Some Estimates of the Effects of Heating on Large Scale Disturbances

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Abstract

Effects of heating on large scale disturbances are studied by assuming that heating is tuned to the disturbances, with rate of heating proportional to individual pressure change but displaced by an arbitrary phase angle. The linearized two-layer baroclinic model reveals that the heating which accompanies condensation in the middle troposphere may lead to marked increase in instability and to shift of maximum instability from five to three thousand kilometer wavelengths, and this calls attention to a serious defect in the two-layer model which must influence the accuracy of numerical calculations using the non-linear equations.

Introduction

MILLER (1946) and MILLER and MANTIS (1947) have shown that the oceanic regions off the east coasts of North America and Asia are highly favored regions for cyclogenesis in winter. These are of course regions of large air-sea temperature difference so that the conclusion is almost inescapable that heat provided by the warm ocean plays an important role in cyclogenesis. For these reasons investigation of the distribution of heating within large scale disturbances and of the effects of this heating is needed.

A general discussion of the distribution of heating and its effects is not possible because the physics of the essential transfer processes as well as the physics of clouds is incompletely understood. Nor is it possible accurately to measure the relevant quantities at many points and so to map heating over volumes of the size of large scale disturbances. It is possible, however, to sit in a comfortable chair and to make certain useful generalizations which can serve as a starting point in investigating the problem. One realizes that the major source of heating in the free atmosphere is the latent heat released in condensation within clouds, one realizes that this heating is directly and rather simply related to vertical displacement, and one can make numerical estimates of heating to better than order of magnitude accuracy. One realizes, too, that there must be an imperfect correlation between meridional wind velocity and transfer of heat and water vapor at the ocean surface. Latent heat is distributed through a deep layer, perhaps the lower half of the atmosphere on the pressure scale, and reaches its maximum value at, say, 700 to 900 mb. Heat transferred to the atmosphere by contact with a warmer surface also is distributed through a deep layer but probably decreases in magnitude with height from the maximum value at the surface.

Qualitative and crude though these generalizations are, I should like to understand how heating of this sort influences the development of large scale disturbances. For this reason I have incorporated heating in a linear treatment of the large scale motion. The objective is to investigate the effects of various distributions of heating on the rate of growth

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of unstable disturbances. Heating is assumed proportional to individual pressure change; this represents the physics of the condensation and evaporation processes with some accuracy, and it may be considered to represent in a purely formal way heating by conduction or by radiation so long as this heating is tuned to the developing disturbance.

The Equation of Growth

The most important theoretical properties of large scale disturbances in a baroclinic atmosphere are now well known for normal middle latitude conditions, at least for the linear case. Rate of growth increases with increasing dimension of the disturbance to an equivalent wavelength of about five thousand kilometres and decreases at greater wavelengths, and disturbances normally require a time period of the order of a day in which to double in amplitude. This is probably the most important result in theoretical meteorology; it reveals that large scale disturbances originate in initial impulses of large spatial dimensions and not in the inaccessible region of very small disturbances. The mathematical tour de force which affords this result is quite formidable (KUO, 1952; GREEN, 1960) and there are aspects of the theory which have not yet been clarified. Nevertheless, it is possible to derive the results stated above by simplified approximate methods. It seems to be true, fortunately, that the approximate methods are most nearly valid for dimensions of the greatest growth rates, and these are the dimensions of greatest interest.

In the following I have extended the development of the linearized two-layer baroclinic model given by ELIASSEN (1952) and PHILLIPS (1954) by including heating in the thermodynamic equation. The assumptions on which the two-layer baroclinic model rests have been discussed widely and they are summarized by PETTERSSEN (1956). The linearized vorticity equation may be written

$$L(\zeta) + v(\beta - \partial^2 U/\partial y^2) + \frac{\partial \omega}{\partial p}(\partial U/\partial y - f) = 0 \quad (1)$$

where the symbols have the commonly accepted meanings as follows:

- x, y coordinates toward east and north respectively
- p pressure
- u, v perturbation velocity components in x and y directions respectively
- ζ $\partial v/\partial x - \partial u/\partial y$
- U undisturbed velocity component in x direction
- ω individual rate of change of pressure with time
- f Coriolis parameter
- β $\partial f/\partial y$
- L $\partial/\partial t + U\partial/\partial x$

The terms $\partial^2 U/\partial y^2$ and $\partial U/\partial y$ will be omitted in subsequent steps, but they may be reintroduced at any point, if desired. The thermodynamic energy equation may be written

$$\frac{1}{CpT}\frac{dq}{dt} = \frac{1}{\Theta}\frac{\partial \theta}{\partial t} + \frac{U}{\Theta}\frac{\partial \theta}{\partial x} + v\frac{\partial ln\Theta}{\partial y} + \omega\frac{\partial ln\Theta}{\partial p} \quad (2)$$

where Θ and θ represent respectively the undisturbed and disturbed potential temperatures, T the undisturbed temperature, q the heat added per unit mass, and C_p specific heat at constant pressure.

In order to express the heating as proportional to the individual rate of pressure change (ω) and to leave the phase relation (δ) between heating and ω arbitrary we may write

$$\frac{1}{C_pT}\frac{dq}{dt} = a\frac{\partial ln\Theta}{\partial p}\omega e^{i\delta} \quad (3)$$

where $a \equiv \left|\frac{dq/dp}{c_pT\partial ln\Theta/\partial p}\right|$. The arbitrary quantities 'a' and 'δ' are chosen to represent any one of several heating distributions. It follows from equation (3) that

$$\omega = -\frac{\Theta^{-1}L(\theta) + v\partial ln\Theta/\partial y}{\partial ln\Theta/\partial p(1 - ae^{i\delta})} \quad (4)$$

From the hydrostatic equation $\theta/\Theta = (\partial z/\partial p)/\partial Z/\partial p$ where z and Z represent, respectively, the perturbation height and undisturbed height of the pressure surface. If this relation is substituted into (4) and if the derivative with respect to p is replaced by a finite difference, then the individual rate of pressure change at a surface z_3 is expressible as

$$\omega_3 = \frac{\dfrac{g p_3}{RT(p_4 - p_2)}\left[\dfrac{\partial}{\partial t}(z_4 - z_2) + U_3 \dfrac{\partial}{\partial x}(z_4 - z_2)\right] + \left[v \dfrac{\partial \ln \Theta}{\partial y}\right]_3}{\left[\dfrac{\partial \ln \Theta}{\partial p}(1 - a e^{i\delta})\right]_3} \quad (5)$$

We make the following assignments of pressure surface:

z_0 : zero pressure
z_1 : 250 mb
z_2 : 500 mb
z_3 : 750 mb
z_4 : 1000 mb

If equation (5) were applied at z_2, only the mean heating of the atmosphere but not its vertical distribution could be incorporated in the problem. Accordingly, the equation is applied at z_1 and z_3. In order to avoid introducing additional dependent variables, however, we shall make the assumption that z_4—z_2 and z_2—z_0 may be replaced in (5) and in the analogous equation for ω_1 by z_3—z_1, that is, that the temperature distributions at z_1, z_2 and z_3 are sufficiently the same to be used interchangeably. Finally we introduce the geostrophic relations

$$v = \frac{g}{f}\frac{\partial z}{\partial x} \quad (6)$$

$$\Delta U = \frac{g}{f}\frac{\partial}{\partial y}(Z_4 - Z_2) \quad (7)$$

and for convenience assume $\partial U/\partial p$ independent of pressure and $T_1 \approx T_3 \approx T$. Equation (5) then may be written

$$\omega_3 = \frac{g p_3 [L_3(z_3 - z_1) + \Delta U \partial z_3/\partial x]}{RT p_2 (\partial \ln \Theta/\partial p)_3 (1 - a_3 e^{i\delta_3})} \quad (8)$$

Similarly

$$\omega_1 = \frac{g p_1 [L_1(z_3 - z_1) + \Delta U \partial z_1/\partial x]}{RT p_2 (\partial \ln \Theta/\partial p)_1 (1 - a_1 e^{i\delta_1})} \quad (9)$$

These relations are to be used to eliminate ω from the vorticity equation applied at p_1 and p_3.

We assume that $\partial\omega/\partial p$ is directly proportional to and has the same phase relations as ω at each of the pressure surfaces. For simplicity the proportionality factors are set equal to $4/3p_2$; this permits an asymmetrical ω distribution in the general case but reduces to the much used parabolic ω profile in the case $\omega_1 = \omega_3$. The finite difference representations for $\partial\omega/\partial p$ may be written for $\omega = 0$ at p_4 and p_0.

$$\left(\frac{\partial\omega}{\partial p}\right)_3 = -\frac{4}{3}\frac{\omega_3}{p_2} \quad (10)$$

$$\left(\frac{\partial\omega}{\partial p}\right)_1 = \frac{4}{3}\frac{\omega_1}{p_2} \quad (11)$$

Finally, the geostrophic relations are introduced into the vorticity equation, so that at p_1

$$\frac{\partial}{\partial t}[\nabla^2 z_1 + \lambda_1^2(z_3 - z_1)] + U_1 \frac{\partial}{\partial x}[\nabla^2 z_1 + \lambda_1^2(z_3 - z_1)] + (\beta + \lambda_1^2 \Delta U)\frac{\partial z_1}{\partial x} = 0 \quad (12)$$

and at p_3

$$\frac{\partial}{\partial t}[\nabla^2 z_3 - \lambda_3^2(z_3 - z_1)] + U_3 \frac{\partial}{\partial x}[\nabla^2 z_3 - \lambda_3^2(z_3 - z_1)] + (\beta - \lambda_3^2 \Delta U)\frac{\partial z_3}{\partial x} = 0 \quad (13)$$

where

$$\lambda_1^2 = -\frac{4}{3}\frac{f^2 p_1}{p_2^2 RT} \div \left(\frac{\partial \ln \Theta}{\partial p}\right)_1 (1 - a_1 e^{i\delta_1}) \quad (14)$$

$$\lambda_3^2 = -\frac{4}{3}\frac{f^2 p_3}{p_2^2 RT} \div \left(\frac{\partial \ln \Theta}{\partial p}\right)_3 (1 - a_3 e^{i\delta_3}) \quad (15)$$

We consider only solutions in which the wavelength, speed, and rate of growth are independent of height. Therefore, let

$$z_1 = \hat{z}_1 e^{i\alpha x - i\sigma t} \quad (16)$$

$$z_3 = \hat{z}_3 e^{i\alpha x - i\sigma t} \quad (17)$$

These may be readily generalized to include y dependence. Upon substituting into equations (12) and (13), a quadratic frequency equation follows:

$$\sigma^2\left(1+\frac{\lambda_1^2+\lambda_3^2}{\alpha^2}\right)+\sigma\alpha\left\{\frac{\beta}{\alpha^2}\left(2+\frac{\lambda_1^2+\lambda_3^2}{\alpha^2}\right)+\right.$$
$$\left.+\Delta U\frac{\lambda_1^2+\lambda_3^2}{\alpha^2}\right\}+\frac{\beta^2}{\alpha^4}+\frac{\beta\Delta U}{2\,\alpha^2}\,\frac{\lambda_1^2-\lambda_3^2}{\alpha^2}-$$
$$-\frac{\Delta U^2}{4}\left(1-\frac{\lambda_1^1+\lambda_3^2}{\alpha^2}\right)=0 \qquad (18)$$

The solution may be written

$$\sigma=-\frac{\beta}{2\,\alpha}\,\frac{2\,\Gamma_1\Gamma_3+\Gamma_1+\Gamma_3}{\Gamma_1\Gamma_3+\Gamma_1+\Gamma_3}\pm\frac{i\alpha\sqrt{\Gamma_1\Gamma_3}}{\Gamma_1\Gamma_3+\Gamma_1+\Gamma_3}$$
$$\left\{\Delta U^2\left(1-\frac{\Gamma_1\Gamma_3}{4}\right)+\frac{\beta\Delta U}{2\,\alpha^2}\,(\Gamma_1-\Gamma_3)-\right.$$
$$\left.-\frac{\beta^2}{4\,\alpha^4}\left(\frac{\Gamma_3}{\Gamma_1}+\frac{\Gamma_1}{\Gamma_3}+2\right)\right\}^{1/2} \qquad (19)$$

where

$$\Gamma_1\equiv\alpha^2/\lambda_1^2 \qquad \Gamma_3\equiv\alpha^2/\lambda_3^2$$

For $\Gamma_1=\Gamma_3$ equation (19) takes the form

$$\sigma=-\frac{\beta}{\alpha}\,\frac{\Gamma+1}{\Gamma+2}\pm\frac{i\alpha}{\Gamma+2}\left\{\Delta U^2\left(1-\frac{\Gamma^2}{4}\right)-\right.$$
$$\left.-\frac{\beta^2}{\alpha^4}\right\}^{1/2} \qquad (20)$$

which is a slightly more general form of the frequency equation developed by Phillips (1954). Equality of Γ_1 and Γ_3 implies that $\lambda_1^2=\lambda_3^2$, and this can occur in a variety of ways. For example, for $a_1=a_3$ and $\delta_1=\delta_3$, Γ_1 equals Γ_3 if the static stability at z_3 is three times that at z_1. Or, the same result may occur for equal static stabilities if $\delta_1=\delta_3=0$ and $1-a_3=3(1-a_1)$; this implies that heating at z_1 exceeds heating at z_3.

For the case of no heating ($a=0$) and for the case of heating in phase with ω, Γ is real; and the second term of equation (20) represents the growth of unstable baroclinic waves (positive root) as well as the reduction in amplitude of a damped wave (negative root). For the case of heating displaced in phase from ω, Γ is complex; and equation (20) is difficult to interpret except by specific examples.

Results

Curve A in Figure 1 represents the case of no heating for the following values:

β: 1.6×10^{-13} sec^{-1} cm^{-1} (lat. 45 deg.)
ΔU: 2.0×10^{3} cm sec^{-1}
λ^2: 1.46×10^{-16} cm^{-2} ($\partial\, ln\,\Theta/\partial z$ at 500 mb: 1.6×10^{-7} cm^{-1})

Maximum instability is reached at a wavelength of 5×10^3 kilometers, and all waves shorter than about 3.6×10^3 and longer than about 7×10^3 kilometers are stable. These results should be compared with the result of integrating the three dimensional linear equations, as has been done by Kuo (1952), for example. The models are not identical, but this probably is of little importance. Curve K is taken from Kuo's paper for $\partial U/\partial z=3\times10^{-3}$ sec^{-1} and $\partial ln\Theta/\partial z=1.5\times10^{-7}$ cm^{-1}. The sharp cut-off to instability on both the short and long wave sides shown by curve A is evidently incorrect, and the wavelength of greatest instability is about 10^3 kilometers too great in the two-layer model. Corresponding defects must be expected in the non-linear treatment of the two-layer model as well.

The effect of heating in the two-layer model may be understood qualitatively from equations (3), (14) and (15). Consider first the case which results from vertical motion in saturated air; condensation and heating occur with upward motion, evaporation and cooling with downward motion. For this case δ vanishes.

It is evident that as 'a' increases Γ decreases and instability increases in magnitude and covers a wider range of wavelengths. For $a=0.7$, which corresponds to heating at the saturated adiabatic rate, equation (20) yields curve B in Figure 1. Maximum value of the exponential factor is more than doubled, the wavelength of maximum instability shifts from 5×10^3 kilometers to 3×10^3 kilometers; and instability is extended to much shorter waves. This result may be achieved also by calculating static stability with respect to saturated air and setting 'a' equal to zero in equation (20).

We may imagine the reverse distribution of heating in subsiding air, cooling in ascending air; although this situation is unrealistic for the normal middle latitude planetary disturbance. For this case $\delta=\pi$ and Γ is increased over the adiabatic value. This results in decreased instability; for $a = 0.7$ maximum value of the exponential factor of only 10^{-6} sec^{-1} occurs at about 6×10^3 kilometers. This is not shown in Figure 1.

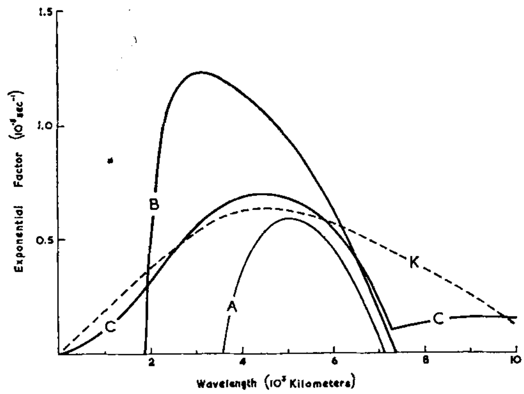

Fig. 1. Exponential factor as a function of wavelength calculated from equation (20) for no heating (curve A), for heating east of pressure troughs (curve B), and from equation (21) for heating in pressure troughs (curve C). Kuo's (1952) result is shown as curve K.

An arbitrary phase difference between heating and individual pressure change may be represented by choosing an arbitrary finite value for δ. For convenience we will choose δ to be $\pi/2$, representing maximum heating where individual pressure change vanishes. For this case we may express Γ by $\Gamma_r\ (1 + ib)$, and equation (20) becomes

$$\sigma_i = \frac{\alpha}{\Gamma_r^2(1+b^2)+4(1+\Gamma_r)}\left\{-\frac{\beta b}{\lambda^2} \pm (\Gamma_r + 2 - ib\Gamma_r)\left[\Delta U^2\left(1 - \frac{\Gamma_r^2(1-b^2)}{4}\right) - \frac{\beta^2}{\alpha^4} - \frac{ib\Gamma_r^2}{2}\Delta U^2\right]^{1/2}\right\} \qquad (21)$$

For $b = -0.7$, which represents heating in the pressure troughs and cooling in the pressure ridges at the same rate referred to earlier, equation (21) yields curve C in Figure 1. This curve exhibits instability at short wave lengths qualitatively similar to the linearized three dimensional wave studied by Kuo. It also exhibits instability at long wavelengths, but here there is considerable difference from Kuo's result.

Heating in the pressure ridges with cooling in the troughs may be represented by setting b equal to $+0.7$. This results in instability very much like that shown by curve C but displaced downward slightly and with no instability at long wavelengths. This is not shown in the figure.

Heating distributions intermediate between those already discussed may be readily calculated for any arbitrary phase angle, δ. Their qualitative nature may be determined by superimposing two curves appropriately weighted, say curves B and C.

Significance of Results

In the following I should like to push interpretation somewhat beyond the valid limits of the preceding analysis; that is, to try to draw insights into the mechanism of the real atmosphere from the behavior of this greatly simplified and restricted model. There is no doubt that this is speculative and potentially dangerous, but often progress is made through initial insights which are largely intuitive.

Let us imagine a series of moving sinusoidal disturbances and a distribution of heating which is fixed with respect to the earth. This condition might represent roughly wintertime heating and evaporation off the east coasts of North America and Asia. The results represented by curve B suggest that when the disturbance is situated so that the middle troposphere pressure trough is located to the west of the warm source (say, along the coast), waves should amplify rapidly. When the pressure trough is located above the warm source, rate of growth is then only slightly greater than with no heating; and when the northwest current lies above the warm source, rate of growth is much less and the disturbance may be neutrally stable. Finally, when the ridge is located above the warm source, rate of growth is similar to the rate with trough above the warm source. So one may imagine energy being fed into a train of moving waves in a series of pulses, each succeeding wave being amplified rapidly as the trough passes the coastline.

Figure 1 indicates that dependence of instability on wavelength is strongly influenced by the distribution of heating. However, this may not be valid, for comparison of curves A and K shows that the two-layer model is seriously deficient in failing properly to describe the dependence of instability on wavelength. Curve A indicates growth sharply limited to a narrow range of wavelengths, and it is easy to realize that this may result in serious errors in prediction. Comparison of curves A, C and K suggests that such errors in prediction arising from the short and long wave stability of the two-layer model might be reduced substantially by arbitrarily introducing heating in the troughs and cooling in the ridges.

The two-layer model requires that maximum ω occur at 500 mb for all wavelengths. This property violates the observed structure of baroclinic disturbances as well as the solution of the relevant differential equation (KUO, 1952); this we must recognize as payment for the simplicity of the method used here. If the restriction is removed (for a simple example, see FLEAGLE, 1957), short waves become unstable though less unstable than waves of about 5000 kilometers length.

Long wave stability arises in equation (19) through the strong effect of the squared β term at long wavelengths. One should not infer from this, however, that long waves are necessarily stable. The long wave instability revealed by curve C in Figure 1 reflects the appearance of a phase difference between the fields of vertical velocity and of individual temperature change at constant pressure. It is not difficult to demonstrate that if other phase differences were present, say between vertical and meridional velocity components, similar long wave instability may result. Phase differences of this sort are always observed in atmospheric disturbances and they must be expected in any but very much oversimplified theoretical treatments of baroclinic waves; there is, therefore, no convincing evidence for "stable wavelengths" in the real atmosphere, a point which I did not comprehend when the 1957 paper was written. There is, however, a considerable difference in growth rate between wavelengths of about 5000 kilometers and all others, and this is the most important point.

Critical Observations

(1) Replacement of the vertical derivative by finite differences violates the physics expressed by the governing differential equations; the maximum vertical velocity should not be prescribed as occurring at an arbitrary surface. As pointed out earlier, this has serious consequences in introducing a false short wave cut-off to instability, but the wavelengths of greatest instability probably are not greatly affected.

(2) Observations indicate a complicated spatial distribution of heating. The most important heating probably results from condensation in the middle troposphere, but this probably is not uniformly distributed between 750 and 250 mb as assumed in the illustrative calculations. Very intense heating may occur at and just above the earth's surface. Heating is distributed horizontally by the geographical distribution of land, sea, ocean currents and surface characteristics, and not, as assumed here, entirely by the dimensions of the growing disturbances.

(3) Linear theory becomes increasingly inaccurate as amplitude increases and is incapable of representing such important features as the development of fronts.

These criticisms are sufficiently serious that one should be wary of drawing generalizations of wide applicability. I believe that the proper attitude toward the results discussed here is that they provide qualitative estimates of certain effects of heating by which one may judge what is and what is not to be expected from adiabatic theory. It also provides tentative relationships, such as the large heating effect when heating and vertical velocity are in phase or the effect of phase shift in introducing instability at all wavelengths, which it may be interesting to verify by independent means.

The results indicate the following:

(1) Heating of the magnitude which accompanies condensation in the middle troposphere may result in a significant change in growth rate of the most unstable waves. If such heating occurs in the ascending air east of the pressure trough, increase in rate of growth achieves its greatest value. If such heating occurs in the descending air to the west of the trough, instability is strongly inhibited.

(2) Heating which occurs in the pressure trough may result in only slight increase in instability at the wavelengths of greatest instability, but it has the effect of introducing into the two-layer model instability at all wavelengths.

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