On the motion of a cyclone embedded in a uniform flow[1]

By JULIÁN ADEM and PEDRO LEZAMA, Instituto de Geofísica, Universidad Nacional de México, México 20, D. F.

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Abstract

Using the barotropic model, an analytical expression for the motion of a cyclone embedded in a uniform flow is obtained. The solution contains as parameters the radius of the vortex, the maximum velocity in the vortex, the velocity of the uniform flow and the variation of the Coriolis parameter.

It is shown that, if the variation of the Coriolis parameter is neglected, a cyclone initially embedded in a uniform flow moves with the velocity of the flow. However, superimposed with that motion there always exists a north-westward translation due to the variation of the Coriolis parameter that in some cases represents a non-negligible percentage of the total displacement of a hurricane.

Let us consider as initial flow the one given by the stream function

$$\psi(r, \Theta, 0) = DF(r/r_0) + Ar\sin\Theta + Br\cos\Theta \quad (1)$$

where (r, Θ) are polar coordinates and A, B, D and r_0 are constants. This stream function represents a vortex embedded in a uniform flow. The term $Ar\sin\Theta$ is the zonal component of the flow that has a velocity $u = -\partial\psi/\partial y = -A$, the term $Br\cos\Theta$ is the meridional component that has a velocity $v = \partial\psi/\partial x = B$, and the term $DF(r/r_0)$ represents a vortex. We choose $F(r/r_0) = 0$ for $r \geqslant r_0$, so that the radius of the vortex is r_0.

Let

$$\left.\begin{array}{ll} DF(r/r_0) = D[1-(r/r_0)^2]^4 & \text{for } r \leqslant r_0 \\ DF(r/r_0) = 0 & \text{for } r \geqslant 0 \end{array}\right\} \quad (2)$$

This stream function represents a vortex of radius r_0; it is a cyclone for D negative and an anticyclone for D positive. The maximum velocity in the vortex is at $(r/r_0) = \sqrt{1/7}$ and D can be expressed as a function of the radius and of the maximum velocity by

$$D = -0.525\, r_0\, v_m \quad (3)$$

The vortex is therefore completely determined by the radius r_0 and the maximum velocity v_m.

From the barotropic vorticity equation and the equations obtained by differentiating the vorticity equation with respect to time, we can compute $\partial\psi/\partial t$, $\partial^2\psi/\partial t^2$, $\partial^3\psi/\partial t^3$, etc. by solving equations of the Poisson type in the same way as described in a previous paper (ADEM, 1956).

From the vorticity equation we obtain

$$\nabla^2\frac{\partial\psi}{\partial t} = -\beta D\cos\Theta\frac{dF}{dr} + AD\cos\Theta\frac{d\nabla^2 F}{dr} - BD\sin\Theta\frac{d\nabla^2 F}{dr} - \beta B$$

For the range of values in which we are interested, the term βB is small and will be neglected.

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If we prescribe $\partial\psi/\partial t = 0$ at the boundary $r = R_0$ we obtain

$$\frac{\partial\psi}{\partial t} = \left[AD\frac{dF}{dr} - \frac{\beta D}{r}\int_0^r rFdr + \right.$$

$$\left. + \frac{\beta Dr}{R_0^2}\int_0^{r_0} rFdr\right]\cos\Theta - BD\frac{dF}{dr}\sin\Theta \qquad (4)$$

We will assume that $R_0 \gg r_0$ so that the term that depends on the boundary can be neglected.

Similarly, by prescribing $(\partial^2\psi/\partial t^2)_{r=R_0} = 0$ and assuming that $R_0 \gg r_0$ we obtain $\partial^2\psi/\partial t^2$.

Using a series expansion with respect to t, the stream function after a sufficiently small interval of time is given by

$$\frac{\psi(r, \Theta, t)}{D} = -\frac{As\sin\Theta}{0.525\, v_m} - \frac{Bs\cos\Theta}{0.525\, v_m} + F(s) +$$

$$+ \left[\frac{A}{r_0}\frac{dF}{ds} - \frac{\beta r_0}{s}\int_0^s F(s)\,s\,ds\right] t\cos\Theta -$$

$$- \left[\frac{B}{r_0}\frac{dF}{ds}\, t + 0.525\,\beta v_m G(s)\, t^2\right]\sin\Theta + R \qquad (5)$$

where we have made use of (3), R is the residue in the series expansion and

$$G(s) = \frac{s}{4}\int_1^s F_1(s)\,ds - \frac{1}{4s}\int_0^s s^2 F_1(s)\,ds$$

is a function of the non-dimensional variable $s = r/r_0$, in which

$$F_1(s) = -\frac{1}{s}\left(\frac{dF}{ds}\right)^2 + \frac{1}{s^2}\frac{d\nabla^2 F}{ds}\int_0^s sF(s)\,ds$$

Solution (5) has been normalized dividing all the terms by D and using the non-dimensional variable $s = r/r_0$. It contains as parameters the components of the velocity of the uniform flow A and B, the variation of the Coriolis parameter β, and the two vortex parameters: The maximum velocity in the vortex v_m and the radius of the vortex r_0. The solution is therefore useful in a study in which any of these five parameters needs to be varied.

Solution (5) has the following physical interpretation:

1) The first three terms represent the initial stream function, namely, a vortex of radius r_0 and maximum velocity v_m immersed in a uniform flow, with velocity components $u = -A$ and $v = B$.

2) The terms

$$\left[\frac{A}{r_0}\frac{dF}{ds} - \frac{\beta r_0}{s}\int_0^s F(s)\,s\,ds\right] t\cos\Theta$$

when added to the initial stream function, displace the vortex in the direction parallel to the x-axis. They are, therefore, the terms responsible for the east-west component of the displacement of the vortex. The first term represents the steering effect of the zonal component of the uniform flow. The second is due to the effect of the variation of the Coriolis parameter on the vortex. For the purpose of comparing their relative value and determining their contribution in the displacement of the vortex, we multiply both terms by r_0. We see then that the first term is proportional to the velocity of the zonal flow (A), while the second one is proportional to the square of the radius of the vortex and to the variation of the Coriolis parameter (βr_0^2).

3) The terms

$$-\left[\frac{B}{r_0}\frac{dF}{ds}\, t + 0.525\,\beta v_m G(s)\, t^2\right]\sin\Theta$$

when added to the initial stream function, displace the vortex in the meridional direction. For the purpose of comparing their relative value we multiply them by r_0. The first one is the steering effect of the meridional component of the uniform flow proportional to B, while the second one, which is due entirely to the variation of the Coriolis parameter, is proportional to $\beta r_0 v_m$.

Before we carry out the numerical computations we shall prove the following result:

Within the framework of the barotropic model, if we neglect the variation of the Coriolis parameter, a circular vortex embedded in a uniform flow moves with the velocity of the flow.

Proof. — Neglecting the β-terms and using the x, y coordinates, solution (5) can be written as

$$\psi^{\star}(x, y, t) = Ay + Bx + F^{\star}(x,y) + + \frac{\partial F^{\star}}{\partial x} At - \frac{\partial F^{\star}}{\partial y} Bt \tag{6}$$

$$\text{where } \psi^{\star}(x, y, t) \equiv \psi(r, \Theta, t)$$

$$\text{and } F^{\star}(x, y) = DF(s)$$

For small t the last three terms of the second member of (6) are the Taylor's expansion of $F^{\star}(x + At, y - Bt)$, and this function represents a vortex equal to the initial one, but with its center at the point $(-At, Bt)$. Therefore, the vortex moves with the velocity $(-A, B)$ of the uniform flow.

The next step is to establish the importance of the β-terms in the translation of the vortex. For that purpose we have numerically computed the different terms that enter in solution (5), using the function given in (2), for $\beta = 2.15 \times 10^{-13}$ sec^{-1} cm^{-1} (that corresponds to latitude 20°), for $t = 1$ hour and different values of A, B, r_0 and v_m in the range of those that appear in the case of tropical cyclones. The results of the computations are shown in Table 1 where normalized quantities are shown. To obtain actual quantities we have to multiply all the values by D.

Table 1

Terms in solution (5) for $\beta = (2.15)\ 10^{-11}$ sec^{-1} m^{-1}, $t = 1$ hour, $A = 10$ m/s, $B = 10$ m/s, $v_m = 50$ m/s and different values of r_0 in Km.

1	2	3	4	5	6
	Cyclone	Uniform flow	$\begin{Bmatrix} At \\ Bt \end{Bmatrix}$ terms	βt-term	βt^2-term
$s = \frac{r}{r_0}$	$F(s)$	$-\frac{s}{0.525\, v_m}\begin{Bmatrix} A \sin\Theta \\ B \cos\Theta \end{Bmatrix}$	$\frac{1}{r_0}\frac{dF}{ds}\begin{Bmatrix} A \cos\Theta \\ B \sin\Theta \end{Bmatrix} t$	$-\frac{\beta r_0}{s}\int_0^s F(s) ds t \cos\Theta$	$\beta v_m G(s)\, t^2 \sin\Theta$
		for $\begin{Bmatrix} \sin\Theta \\ \cos\Theta \end{Bmatrix} = -1$	for $\begin{Bmatrix} \cos\Theta \\ \sin\Theta \end{Bmatrix} = -1$	for $\cos\Theta = -1$	for $\sin\Theta = 1$
		$v_m = 50$	$r_0 = 1000$	$r_0 = 1000$	$v_m = 50$
0	1	0	0	0	0
.2	.8493	.0762	.0510	.0076	.0011
.4	.4974	.1524	.0680	.0114	.0015
.6	.1678	.2290	.0450	.0121	.0009
.8	.0178	.3048	.0100	.0096	.0003
1.0	0	.3800	0	.0082	0
			$r_0 = 350$	$r_0 = 350$	
0			0	0	
.2			.1460	.0027	
.4			.1940	.0038	
.6			.1290	.0044	
.8			.0286	.0034	
1.0			0	.0029	
			$r_0 = 200$	$r_0 = 200$	
0			0	0	
.2			.2550	.0015	
.4			.3400	.0023	
.6			.2750	.0024	
.8			.0500	.0019	
1.0			0	.0016	

Comparison of the values of columns 4 and 5 of Table 1 shows that a cyclone of radius $r_0 = 1000$ Km is translated to the west due to the variation of the Coriolis parameter, approximately at the rate of 1.5 m/sec; for $r_0 = 500$ Km at the rate of 0.4 m/sec; for $r_0 = 350$ Km at 0.2 m/sec; and for $r_0 = 200$ Km at 0.1 m/sec. If we consider that the effective radius of our vortex model is practically 0.7 r_0, we conclude that in a hurricane of effective radius of 700 Km embedded in an easterly flow of 5 m/sec, the β-term contributes 23 % of the easterly translation of the hurricane, while for effective radius of 350 Km and 140 Km, its contribution is of 7 % and 4 %, respectively.

The northward translation of the cyclone, due to the variation of the Coriolis parameter that is given in column 6 of Table 1, contains t^2 as factor and therefore cannot easily be compared with the translation due to the meridional flow which contains only t as factor.

From the values of column 6 and its comparison with column 4 it is clear that the importance of this term increases when we deal with cyclones with a large radius and strong-wind velocities.

When the meridional component of the flow is zero ($B = O$) the βt^2-term is the only one responsible for the northward displacement of tropical cyclones (in the Northern Hemisphere).

Regarding the importance of the β-terms, it can be stated that while the steering effect due to the uniform flow can reverse the sign or be zero, the northwestward translation due to the β-terms is always present and, therefore, in the long run contributes a non-negligible part to the displacement of a cyclone.

Besides the βt^2-term, the computation of $\partial^2\psi/\partial t^2$ yields additional terms in solution (5) of the form

$$\frac{A^2+B^2}{4\,r_0^2}\,\nabla^2 F(s)\,t^2,$$

$$\frac{1}{4\,r_0^2}\left\{\frac{1}{2}\,(A^2-B^2)\cos 2\Theta\right\}\left\{\frac{d^2F}{ds^2}-\frac{1}{s}\,\frac{dF}{ds}\right\}t^2,$$

$$\frac{1}{4\,r_0^2}\left\{-AB\sin 2\Theta\right\}\left\{\frac{d^2F}{ds^2}-\frac{1}{s}\,\frac{dF}{ds}\right\}t^2$$

and negligible $\beta A \cos 2\Theta$-term, $\beta B \sin 2\Theta$-term and β^2-terms.

As A or B increase, these terms become of the order of magnitude equal and even greater than the βt^2-term. This is due to the fact that for large A or B the motion of the vortex depends more on the steering effect of the uniform flow than on the variation of the Coriolis parameter. However, these terms do not contribute to the motion of the cyclone. In fact, the terms that have factors cos 2 Θ and sin 2 Θ have the effect of deforming the circular cyclone in an elliptical one, and the terms that depend only on the radius represent a change in the initial axially-symmetric pattern of the cyclone.

As A or B increase, the terms containing these parameters become the dominant ones; however, since the scale of motion decreases when A or B increase, we have to reduce the time step, reducing, hence, the contribution of the t^2-terms which become negligible compared with the At- and Bt-steering terms.

The t^2-terms containing A or B can in fact be considered as due to the neglection of the residue in the series expansion, and similar terms should appear in the higher derivatives that cancel out their contribution to the solution.

REFERENCE

ADEM, J., 1956: A series solution of the barotropic vorticity equation and its application in the study of atmospheric vortices. *Tellus* **8**, pp. 364—372.