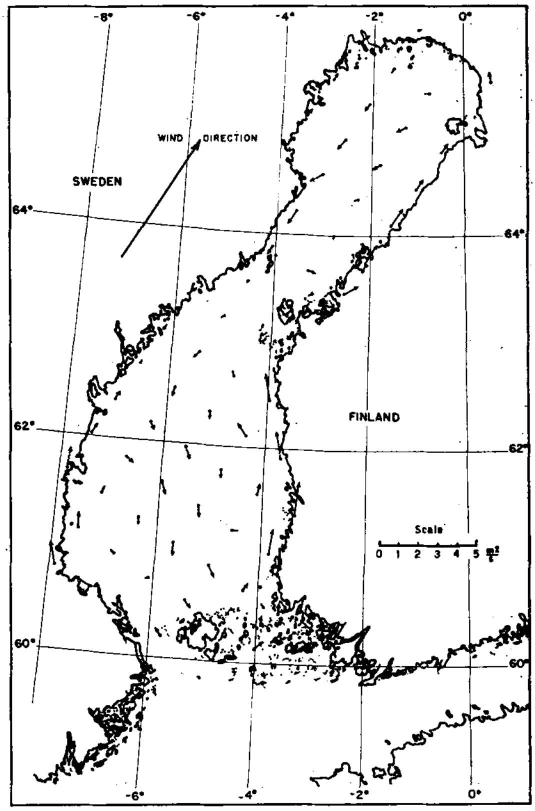


Fig. 1 e, f. Water elevations in cm after 3^h wind of 10 m/s from a southwesterly direction and the corresponding water transports.

the most irregular shape, tending later on to become smoother. A survey of the situation after 12 hours showed a clockwise circulation in the Sea of Bothnia. This circulation would however disappear later, and after 37 hours, when the currents had become steady, a northeasterly water transport was to be seen along the coast, and a counter transport in the centre of the sea. The sea surface is not a plane, but the general character is that the tilting is stronger in such areas, where the sea is shallower (P. WELANDER, 1957).

The calculated magnitude of the difference in water level elevations between the extreme ends of the Gulf of Bothnia was checked against an actual case on 16 December 1953. On the 10th December a weak southwesterly wind grew to a value of 9 m/s and lasted for a few days. The observed difference of 29 cm compared well with a calculated difference of 32 cm for a wind of 10 m/s; a calculated difference of 26 cm would have been obtained if a wind velocity of 9 m/s had been used in the calculations.

In the basins there occur regular oscillations, seiches. They show themselves most clearly in the Sea of Bothnia, where oscillations between the Swedish coast (Gävlebukten) and the Finnish coast can be observed. (See Fig. 2.) According to our calculations, the oscillations have a period of 5.4^h. This oscillation time was checked against that obtained from the formula

$$T = 2 \int_0^l \frac{ds}{\sqrt{gh}}$$

of du Boys (FOREL, 1894), which yielded 5.3^h. Here l is the distance between the two points of observations.

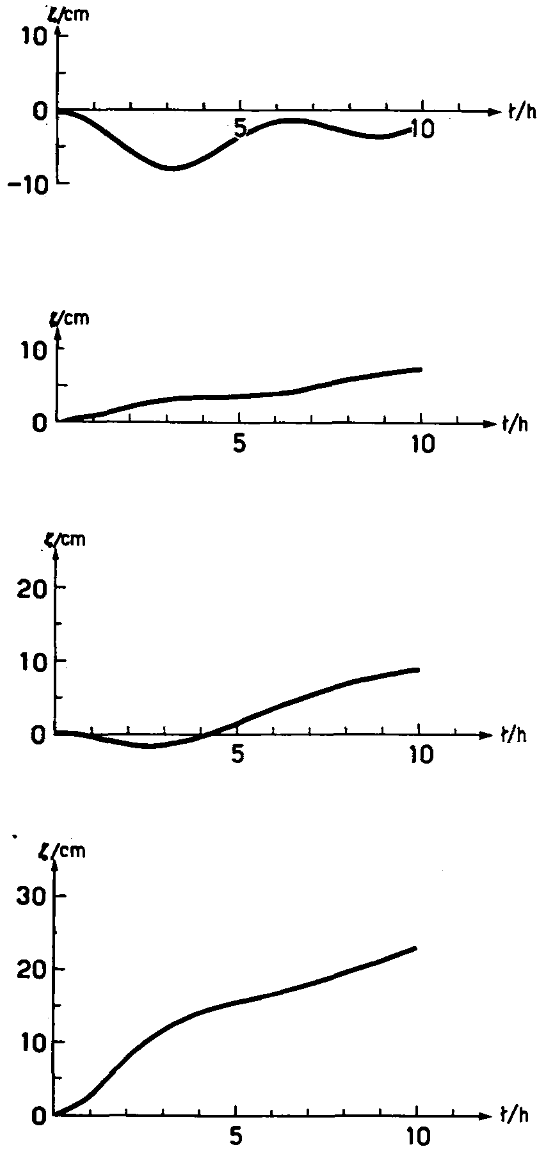


Fig. 2. Water elevations as a function of time in the southwesterly and northeasterly ends of the Sea of Bothnia, and in the southwesterly and northeasterly ends of the Bay of Bothnia.

The water elevation in the Bay of Bothnia also shows slow variations similar to seiches, but they are heavily damped and seem to be caused only by the filling of the basin.

It is still to be observed that in our calculations no seiches which belong to the whole basin have been obtained.

An analysis of some mareograms from mareograph stations on the Gulf of Bothnia yielded some oscillations which were well in accord with the assumption of seiches. Nevertheless, such a large number of local disturbances and parasitical oscillations confused the picture that no definite conclusions can at present be drawn.

II

The development of the case in which the wind blows across the Gulf of Bothnia generally followed the same lines as the foregoing. (See Fig. 3.) The maximum irregularity was here attained 3 hours after the start of the wind. At this stage the water was running out from the Gulf of Bothnia, due to the fact, that the strait to Baltic lies to the southeast of the southwesterly end of the Gulf of Bothnia. After the situation had become stable, there was a circulation of water, but even though the phenomenon was similar to the first case, it was not so clearcut. The water transport was over shallow areas with the wind direction, but hampered here by the direction of the coast line.

In this case seiches would also occur. The oscillation time in the Bay of Bothnia was found to be 3.0—4.0^h and in the Sea of Bothnia about 3.8^h. This is much less than the real seich-periods, and thus one may expect that the oscillations found may be spurious and caused by the finite difference approximation.

5. The Influence of Density Variations and Friction

When the water is not homogeneous in the whole basin, some complications occur. To illustrate what can happen, we assume that we have initially a two-layer system with a horizontal interface. (See Fig. 4.) The water beneath this interface is denser than the upper strata. For the sake of simplicity, we assume that the basin runs east-west and the wind starts blowing from the west. The water near the surface is then transported to the east and piles up there. The increasing pressure therefore pushes down the layers underneath, and contrariwise, due to the decreasing pressure at the western side, the water rises there. In approaching the position of equilibrium, the interface is much more tilted than the surface of the sea because

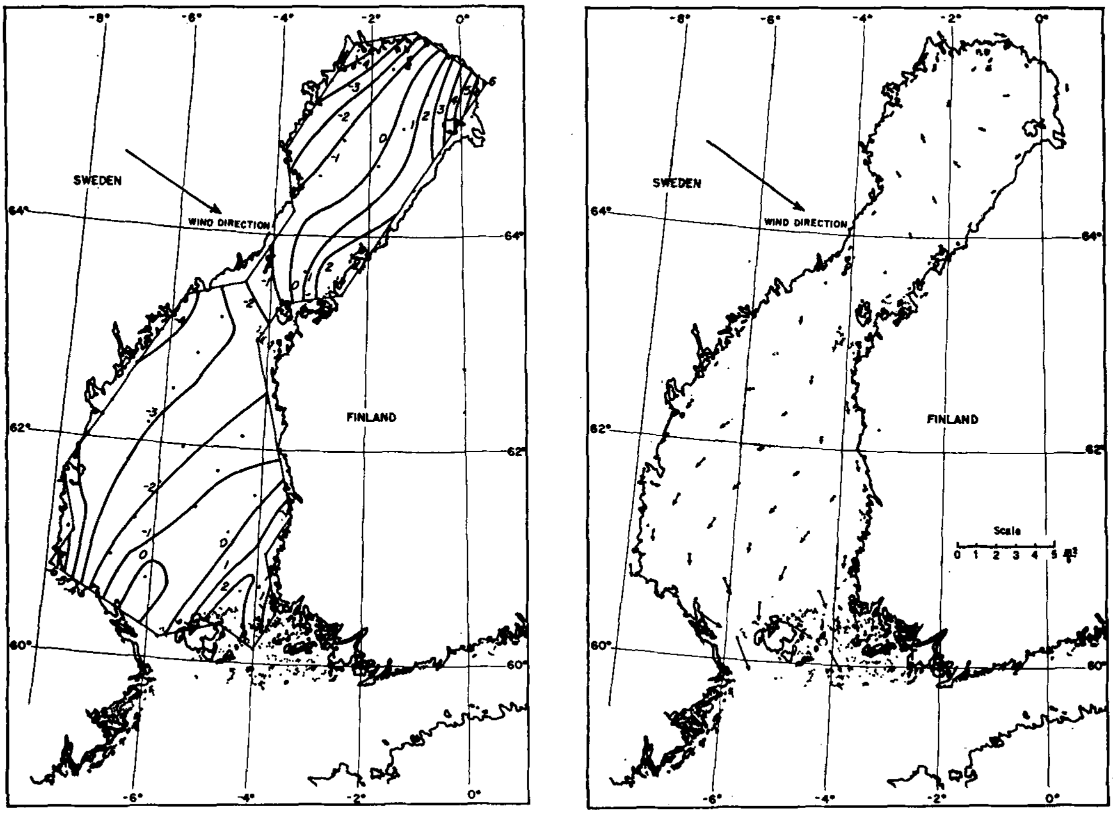


Fig. 3 a, b. Water elevations in cm after 3^h wind of 10 m/s from a northwesterly direction and the corresponding water transports.

$$\frac{\Delta h}{\Delta h'} = -\frac{\varrho' - \varrho}{\varrho}$$

where Δh and $\Delta h'$ are the differences in elevation of the surface of the sea and of the interface, respectively

ϱ and ϱ' are the densities of the lighter and denser water respectively.

This condition serves wholly to disturb our friction calculations. Instead of taking the total depth of the sea for the depth in computing the friction, perhaps the depth of the pycnocline should have been taken. The situation is indeed far more complicated because circulation is taking place not only in the upper layer, but also in the water masses below the pycnocline. When this reaches the sea surface, the situation becomes even more intricate.

When the density distribution is more complicated than described here, it is clear that there is no simple method of correcting our calculations.

The bottom friction has been calculated according to the Ekman theory, which is valid for a special kind of circulation. It is an easy matter to determine the size of the errors introduced in special cases. Let us assume that the wind is beginning to blow over an undisturbed sea surface. Then for different stages of development the velocity-depth curve can assume the forms given in Figure 5. The velocity curve which corresponds to the Ekman theory is also plotted. The abscissae of the dots seen in the figures give the computed frictions for different cases. The crosses, again, show the correct friction. The separation between the dot and the cross in each of the several sets represents the error. As seen in the figure, the friction computed can even have a sign which is opposite to the correct one.

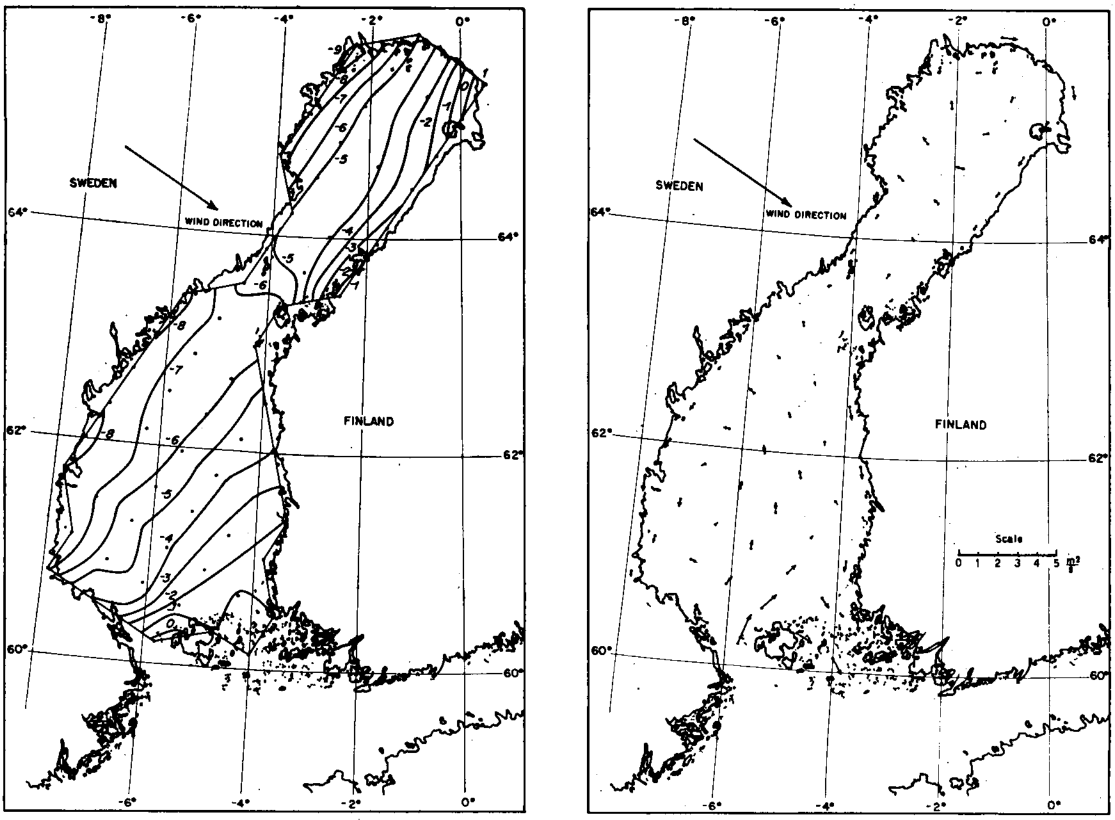


Fig. 3 c, d. Water elevations in cm after 28[h] wind of 10 m/s from a northwesterly direction and the corresponding water transports.

Even though this might seem to invalidate the method of calculation, the effect on the results is after all minor. This is because in the deep sea the friction is rather small, and in shallow regions, where the friction is noticeable, the law of friction is in rather close agreement with the Ekman theory.

6. Combination of the two Principal Solutions

Assume that at the time t', when $U = V = \zeta = 0$ a constant wind starts to blow over the whole region, giving rise to the constant wind stress components τ_x, τ_y. Then the solution of the equation system is

$$U = \tau_x \overline{U}_x(t - t') + \tau_y \overline{U}_y(t - t')$$
$$V = \tau_x \overline{V}_x(t - t') + \tau_y \overline{V}_y(t - t')$$
$$\zeta = \tau_x \overline{\zeta}_x(t - t') + \tau_y \overline{\zeta}_y(t - t'),$$

where $\overline{U}_x$, $\overline{V}_x$, $\overline{\zeta}_x$ and $\overline{U}_y$, $\overline{V}_y$, $\overline{\zeta}_y$ are the solutions of this equation system when the wind-stress components τ_x, τ_y are replaced by 1, 0 and 0, 1 respectively.

If the wind changes but remains uniform over the whole region, then the increments of the wind stresses and the corresponding solutions may be calculated and added together. (See Fig. 6.) We obtain

$$U = \sum_\nu \left[\Delta\tau_{x\nu} \overline{U}_x(t - t_\nu) + \Delta\tau_{y\nu} \overline{U}_y(t - t_\nu)\right]$$
$$V = \sum_\nu \left[\Delta\tau_{x\nu} \overline{V}_x(t - t_\nu) + \Delta\tau_{y\nu} \overline{V}_y(t - t_\nu)\right]$$
$$\zeta = \sum_\nu \left[\Delta\tau_{x\nu} \overline{\zeta}_x(t - t_\nu) + \Delta\tau_{y\nu} \overline{\zeta}_{y\nu}(t - t_\nu)\right].$$

These equations are suitable for practical calculations.

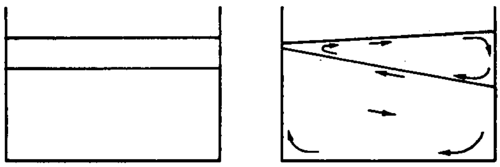


Fig. 4. Influence of the pycnocline.

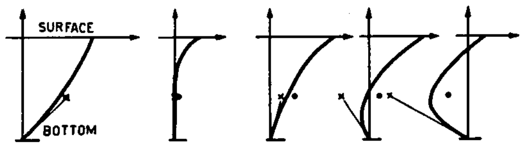


Fig. 5. Influence of the law of friction employed.

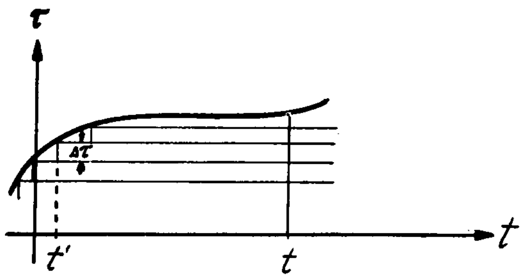


Fig. 6. The summation of constant wind stresses to obtain variable ones.

From the theoretical point of view, it can be noted that these solutions tend to follow limiting values when $\Delta\tau$ is approaching zero.

$$U = \int_{-\infty}^{t} \frac{d\tau_x(t')}{dt'} \, \bar{U}_x(t-t')\,dt' + \int_{-\infty}^{t} \frac{d\tau_y(t')}{dt'} \, \bar{U}_y(t-t')\,dt'$$

$$V = \int_{-\infty}^{t} \frac{d\tau_x(t')}{dt'} \, \bar{V}_x(t-t')\,dt' + \int_{-\infty}^{t} \frac{d\tau_y(t')}{dt'} \, \bar{V}_y(t-t')\,dt'$$

$$\zeta = \int_{-\infty}^{t} \frac{d\tau_x(t')}{dt'} \, \bar{\zeta}_x(t-t')\,dt' + \int_{-\infty}^{t} \frac{d\ _y(t')}{dt'} \, \bar{\zeta}_y(t-t')\,dt'$$

This method was proposed by P. Welander in May 1957. We hope to test it on some meteorologically interesting cases in the near future.

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