# On Critical Regimes and Horizontal Concentration of Momentum in Ocean Currents with Two-Layered System

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## Abstract

The principles of minimum momentum transport and minimum energy which are commonly used in hydraulics of an open channel are applied to the current in a two-layered ocean, in which the geostrophic relation holds. For a flow of a finite width over the motionless lower layer these principles with the constancy of total volume transport predict the critical values of current width and upper layer thickness which will be attained by the flow after losing momentum or energy by friction. The two principles lead to critical regimes with almost similar order of magnitude. The critical width thus determined is less than 100 km and the upper layer disappears at the left edge of the flow in the critical regime, being in good agreement with the Gulf Stream and the Kuroshio. The analysis is also applied to a model of vertically continuous density stratification. Further the momentum transport of double jets often observed in actual currents is shown to be the minimum among multiple jets, next to a single jet corresponding to the critical regime.


## 1. Introduction

Many synoptic surveys done recently have confirmed the fact that ocean currents such as the Gulf Stream and the Kuroshio remain narrow and well defined not only along the coast but also as far offshore as at least 3,000 km from the continent. (FUGLISTER and WORTHINGTON, 1951; ICHIYE, 1956). This is rather remarkable when it is recalled that frictional forces due to strong shear at the boundaries have a tendency to dissipate such narrow currents. In order to explain this fact, the late C. G. ROSSBY developed a novel theory of the concentration of momentum (1951) based on the momentum principle which has long been used in hydraulics of an open channel. As a remarkable result of his theory, are obtained vertical density and velocity distribution of ocean currents which agree well with observed ones in the Gulf Stream in spite of the very simple basic assumption. Thus he explained the strong current at the upper layer of the Gulf Stream as the result of vertical concentration of momentum transport. However, his promising theory had not yet the final goal, when his sudden death in 1957 prevented appearance of the second part of his paper which was expected to discuss the horizontal concentration of momentum.

The author does not pretend to succeed to Rossby's first paper, but does an application of the momentum principle and similar energy principle, to explain this horizontal concentration of the current as a critical regime of a jet-like current. In Rossby's original paper and Craya's subsequent paper (CRAYA, 1951), the density stratification of ocean currents is derived from the principle of minimum momentum or energy flux, which gives the vertical velocity distribution similar to the observed one in the Gulf Stream as a final stage of the current. The basic idea of their argument which has introduced these principles primarily used in hydraulics to oceanography is that the ocean current may reach the final stage or critical regime specified by these minimum principles after it loses momentum or energy through dissipative forces. The critical regime is determined from the condition of minimum momentum or energy with the assumption of constant volume transport, specifying a critical velocity in an open channel with a uniform density distribution. When the density of fluid is variable and thus the velocity is not uniform, the minimum condition leads to a critical regime which specifies a critical density distribution and corresponding current.

Both authors considered gravity currents as a basic flow, in which a pressure gradient, if any, exists only in the direction of the flow. However, actual ocean currents are strongly geostrophic, that is, a pressure gradient across the flow has to balance the Coriolis' force. Further, they are mostly confined in a relatively shallow layer of a finite width. Thus a two-layered system with a motionless lower layer well represents a model of the actual ocean in spite of its simplicity. When each layer has a different uniform density, the velocity of the upper layer is proportional to a gradient of thickness of the upper layer across the current. Then the total volume transport of the current with a finite width becomes proportional to the difference of the square of thickness at the sides. The energy or momentum transport is a function of the width of the current, thickness of the upper layer and its lateral gradient. Therefore these quantities, width and thickness instead of density distributions in the previous authors' argument, are considered as variables in the variational problem for momentum or energy transport. Their critical values specified by the minimum condition of energy or momentum transport under the constancy of total volume transport correspond to a critical regime which will be attained by the current after its energy or momentum is dessipated by frictional forces.

The critical regime determined from the minimal energy principle are generally different from those from the minimal momentum principle, as discussed by CRAYA (1951). However, critical values of width and thickness by two principles are shown to be of a comparable order of magnitude. Further a critical regime in ocean currents "may be only hypothetical terminus" which indicates only the sense of direction in future change of the state, as pointed out by CRAYA (1951). Therefore it is useless to discuss the difference of critical regimes determined by the two principles in details.

## 2. Momentum transport of a two-layered ocean and minimal momentum condition in a uniform jet

A two-layered system is considered, in which the current flows only in the upper layer with the depth $D$ (Fig. 1). Density of the upper and lower layer is assumed to be constant and is put to $\varrho$ and $\varrho_0$, respectively. The vertical coordinate $z$ is positive downwards from the equilibrium sea surface and $\zeta$ means the deviation of the surface from the equilibrium state. According to CRAYA (1951) the thrust $P$ of the current per unit width is given by

$$P = (D+\zeta)^2 g\varrho(\varrho_0-\varrho)/2\varrho_0 + (D+\zeta)\varrho v^2 \quad (1)$$

for a flow which has a uniform velocity $v$ at the upper layer with thickness $D+\zeta$ and is motionless at the lower layer (in Craya's original equation, flux $v(D+\zeta)=q$ is used instead of $v$). This equation can be derived directly from the definition of the thrust which is the variable part of the integral of the sum of pressure and momentum transport from the bottom to the surface. The pressure of upper and lower layer $p$ and $p_0$ are given by

$$p = g\varrho(z+\zeta), \quad p_0 = g\varrho(D+\zeta) + g\varrho_0(z-D) \quad (2)$$

Since there is no current in the lower layer, the

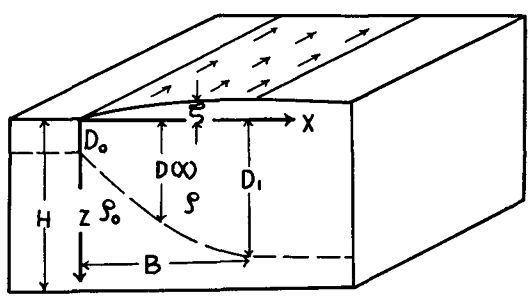


Fig. 1. Block Diagrams of a Current in a Two-layered Ocean

pressure at the bottom $p_b$ is constant and is given by putting $z = H$ in (2), where $H$ is the total depth.

The integration of pressure from the surface to the bottom gives

$$\int_{-\zeta}^{D} p\,dz + \int_{D}^{H} p_0\,dz = \frac{1}{2} g\varrho (D+\zeta)^2 + g\varrho (D + \zeta)(H - D) + \frac{1}{2} g\varrho_0 (H-D)^2 = p_b^2/2g\varrho_0 + (D+\zeta)^2 g\varrho(\varrho_0 - \varrho)/2\varrho_0 \quad (3)$$

in which the first term of the last expression is constant. Thus the thrust of the equation (1) is the sum of the momentum transport and the second term with $(D+\zeta)^2$, as shown in the equation (1).

From the geostrophic relation the velocity of the upper layer is

$$v = (\partial p/\partial x)/f\varrho = (g/f)(\partial\zeta/\partial x)$$

in which $x$ is taken across the current. The assumption that the lower layer is motionless gives $\partial p_0/\partial x = 0$ or $\partial\zeta/\partial x = (\Delta\varrho/\varrho)\partial D/\partial x$, in which $\Delta\varrho = \varrho_0 - \varrho$.

Therefore the equation (1) becomes

$$P = (D+\zeta)^2 \varrho g \Delta\varrho/2\varrho_0 + (D+\zeta)\varrho(g\Delta\varrho/f\varrho)^2(\partial D/\partial x)^2 \quad (4)$$

Since $\zeta$ is of an order of one meter and $D$ is of hundreds meters, we can put $D + \zeta \approx D$. Further $\Delta\varrho \approx 2\times 10^{-3}$ and also we can take

$$g' = g\Delta\varrho/\varrho_0 \approx g\Delta\varrho/\varrho.$$

The equation (4) becomes

$$P = \varrho D\left[\frac{1}{2} g' D + (g'/f)^2(\partial D/\partial x)^2\right] \quad (5)$$

When the current flows only in a belt extending from $x = 0$ to $B$ as is the case in ocean currents, the total thrust is given by

$$M = \int_0^B P\,dx \quad (6)$$

which tends to decrease always through the loss of momentum. On the other hand the total volume transport $V$ by the upper current remains constant and is given by

$$V = \int_0^B \varrho v D\,dx = (\varrho g'/2f)(D_1^2 - D_0^2) \quad (7)$$

by substituting the geostrophic relation for velocity $v$, where $D_0$ and $D_1$ are the values of $D$ at $x = 0$ and $B$, respectively. The equation (7) indicates that $V$ is independent of the width $B$. The critical regime for the momentum principle can be obtained by making $M$ minimum under the condition of constant $V$.

When the velocity $v$ is constant within the belt, $D$ becomes a linear function of $x$ and is given by

$$D = D_0 + (D_1 - D_0)x/B$$

Then $M$ becomes

$$M = \frac{1}{2}\cdot \varrho B\left[(g'/fB)^2 D_s^2 \Delta D + g'\left(D_0 D_1 + \frac{1}{3}\Delta D^2\right)\right] \quad (8)$$

in which $\Delta D = D_1 - D_0$ and

$$D_s^2 = D_1^2 - D_0^2 = 2fV/\varrho g'$$

In ROSSBY's or CRAYA's model (1951) the minimum condition $\delta P = 0$ leads to the critical velocity $(g'D)^{\frac{1}{2}}$ under the constant transport. On the other hand, in the present minimum problem the variables are $B$ and $D_0$ (or $D_1$). Thus the total thrust $M$ is a function of these two variables and its minimum can be determined from two conditions.

Since $D_s$ is constant, we can introduce non-dimensional variables such as

$$\xi = B/D_s \text{ and } \eta_i = D_i/D_s, \quad (i = 0, 1)$$

where $\eta_0$ and $\eta_1$ are related with

$$\eta_1^2 - \eta_0^2 = 1 \quad (9)$$

Then the equation (8) is expressed by

$$M = \frac{1}{6}\varrho g' D_s^3 \xi \cdot \{\eta_0^2 + \eta_1^2 + \eta_0\eta_1 + 3\gamma(\eta_1 - \eta_0)\xi^{-2}\} \quad (10)$$

in which

$$\gamma = g'/f^2 D_s$$

The total thrust $M$ is a function of $\xi$ and $\eta_0$ since $\eta_1$ and $\eta_0$ are related by (9). Thus the minimum condition for $M$ becomes

$$\partial M/\partial\xi = 0 \quad \text{or} \quad \xi^2(\eta_0^2 + \eta_1^2 + \eta_0\eta_1) = 3\gamma(\eta_1 - \eta_0) \qquad (11)$$

and

$$\partial M/\partial\eta_0 = 0 \quad \text{or} \quad \xi^2(\eta_0^2 + \eta_1^2 + 4\eta_0\eta_1) = 3\gamma(\eta_1 - \eta_0) \qquad (12)$$

These two relations (11) and (12) with the condition (9) give the critical values of $\xi$ and $\eta_i$. If the width or the depth of the upper layer is specified, (12) or (11) separately leads to critical depth or width with (9). In general cases, (11) and (12) should be satisfied simultaneously. Therefore we have $\eta_0\eta_1 = 0$, which gives $\eta_0 = 0$ with the condition (9). Thus in critical regimes the lower layer comes up to the surface at the left edge as often observed in the Gulf Stream and the Kuroshio (ICHIYE, 1956). The critical width $B_c$ is given by

$$B_c = (3g' D_s)^{\frac{1}{2}} f^{-1} \quad \text{or} \quad \xi_c = B_c/D_s = (3\gamma)^{\frac{1}{2}} \qquad (13)$$

The equation (11) is expressed by $\xi_c$ as

$$\xi_c^2 = \xi^2(\eta_0 + \eta_1)(1 + 2\eta_0^2 + \eta_0\eta_1)$$

This gives $\xi_c \geqslant \xi$, indicating that $\xi_c$ is the maximum of the critical $\xi$ under the condition that the depth of the current is constant. Similarly, when $\xi_c$ is put into (12), it is concluded that the given width $\xi$ must be smaller than $\xi_c$ in order that real critical value of $\eta_0$ or $\eta_1$ may exist. Using numerical constants such as

$$\Delta\varrho = 2 \times 10^{-3}, \; f = 10^{-4}\,(\text{sec}^{-1}) \text{ and } D_s = 800\,(m) \qquad (14)$$

which correspond to the volume transport of $64 \times 10^6$ $M^3$/sec. about equal to that of the Gulf Stream, the critical width becomes 70 km. This width almost agrees with that of the swiftest part of the Gulf Stream and the Kuroshio as revealed by recent surveys (VON ARX, 1952; ICHIYE, 1956).

### 3. Variational problem for the minimum momentum condition

In the preceding section a constant velocity is assumed in a belt of width $B$. Generally the velocity is a function of $x$. Introducing non-dimensional variables $\chi = x/D_s$ and $\eta = D/D_s$, the minimal condition for $M$ requires the integral

$$I = \int_0^{\xi} \{\eta^2 + 2\gamma\eta(d\eta/dx)^2\}\, d\chi \qquad (15)$$

to be minimal by taking a suitable function $\eta(\chi)$ and upper boundary $\xi$ under the condition (9). In (15) $\xi$ and $\gamma$ are as defined in Section 2.

In order to avoid the bothersome variability of $\xi$, the funtion $\eta$ and an argument $\chi$ are interchanged. The equation (15) becomes

$$I = \int_{\eta_0}^{\eta_1} \eta\chi' \{\eta + 2(\chi')^{-2}\}\, d\eta \qquad (16)$$

in which $\chi' = d\chi/d\eta$. The variation of the integral (16) is given by

$$\delta I = \int_{\eta_0}^{\eta_1} [F]_\eta\, \delta\chi\, d\eta + F_{\chi'}\, \delta\eta \Big|_{\eta_0}^{\eta_1} \qquad (17)$$

(COURANT and HILBERT, 1953), in which $F = F(\eta, \chi')$ is the integrand of the equation (16), $F_{\chi'}$ is a partial derivative of $F$ with $\chi'$ and $[F]_\eta$ is an Euler's differential form. The condition that $\delta I = 0$ leads to

$$[F]_\eta = 0 \quad \text{or} \quad F_{\chi'} = \eta^2 - 2\gamma\eta(\chi')^{-2} = c \qquad (18)$$

Further the second term of the right side of (17) must vanish and this boundary condition gives

$$c(\delta\eta_1 - \delta\eta_0) = c(\eta_0\eta_1^{-1} - 1)\,\delta\eta_0 = 0 \qquad (19)$$

using the condition (9) and the relation (18). When the boundary value of $\eta_0$ (or $\eta_1$) is specified, $\delta\eta_0 = 0$ and any value of $c$ is possible. However, when it is not so, $c$ must vanish. Thus integration of (18) with $\eta$ gives

$$\chi = 2(2\gamma)^{\frac{1}{2}}(\eta^{\frac{1}{2}} - \eta_0^{\frac{1}{2}}) \qquad (20)$$

The width of the current $\xi$ is expressed by

$$\xi = 2(2\gamma)^{\frac{1}{2}}(\eta_1^{\frac{1}{2}} - \eta_0^{\frac{1}{2}}) \qquad (21)$$

The equation (20) indicates that the thickness $D$ is represented by a quadratic function of the distance $x$. When the relation (9) is substituted, the critical width $\xi_c$ becomes a function of $\eta_0$

only. It reaches maximum $2(2\gamma)^{\frac{1}{2}}$ for $\eta_0=0$, giving

$$B_c = 2(2g' D_s)^{\frac{1}{2}} f^{-1} \tag{22}$$

This is equal to 113 km for the numerical constants of (14) and is larger than that of the previous section.

In more restricted case of fixed $\eta_0$ the critical depth distribution is given by

$$\chi = (2\gamma)^{\frac{1}{2}} \int_{\eta_0}^{\eta} \{\eta/(\eta^2 - c)\}^{\frac{1}{2}} d\eta \tag{23}$$

The critical width $\xi$ is obtained by putting $\eta=\eta_1$ in this equation. It should be noted that $\xi$ in this case reaches maximum for $c=\eta_0{}^2$ because for any $c$ which gives positive $\eta^2 - c$ we have

$$\eta^2 - c \geqslant \eta^2 - \eta_0^2$$

When a function $\eta^{\mathrm{I}}(\chi)$ and $\eta^{\mathrm{II}}(\chi)$ are assumed to correspond to different integral constant $c^{\mathrm{I}}$ and $c^{\mathrm{II}}$ in the integral (23), $\xi^{\mathrm{I}}$ and $\xi^{\mathrm{II}}$ are determined from (23) for $c^{\mathrm{I}}$ and $c^{\mathrm{II}}$. Further assume that $\xi^{\mathrm{I}} > \xi^{\mathrm{II}}$. The variational problem which requires making the integral (15) minimal in an interval $(0, \xi^{\mathrm{I}})$ has a solution $\eta^{\mathrm{I}}(\chi)$. On the other hand, the function $\eta^{\mathrm{II}}(\chi)$ satisfies the Euler's equation only in the partial interval $(0, \xi^{\mathrm{II}})$. Then the integral I given by $\eta^{\mathrm{I}}(\chi)$ is smaller than that of $\eta^{\mathrm{II}}(\chi)$ in the range $(0, \xi^{\mathrm{I}})$ (Courant and Hilbert, pp. 199—200). Therefore the function corresponding to $c=\eta_0^2$ is the real solution of the variational problem, because this gives the maximum value of $\xi$.

The integral $\chi(\eta)$ of (23) with $c=\eta_0{}^2$ is expressed by the elliptic integral of the first and second kind, $F(\varphi, k)$ and $E(\varphi, k)$ (Byrd and Friedman, 1954, p. 82) as

$$\chi(2\gamma)^{-\frac{1}{2}} = J(\eta) = (2\eta_0)^{\frac{1}{2}} \{F(\varphi, k) - 2E(\varphi, k) + 2\tan\varphi(1 - k^2\sin^2\varphi)^{\frac{1}{2}}\} \tag{24}$$

in which

$$k^2 = \frac{1}{2}, \quad \varphi = \sin^{-1}(\eta - \eta_0/\eta)^{\frac{1}{2}}$$

Corresponding critical width is given by

$$B_c = (2g' D_s)^{\frac{1}{2}} f^{-1} J(\eta_1) \tag{25}$$

in which $(2g' D_s)^{\frac{1}{2}} f^{-1}$ is equal to 57 km for the constants of (14).

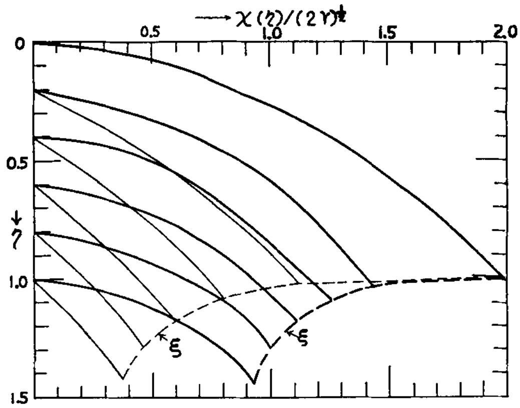


Fig. 2. Curves of $\chi(\eta)(2\gamma)^{-\frac{1}{2}}$ for different values of $\eta_0$ (in full lines) and corresponding $\xi(2\gamma)^{-\frac{1}{2}}$ (in broken lines) obtained by the momentum principle. (Heavy lines are for the function $J(\eta)$ and corresponding $J(\eta_1)$. Fine lines are for the function of (20) and $\xi(2\gamma)^{-\frac{1}{2}}$ of (21).

The curves of $J(\eta)$ are plotted for different values of $\eta_0$ in Fig. 2. The curve for $\eta_0=0$ corresponds to the solution (20). The values of $J(\eta_1)$ are proportional to the critical width $B_c$ as shown in (25) and the curve for $\eta_0=0$ gives the maximum value 2 of $J(\eta_1)$ as expected. In this figure the curves of $\chi(2\gamma)^{-\frac{1}{2}}$ and $\xi(2\gamma)^{-\frac{1}{2}}$ from (20) and (21) are also indicated. When $\eta_0=0$, $J(\eta)$ becomes equal to the curve of (20). When $\eta_0$ is finite, $J(\eta)$ is larger than the corresponding function of (20) for the same $\eta$, and thus the critical width $\xi$ from (24) is larger than that of (21). The curves $J(\eta)$ are interpreted as the thickness of the upper layer which is attained by the current after the loss of momentum under the condition that the depth of edge of the current $\eta_0$ (or $\eta_1$) is maintained constant by some mechanism. On the other hand, all curves from (20) except one for $\eta_0=0$ do not represent the critical regime which gives the minimum momentum transport. Thus even if the current system eventually has a thickness distribution expressed by one of these curves, it will change further downstream through the loss of momentum and tend to approach the regime corresponding to the curve for $\eta_0=0$.

## 4. Energy principle

According to Craya (1951) momentum principle and energy principle lead to a different critical regime for a stratified current. It is necessary to compare the results obtained from both principles in the present model. The energy flux per unit width $\varepsilon$ is defined by Craya (1951) as

$$\varepsilon = p - \varrho g z + \frac{1}{2}\varrho u^2$$

in which $p$ and $u$ are pressure and velocity in the general sense. This is given in the two-layered system by

$$\varepsilon = g\varrho\Delta\varrho(D+\zeta)/\varrho_0 + \frac{1}{2}\varrho v^2 \approx g'\varrho D + \frac{1}{2}\varrho v^2 \quad (26)$$

in which $v$ is the velocity at the upper layer as defined in section 2.

It is the total energy flux defined like $M$ that should become minimum. In a model of uniform current within a belt $x = 0$ to $B$, computation similar to (5) gives

$$2E = 2\int_0^B \varepsilon dx = g'\varrho(D_0 + D_1)B + \varrho(g'/f)^2(\Delta D)^2/B \quad (27)$$

By introducing non-dimensional parameters, $E$ is expressed by

$$E = \frac{1}{2}\varrho g' D_s^2 \xi\{\eta_0 + \eta_1 + \gamma(\eta_1 - \eta_0)^2 \xi^{-2}\} \quad (28)$$

The minimum conditions about two variables $\xi$ and $\eta_0$ lead to

$$\partial E/\partial\xi = 0 \quad \text{or} \quad (\eta_0 + \eta_1)\xi^2 = \gamma(\eta_1 - \eta_0)^2 \quad (29)$$

$$\partial E/\partial\eta_0 = 0 \quad \text{or} \quad (\eta_0 + \eta_1)\xi^2 = 2\gamma(\eta_1 - \eta_0)^2 \quad (30)$$

Therefore there is no critical regime which satisfies both of these conditions. However, the relation (29) gives the critical width $\xi_c$ for a fixed $\eta_0$ such as

$$\xi_c = (\eta_1 - \eta_0)\{\gamma/(\eta_1 + \eta_0)\}^{\frac{1}{2}}$$

This attains maximum $\gamma^{\frac{1}{2}}$ for $\eta_0 = 0$, giving an actual critical width $B_c = 40$ (km) which is smaller than that derived from the momentum principle. Further the equation (30) gives the critical depth $\eta_0$ or $\eta_1$ under the constant value of $\xi$.

In case of variable $v$ corresponding to Section 3 the total energy $E$ is given by

$$E = g'\varrho\int_0^B D dx + \frac{1}{2}\varrho(g'/f)^2\int_0^B(\partial D/\partial x)^2 dx \quad (31)$$

Introducing non-dimensional parameters, this quantity is proportional to

$$L = \int_0^\xi\{2\eta + \gamma(d\eta/d\chi)^2\}d\chi \quad (32)$$

Interchange of $\eta$ and $\chi$ leads to

$$L = \int_{\eta_0}^{\eta}\frac{d\chi}{d\eta}\left\{2\eta + \gamma\left(\frac{d\chi}{d\eta}\right)^{-2}\right\}d\eta \quad (33)$$

The variational problem becomes $\delta L = 0$ under the condition $\eta_1^2 - \eta_0^2 = 1$. The first integral of the Euler's equation is given by

$$2\eta - \gamma(d\chi/d\eta)^{-2} = d \quad (34)$$

with an integral constant $d$.

In a case of unfixed boundary values $\eta_0$ and $\eta_1$, $d = 0$ and we have

$$\chi(\eta) = (2\gamma)^{\frac{1}{2}}(\eta^{\frac{1}{2}} - \eta_0^{\frac{1}{2}}) \quad (35)$$

This equation indicates that the thickness $D$ is also a quadratic function of $x$, but it gives a half of the critical width given by (20).

In a case of fixed $\eta_0$, $\xi = \chi(\eta_1)$ becomes maximum for $d = 2\eta_0$. Thus the criterion about the comparison of functions satisfying the Euler's equation in different ranges of $(0, \xi)$ leads to the critical depth distribution

$$\chi(\eta) = (2\gamma)^{\frac{1}{2}}(\eta - \eta_0)^{\frac{1}{2}} \quad (36)$$

corresponding to (24). This equation also gives the maximum critical width $\xi_c$ for $\eta_0 = 0$ as in Section 3. The curves of $\chi\,(2\gamma)^{-\frac{1}{2}}$ from (35) and (36) and corresponding $\xi\,(2\gamma)^{-\frac{1}{2}}$ are shown in Fig. 3 in the same way as in Fig. 2.

Comparing these results with those of Section 3, it is found that the critical widths derived by the energy principle are smaller than those derived by the momentum principle. However, critical regimes themselves are highly idealized state for the ocean currents as pointed out by CRAYA (1951). Therefore it

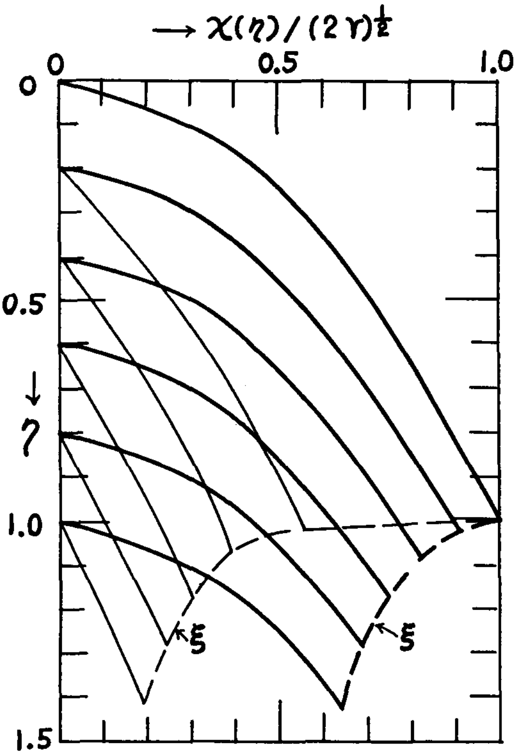


Fig. 3. Curves of $\chi(\eta)\ (2\gamma)^{-\frac{1}{2}}$ (in full lines) and $\xi\ (2\gamma)^{-\frac{1}{2}}$ (in broken lines) obtained by the energy principle. (Heavy lines are for the function of (36) and fine lines are for (35).)

is almost impracticable to compare them with actual ocean currents.

### 5. A system with continuous density distribution

The minimum momentum principles can be applied to a two-layered ocean with continuous density distribution. This is more realistic than the previous models. The most appropriate distribution in middle latitudes is given by STOMMEL (1954) as

$$\varrho = \varrho_s \text{ (const.)}: \quad \text{for } D > z > 0$$

$$\varrho_0 = \varrho_b - \delta\varrho \exp\{\nu(D-z)\}: \quad \text{for } H > z > D \tag{37}$$

in which $\delta\varrho = \varrho_b - \varrho_s$ and $\nu$ is a constant equal to $3 \times 10^{-5}$ ($\text{cm}^{-1}$). Since $H$ is assumed to be very large, we can put

$$\exp\{\nu(D-H)\} \approx 0$$

The pressure at the lower layer $p_0$ is given by

$$\begin{aligned} p_0/g &= \varrho_s(D+\zeta) + \varrho_b(z-D) + \\ &+ \delta\varrho\,[\exp\{\nu(D-z)\} - 1]\,\nu^{-1} = \\ &= p_b/g - \varrho_b(H-z) + \delta\varrho \exp\{\nu(D-z)\}\,\nu^{-1} \end{aligned} \tag{38}$$

in which $p_b$ is the bottom pressure assumed constant and is equal to

$$p_b = \varrho_s(D+\zeta) + \varrho_b(H-D) - \delta\varrho\nu^{-1}$$

The integral of pressure from the surface to the bottom is given by

$$\begin{aligned} \int_0^D p\,dz + \int_D^H p_0\,dz &= \frac{1}{2} g\varrho_s\,\delta\varrho D^2/\varrho_b + \frac{1}{2} p_b^2/\varrho_b g + \\ &+ g\delta\varrho\nu^{-2}(1 - \delta\varrho/2\varrho_b) \end{aligned} \tag{39}$$

in which we put $\zeta \ll D$ as in Section 2.

Owing to the condition that there is no current at the bottom or that $p_b$ is constant, we have

$$\varrho_s(\partial\zeta/\partial x) = \delta\varrho(\partial D/\partial x)$$

The geostrophic relation with this equation gives the velocity at the upper and lower layer $v$ and $v_0$ such as

$$\left.\begin{aligned} v &= (g''/f)(\partial D/\partial x), \\ v_0 &= (g''/f)(\partial D/\partial x)\exp\{\nu(D-z)\} \end{aligned}\right\} \tag{40}$$

in which $g'' = g\delta\varrho\,/\,\varrho_s$. By the approximation similar to the Section 2 we can put $g'' \approx g\delta\varrho/\varrho_b$ in the equation (39). Further, with the approximation $\zeta + D \approx D$, the integration of momentum transport from the surface to the bottom gives

$$\begin{aligned} &\int_0^D \varrho v^2\,dz + \int_D^H \varrho_0 v_0^2\,dz = \\ &= \varrho(g''/f)^2(\partial D/\partial\chi)^2\left(D + \frac{1}{2}\nu\right) \end{aligned} \tag{41}$$

This relation is different from the one in the previous model only by the term $(2\nu)^{-1}$ which is due to the finite velocity at the lower layer.

Since the first term of the right hand side of (39) is variable, the thrust becomes the sum of this term and (41). The total thrust expressed by non-dimensional parameters is then proportional to

$$I = \int_0^\xi \{\eta^2 + 2\gamma'(\eta+\alpha)(d\eta/d\chi)^2\}\,d\chi \tag{42}$$

in which $\alpha = (2\nu\ D_s)^{-1}$ and $\gamma' = g''/f^2 D_s$.

The total volume transport corresponding to (7) is given by

$$V = \frac{1}{2} g'' (D_1 - D_0)(D_1 + D_0 + 2/\nu)/f = \\ = \frac{1}{2} g'' D_s^2 / f \qquad (43)$$

or in non-dimensional form

$$\eta_1^2 - \eta_0^2 + 4\alpha(\eta_1 - \eta_0) = 1 \qquad (44)$$

The variation problem which makes $\delta I = 0$ for the integral of (42) under the condition (44) is solved by the method similiar to the previous cases. In the case of non-fixed $\eta_0$, the solution of this variation problem is given by

$$\chi(2\gamma')^{-\frac{1}{2}} = \int_{\eta_0}^{\eta} (\eta + \alpha)^{\frac{1}{2}} \eta^{-1} d\eta = \\ = 2\{(\eta + \alpha)^{\frac{1}{2}} - (\eta_0 + \alpha)^{\frac{1}{2}}\} + \\ + \alpha^{\frac{1}{2}} [\log\{(\eta + \alpha)^{\frac{1}{2}} - \alpha^{\frac{1}{2}}\} - \log\{(\eta_0 + \alpha)^{\frac{1}{2}} - \alpha^{\frac{1}{2}}\} - \\ - \log\{(\eta + \alpha)^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\} + \log\{(\eta_0 + \alpha)^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\}] \qquad (45)$$

When $\eta_0$ or $\eta_1$ is fixed, the same discussion about the integral constant of the Euler's integral as in Section 3 and 4 leads to the solution such as

$$\chi(2\gamma')^{-\frac{1}{2}} = K(\eta) = \int_{\eta_0}^{\eta} (\eta + \alpha)(\eta^2 - \eta_0^2)^{-\frac{1}{2}} d\eta \qquad (46)$$

The critical width $\xi_c$ can be obtained by taking $\eta = \eta_1$ in this equation. The function $K(\eta)$ is also expressed by elliptic integrals (BYRD and FRIEDMAN, 1954, p. 82) as

(For $\eta_0 < \alpha$)

$$K(\eta) = 2(\eta_0 + \alpha)^{\frac{1}{2}} \{F(\varphi, k) - E(\varphi, k) + \\ + \tan\varphi(1 - k^2 \sin^2\varphi)^{\frac{1}{2}}\} \qquad (47)$$

where

$$k^2 = (\alpha - \eta_0)/(\alpha + \eta_0), \ \sin\varphi = \\ \{(\eta - \eta_0)/(\eta + \eta_0)\}^{\frac{1}{2}}$$

(For $\eta_0 > \alpha$)

$$K(\eta) = (\eta_0 + \alpha)(2/\eta_0)^{\frac{1}{2}} (k')^{-2} \{(k')^2 F(\varphi, k) - \\ - E(\varphi, k) + \tan\varphi(1 - k^2 \sin^2\varphi)^{\frac{1}{2}}\} \qquad (48)$$

where

$$k^2 = (\eta_0 - \alpha)/2\eta_0, \quad (k')^2 = 1 - k^2 \\ \sin\varphi = \{(\eta - \eta_0)/(\eta + \alpha)\}^{\frac{1}{2}}$$

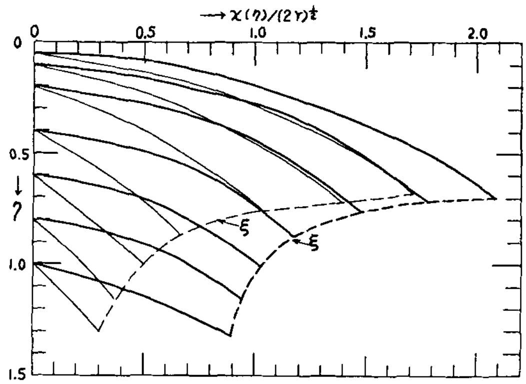


Fig. 4. Curves of $\chi(\eta)$ $(2\gamma')^{-\frac{1}{2}}$ for different $\eta_0$ (in full lines) and $\xi$ $(2\gamma')^{-\frac{1}{2}}$ (in broken lines) for the system with continuous density distribution. (Heavy lines are for $K$ $(\eta)$ and $K$ $(\eta_1)$ and fine lines are for the function of (45).)

The function $K(\eta)$ and factor $K(\eta_1)$ of the critical width $\xi$ are plotted in Fig. 4. It is seen that for the same value of $\eta$ and $\eta_0$ $K(\eta)$ is larger than $J$ $(\eta)$. However, the values of $K$ $(\eta_1)$ are smaller than $J$ $(\eta_1)$, except for the curves corresponding to $\eta_1$ less than 0.1, since values of $\eta_1$ determined from (44) are smaller than those from (9). As expected from the equation (46) the integral $K(\eta)$ becomes infinite for $\eta_0 = 0$. In this case there is no critical width and only the critical depth distribution can be obtained by taking suitable integral constant, say $b$, instead of $\eta_0^2$ in the integrand of (46) for a given width $\xi$. This indetermination of critical width for a limiting case $\eta_0 = 0$ is due to the finite velocity at the lower layer. When $\eta_0 \approx 0$ the slope of the upper layer $d\eta/d\chi$ near $\eta = \eta_0$ has to be proportional to $\eta_0$ in the present model, whereas it is proportional to $\eta_0^{\frac{1}{2}}$ in the model of motionless lower layer. This requirement leads to an infinite value of $\chi$, since

$$d\chi/d\eta \sim \eta_0^{-1} \text{ or } \chi \sim \log \eta_0$$

In Fig. 4 the curves expressed by the equation (45) are also shown. The values of $K(\eta)$ and $K(\eta_1)$ for the same $\eta_0$ and $\eta$ are larger than those of the integral of (45). This integral also becomes infinite for $\eta_0 = 0$. However, the critical depth distribution can be determined even for $\eta_0 = 0$. Since $\chi$ becomes infinite very slowly near $\eta_0 = 0$ according to its asymptotic equation $\chi \sim \log \eta_0$, the critical width can be determined for $\eta_0$ very close to 0.

## 6. Momentum principle applied to a multiple current system

It is assumed that the energy or momentum is dissipated from the current in the present model. However, there is an evidence that the ocean current also receives momentum or energy from the surrounding water (ICHIYE, 1957). Therefore the results of the present theory cannot be applied directly to the actual condition of ocean currents.

For instance, it is shown by FUGLISTER (1951, 1956) that the Gulf Stream and the Kuroshio have a multiple structure consisting of a series of overlapping currents after it leaves the continent. In such a system the theory has to be applied to each branch of the multiple current. Then it is understood that each branch has a critical width and depth distribution predicted by the theory.

On the other hand, it was observed often in the Kuroshio that the split of the current into two branches was caused by the intense dissipation of the momentum (or energy) of the current due to storms traveling along the current (ICHIYE, 1954, 1955). Further in some cases it was confirmed that the total energy of the current decreased after the split but its volume transport was almost unchanged (ICHIYE, 1958). Such examples suggest to us that the momentum or energy principle may be applied to the possibility of occurrence of the multiple structure.

In the application of the momentum principle, the distribution of the thickness of the upper layer is assumed to be given by

$$D = D_0 + \Delta D \cdot x/B + D_n \sin(n\pi x/B) \qquad (49)$$

in which $0 \leqslant x \leqslant B$.
When $n$ is an integer, this distribution yields the same volume transport for fixed values of $D_0$ and $D$ owing to the equation (7). Yet the velocity distribution has a multiple structure for $n$ larger than 2. Then the total momentum for different $n$ is computed and is compared with each other. The argument similar to the previous sections leads to the conclusion that the most probable mode of the thickness distribution is realized by the one which has the minimum value of momentum transport.

The momentum transport for each mode of $n$ can be computed by elementary calculation similar to (8). When $M_0$ means the value corresponding to $n = 0$ or that given by (8), the value of $M_n$ for general $n$ can be expressed by

$$M_n - M_0 = A\left\{\frac{1}{4}(r_1 n\pi)^2(1 + 2r_0) + \right.$$
$$+ \phi(n)\left(\frac{1}{3}r_1^3 n\pi - r_1/n\pi\right) +$$
$$+ (\Delta D/B)^2\left[\frac{1}{2}r_1^2 + 2r_1 r_0 \phi(n)/n\pi - \right.$$
$$\left.\left. - 2r_1(-1)^n/n\pi\right]\right\} \qquad (50)$$

in which

$$A = 2g'(\Delta D)^3 f^{-2} B^{-4}, \quad r_1 = D_n/\Delta D$$
$$r_0 = D_0/\Delta D$$
$$\phi(n) = 0 \quad (\text{for even } n)$$
$$\phi(n) = 2 \quad (\text{for odd } n) \qquad (51)$$

Since only the relative magnitude of $M_n$ is important, the values of $(M_n - M_0)A^{-1}$ are shown for different $n$ in the next table, in which the following numerical constants are used

$$B = 500\,(\text{km}), \quad D_0 = 0{,}1\,(\text{km}), \quad \Delta D = 1\,(\text{km})$$
$$D_n = \Delta D \quad \text{or} \quad -\Delta D \qquad (52)$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| I | 5.06 | —1.56 | 2.16 | —0.21 | 2.01 | 0.72 | 2.45 | 1.71 |
| II | —4.38 | 2.42 | —1.00 | 1.78 | 0.00 | 2.04 | 1.08 | 2.70 |

The first line corresponds to the case $D_n = \triangle D$ and the second to $D_n = -\triangle D$. In the first case the minimum is attained for $n = 2$ and $M_n$ for even $n$ is smaller than for odd $n$. On the other hand, in the second case $M_1$ is minimum and $M_n$ for odd $n$ is larger than for even $n$. The thickness for $n = 1$ in II gives the minimum of all $M_n$, because this is very similar to that of the critical regime determined in Section 3. The modes $n = 2$ in I and $n = 3$ in II yielding the velocity distribution consisting of two jets give the minimum $M_n$ next to this. Therefore it is concluded that the double jets are the most probable state except a single jet corresponding to a critical regime. The result obtained from the energy principle is almost similar. Thus the split of the current into double jets can be attributed to the condition that they represent the second critical regime which the current reaches after the loss of momentum or energy.

## 7. Conclusion

The principle of minimum momentum or energy applied to the geostrophic current in a two-layered ocean gives the critical regime which will be attained by the current as it loses momentum or energy but conserves volume transport. The critical width corresponding to this regime is less than 100 km. This effect explains qualitatively the fact that the Gulf Stream or the Kuroshio remains narrow and well-defined far from the coast. Further the critical depth distribution of the current determined from the momentum principle predicts the vanishment of the depth at the left edge, being in good accordance with the observed structure of ocean currents.

The multiple structure of ocean currents often observed off the coast is also principally attributed to the second critical regime which is represented by double jets, because this state has the second minimum energy or momentum transport. However, the present theory indicates only the probable state of the current. Thus, for instance, it gives no information about the condition for occurrence of double jets, which may be treated from a different approach about the stability of the current.

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