Further Examples of Stationary Planetary Flow Patterns in Bounded Basins[1]

By ALAN J. FALLER, Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

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Abstract

Laboratory experiments which supplement those presented by STOMMEL, ARONS and FALLER (1958) are described. Extension of the analysis to include the effects of bottom friction shows *that the characteristic spreading of zonal currents, apparent in the original experiments, is due* to bottom friction combined with the radial depth variation. The experiments confirm the theoretical prediction of an intense "recirculation" from the western boundary (similar to the "wind-spun vortex" of MUNK, 1950) when a localized interior source or sink is present. Application of the basic concepts of the earlier paper to the geometrically more complicated case with a partial radial barrier gave good agreement between theory and experiment, but the limiting effects of bottom friction were made evident in other examples.

An eastern boundary current was found to occur in a situation geometrically similar to the intrusion of Mediterranean water into the Atlantic Ocean. Simple transformations illustrate the application of the equations developed in polar coordinates to the flow of a uniform fluid on a sphere, and examples of "recirculation" and the spread of zonal flow are worked out for a barotropic ocean of uniform depth.

1. Introduction

A recent paper by STOMMEL, ARONS, and FALLER (1958) hereafter referred to as (A) set forth qualitative and quantitative *a priori* predictions of the circulation in a bounded basin in which the source of energy for the flow was the pressure head set up by the introduction and/or extraction of mass at various locations in the fluid. Specifically the analysis pertained to the circulation of a homogeneous layer of fluid in a pie-shaped basin rotating about the apex of the sector. Because of the rotational constraint the flow could be considered geostrophic and two-dimensional except for certain boundary regions in which the viscous and/or inertia forces could produce appreciable departures from geostrophic flow. The predictions of the simple theory were borne out by subsequent laboratory experiments.

This paper presents the results of some additional related experiments and expands upon details of the flow patterns of the earlier work. As in (A) the experiments are restricted to depths of fluid and rotation rates such that the flow is primarily geostrophic, but in addition to departures from geostrophy near the side boundaries the specific effects of the bottom frictional boundary layer are considered. It is shown that bottom friction is a most important consideration for situations where there are no unbroken radial barriers, and indeed the effects of bottom friction may be observed and are of some significance in all of the experiments.

The analysis described in (A) is extended here to somewhat more complicated regimes of ge-

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ophysical interest, namely the circulation with a partial radial barrier and the "recirculation" associated with localized interior sources or sinks of fluid.

An unanticipated development was the occurrence of *eastern* boundary currents in certain circumstances. It was asserted in (A) that only *western* boundary currents would occur and this assertion seemed to be a justified one on the basis of the original experiments. It now appears that eastern boundary currents of inertial character are possible when fluid is introduced through a slot in the eastern boundary in such a manner that viscosity may generate negative absolute vorticity.

Finally the theory for the flat rotating tank is extended to a spherical shell of fluid. The recirculation phenomenon and the spread of zonal flow are computed as examples of what might be expected for barotropic flow in an ocean basin of constant depth.

2. Equations for the Interior Circulation

The polar equations of motion and the continuity equation for a uniform fluid with an arbitrary distribution of sources (or sinks) are (DRYDEN, MURNAGHAN, and BATEMAN, 1956):

$$\frac{dv_r}{dt} - \frac{v_\Theta^2}{r} - \underline{2\Omega v_\Theta} = \underline{-\frac{1}{\varrho}\frac{\partial p}{\partial r}} +$$

$$+ \nu \left\{ \underline{\frac{\partial^2 v_r}{\partial z^2}} + \frac{1}{r}\frac{\partial^2 r v_r}{\partial r^2} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \Theta^2} - \frac{2}{r^2}\frac{\partial v_\Theta}{\partial \Theta} \right\} \quad (1)$$

$$\frac{dv_\Theta}{dt} + \frac{v_r v_\Theta}{r} + \underline{2\Omega v_r} = \underline{-\frac{1}{\varrho r}\frac{\partial p}{\partial \Theta}} +$$

$$+ \nu \left\{ \frac{\partial^2 v_\Theta}{\partial z^2} + \frac{1}{r}\frac{\partial^2 r v_\Theta}{\partial r^2} - \frac{v_\Theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\Theta}{\partial \Theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \Theta} \right\} \quad (2)$$

$$\frac{dw}{dt} = \underline{-\frac{1}{\varrho}\frac{\partial p}{\partial z}} - g +$$

$$+ \nu \left\{ \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \Theta^2} \right\} \quad (3)$$

$$\underline{\frac{\partial w}{\partial z}} + \underline{\frac{1}{r}\frac{\partial r v_r}{\partial r}} + \underline{\frac{1}{r}\frac{\partial v_\Theta}{\partial \Theta}} = P \quad (4)$$

where the r direction is radially outward, Θ is in direction of rotation (counterclockwise) and z is vertically upward. P is a source of fluid per unit of volume. Dimensional arguments may be used to show that for: 1) small flow rates, 2) small horizontal shear, 3) a steady state, and 4) large radius the equations reduce to those terms which are underlined and which were the equations used by EKMAN (1905) in his studies of wind driven ocean currents. Again from dimensional arguments or from the solutions given below one may see that the depth of a viscous boundary layer at a rigid horizontal surface should be characterized by the Ekman depth $D_E = \pi\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}$, so that far from the boundaries ($z \gg D_E$) we may omit the vertical derivatives in 1) and 2) to obtain the geostrophic equations

$$v \equiv v_{\Theta g} = \frac{1}{2\Omega\varrho}\frac{\partial p}{\partial r} \quad (5)$$

$$u \equiv v_{rg} = -\frac{1}{2\Omega\varrho r}\frac{\partial p}{\partial \Theta} \quad (6)$$

$$\frac{\partial r u}{\partial r} + \frac{\partial v}{\partial \Theta} = 0. \quad (7)$$

Therefore for a total depth of fluid $h \gg D_E$ with a free surface the following lower and upper boundary conditions apply:

$$\text{at } z = 0: \quad v_{r_0} = v_{\Theta_0} = w_0 = 0$$

$$\text{at } z = h: \quad v_{\Theta_h} = v$$

$$v_{r_h} = u$$

$$w_h = u\frac{\partial h}{\partial r} + \frac{v}{r}\frac{\partial h}{\partial \Theta} + \frac{\partial h}{\partial t} \quad (8)$$

where $\frac{\partial h}{\partial t}$ is the depth variation associated with a steady rise or fall of the free surface. The depth variation h' due to the steady geostrophic currents need not be considered in the expression for w_h since $\vec{V}_g \cdot \nabla h' = 0$.

Since the boundary layer circulation decreases exponentially with height, we may safely take the upper limit of integration in exponential expressions as $h = \infty$ and so obtain as solutions to 1) and 2):

$$v_r = u - e^{-\pi\frac{z}{D_E}}\left(u \cos \pi\frac{z}{D_E} + v \sin \pi\frac{z}{D_E}\right) \quad (9)$$

$$v_\Theta = v - e^{-\pi\frac{z}{D_E}}\left(u \sin \pi \frac{z}{D_E} - v \cos \pi \frac{z}{D_E}\right) \cdot \quad (10)$$

By substitution of these expressions into equation 4) we find:

$$\frac{\partial w}{\partial z} + e^{-\pi\frac{z}{D_E}} \sin \pi \frac{z}{D_E}\left(\frac{1}{r}\frac{\partial u}{\partial \Theta} - \frac{1}{r}\frac{\partial rv}{\partial r}\right) = P \cdot \quad (11)$$

Vertical integration of (11) from $z=0$ to $z=h(\infty)$ together with the boundary conditions (8) yields the following equation for the geostrophic flow:

$$u\frac{\partial h}{\partial r} + \frac{v}{r}\frac{\partial h}{\partial \Theta} + \frac{\partial h}{\partial t} + \frac{D_E}{2\pi}\left(\frac{\partial u}{r\partial \Theta} - \frac{\partial rv}{r\partial r}\right) = Q \quad (12)$$

where $Q = \int_0^h P dz$ is the source of fluid per unit area. Equations (7) and (12) are sufficient to solve for the geostrophic flow given Q, $\frac{\partial h}{\partial r}$, $\frac{\partial h}{\partial \Theta}$, $\frac{\partial h}{\partial t}$ and suitable lateral boundary conditions. Introduction of a stream function defined by $u = \frac{\partial \psi}{r\partial \Theta}$, $v = -\frac{\partial \psi}{\partial r}$ into (7) and (12) and considering only radial depth variations yields:

$$\frac{\partial h}{\partial r}\frac{1}{r}\frac{\partial \psi}{\partial \Theta} + \frac{D_E}{2\pi}\nabla^2\psi = Q - \frac{\partial h}{\partial t}. \quad (13)$$

Equation (13) is of the same form as that derived by STOMMEL (1948) with the following differences: 1) the radial depth variation enters in place of the latitudinal variation of the Coriolis parameter, 2) the effects of bottom friction have been derived more completely rather than to assume a linear friction law, and 3) no specific functional form for the source term has been considered here, whereas STOMMEL assumed a specific wind stress distribution in order to obtain an analytical solution. The above equations will now be specialized for specific examples of interest.

3. Transport across an Annular Ring

As a first test of the adequacy of the Ekman theory as developed above a circular channel with inner radius 50 cm and outer radius 100 cm was set up on the same rotating apparatus as had been used in the earlier experiments. Provision was made for a source of mass continuously distributed around the outer rim and a sink at the inner wall as indicated in Figure 1. In this manner a steady mass transport (S_0) across the channel was forced, and since it was anticipated that this transport would take place entirely in an Ekman layer, the source and sink were arranged to feed the bottom boundary layer directly. If we assume circular symmetry ($u=0$) and take $Q = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \Theta} = 0$ we immediately obtain from (12) the solution $rv = C$ where the constant C must be determined by the condition on the total mass transport,

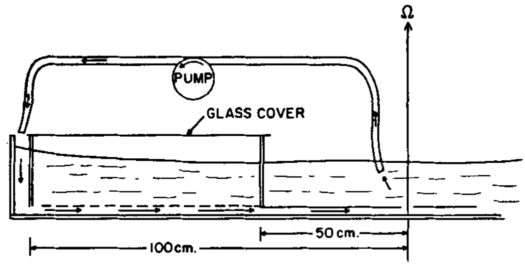

Fig. 1. A schematic diagram of a source-sink experiment in an annular ring with the source continuously distributed around the outer rim and the sink at the inner rim. For relatively slow flow rates the radial mass transport occurs entirely in an Ekman layer.

$$\int_0^{2\pi}\int_0^h r v_r d\Theta\, dz = S_0. \quad (14)$$

Substitution of equation (9) into (14) gives directly:

$$rv = \frac{S_0}{D_E}. \quad (15)$$

The accuracy of this expression, which implies a zonal flow inversely proportional to radius, was examined experimentally with a source strength $S_0 = 8.67$ cm³/sec. The following values of the zonal geostrophic flow were obtained from successive photographs showing the displacements of patches of dye at various radii:

Table 1.

Radius	v	S_p
62.5 cm	.272 cm/sec	7.35 cm³/sec
74	.232	7.31
87	.194	7.45

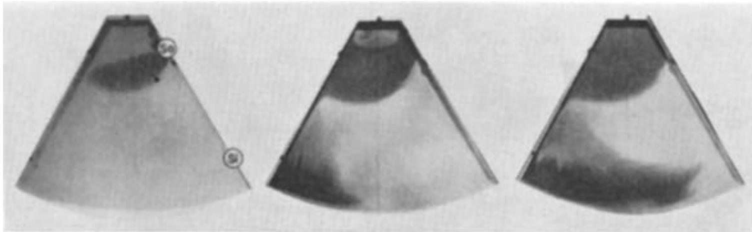

Fig. 2. Reproduced from Stommel, Arons and Faller (1958) to illustrate the spread of zonal flow from the source S_0 as it moved westward and the convergence of the flow as it moved eastward toward the sink S_i.

Values of transport predicted from the zonal flow (S_p) averaged 15 % below the imposed transport $S_0 = 8.67$ cm^3/sec.

Estimates of the neglected terms of the equations of motion show these to be three orders of magnitude lower than those of the Ekman equations, (2) and (3), which fact is indicated directly by the Rossby number, $R_0 = \dfrac{\nu}{\Omega r} \approx .004$. Possible explanations of the discrepancy are: 1) that the bottom boundary surface which was a painted metal surface had some roughness elements which tended to introduce irregular flow and an eddy viscosity above the molecular value $\nu = .01$ at 20° C; 2) that a free surface stress existed because of some properties of the surface film. Other possibilities such as a wind stress or a thermal circulation were eliminated by a sealed glass cover and insulation of the tank. A systematic study of the effects of boundary roughness is planned for the near future, but it is believed that a surface film could easily have caused the observed discrepancy between theory and experiment. Several cases have been observed where contaminants have caused the surface to act as a stationary membrane even in the presence of active (1 cm/sec) circulations within the fluid. Other experiments have shown that flow at the free surface is always somewhat slower than the interior flow. In the present example the Ekman transport produced by a membrane would be radially inward and would augment the transport due to the bottom boundary layer with the result that the required values of ν would be less than with the bottom boundary layer alone.

More extensive experiments of a similar nature have recently been performed by Arons and Bryan at the Institute of Meteorology in Stockholm (personal communication) and by Green and Arons (Green, 1959) at Amherst College.

4. The Spread of Zonal Flow from a Point Source due to Bottom Friction.

Some of the experiments illustrated in (A) and subsequent work have shown a rather peculiar spreading of westward flowing zonal currents. Conversely, currents directed eastward toward a point sink seem to broaden near the western boundary where they originate and then converge smoothly toward the sink. Figure 2 reproduced from (A) illustrates the spread of the westward flowing current from the source at the eastern wall near the apex and an orderly convergence of the eastward flow toward the sink. In Figure 6b the stream from the western boundary to the tip of the partial barrier definitely indicated radial spread near the western boundary and convergence as it moved eastward. Other experiments have shown this phenomenon more strikingly.

We consider the flow from a point source in the eastern wall toward the western boundary where we regard the western boundary current as a sink. The circulation in the immediate vicinity of a source is exceedingly complicated and the analytical solution to be obtained can be valid only some distance from the source itself. Equations (7) and (12) are specialized by letting

$$Q = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \Theta} = 0, \quad \frac{\partial h}{\partial r} = \text{const.},$$

$$\frac{\partial}{\partial y} \equiv \frac{\partial}{r\partial \Theta}, \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial r}.$$

These conditions represent 1) no interior sources or sinks of fluid, 2) a constant radial depth variation, and 3) conversion to rectangular coordinates. Then after cross-differentiation to eliminate u we obtain:

$$\frac{\partial h}{\partial x}\frac{\partial v}{\partial y}+\frac{D_E}{2\pi}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right)=0 \cdot \qquad (19)$$

The term $\frac{\partial^2 v}{\partial y^2}$ will be omitted since it may be shown in terms of the resultant solution that $\frac{\partial^2 v}{\partial y^2}$ is comparable to $\frac{\partial^2 v}{\partial x^2}$ only near the source.

The equation to be considered is therefore:

$$\frac{\partial^2 v}{\partial x^2}+\frac{2\pi}{D_E}\frac{\partial h}{\partial x}\frac{\partial v}{\partial y}=0 \qquad (20)$$

subject to the integral condition that the transport be constant at any value of y,

$$\int_{-\infty}^{+\infty}\int_{0}^{h} - v\,dz\,dx = S_0. \qquad (21)$$

In this integration we take h as independent of x as a first approximation.

Equation (20) is similar to the one-dimensional heat flow equation $\frac{\partial^2 T}{\partial x^2} - K\frac{\partial T}{\partial t} = 0$ where K is the thermometric conductivity. Correspondingly, the integral condition on mass transport (21) is equivalent to supplying a given amount of heat in a pulse at $t=0$ and $x=0$ and thereafter allowing the heat to spread along the rod toward $x=+\infty$ and $x=-\infty$. By analogy, the solution to (20) and (21) is (Carlslaw and Jaeger, 1947):

$$v = \pm \frac{S_0}{h(2\pi)^{\frac{1}{2}}(2Ay)^{\frac{1}{2}}} e^{-\left[\frac{x^2}{2(2Ay)}\right]} \qquad (22)$$

where $A = -\frac{D_E}{2\pi\frac{\partial h}{\partial x}}$. Inasmuch as A is negative, this solution has meaning only for negative values of y although v may be positive or negative corresponding to a sink or a source respectively at $x=0$, $y=0$. Equation (22) represents a Gaussian distribution of v in x for any distance y from the source and with a standard deviation $\sigma_x=(2Ay)^{\frac{1}{2}}$.

Figure 3 illustrates the derived distribution of zonal and radial flow, of Ekman flow, and the vertical motions for a case with $A=-1$.

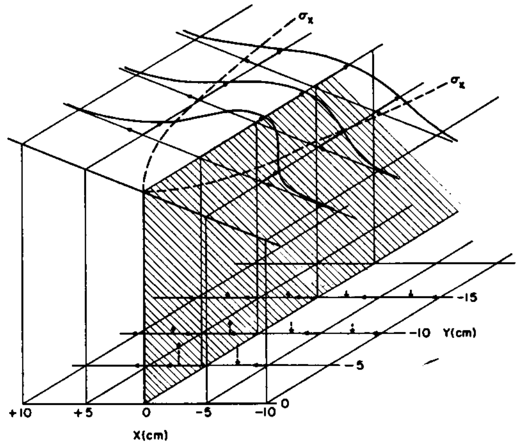

Fig. 3. The theoretical circulation associated with the spread of a westward flowing current from a source at $x=0$, $y=0$ for the value of the parameter $A=-1$.

For negative values of x there exists divergence in the Ekman layer and a loss of mass to the fluid directly above, so that in accordance with the depth variation the interior geostrophic current must have a component toward negative x. The opposite is true for positive values of x, and as a result there is a systematic spreading of the flow. Along the axis of the stream there is clearly a net transport of mass toward positive x by the Ekman flow. Qualitatively this mass is absorbed on the left hand side of the current which is spreading to greater depths and therefore progressively transporting a greater fraction of the mass than the right hand side which is spreading to shallower depths. However, this balance is not quantitatively correct because of the approximation $h=$const. in equation (21) which led to a symmetrical solution. Inspection of $\frac{\partial^2 v}{\partial y^2}$ in comparison with $\frac{\partial^2 v}{\partial x^2}$ indicates that inclusion of this term would tend to retard the divergence of the stream although the effect is already small 10 cm from the source for $A=-1$ (an average value for the experiments).

Quantitative experimental data to support these conclusions are not yet available, but we may estimate the spread of the westward flowing current in Figure 2 and compare this with the theory. The equation of the free surface is:

$$h = h_0\left(1 + l\frac{r^2}{a^2}\right) \qquad (23)$$

Fig. 4. Photographs at 20, 80 and 136 minutes after the introduction of dye showing the existence of both eastern and western boundary currents for the case where the source water was injected through a slot in the eastern wall.

where $l = \frac{\Omega^2 a^2}{2gh_0}$ and for the experimental values $h_0 = 5$ cm, $l = 1$, $a = 100$ cm, $\Omega = 1$ sec^{-1}, and $\nu = .01$ cm^2 sec^{-1} it is found that $A = -\frac{50}{r}$. The radius of the source in Figure 2 is approximately $r = 20$ cm which gives $A = -2.5$ and $\sigma_x = (5y)^{\frac{1}{2}}$. If we consider that most of the dyed source water will be contained within the region $\pm 2\ \sigma_x$ the rimward limit of the dye outline is in quite good agreement for various distances y from the eastern wall. Dye fronts are not adequate for quantitative comparison for several reasons including the fact that the cross stream Ekman flow causes the dye to be systematically displaced toward the left of the axis of the flow.

5. Eastern Boundary Currents

In (A) it was asserted as one of the essential elements which defined the regime to be considered that departures from geostrophic flow were to be permitted only at western boundaries where narrow and rapidly flowing currents could occur. The function of these currents was to preserve continuity of mass in the basin as a whole by providing means for radial transport but the dynamics of the boundary currents were not explicitly considered. In nearly all experiments the above assertion was found to be a justified one, but in one particular situation a pronounced *eastern* boundary current was observed.

Figure 4 is a sequence of photographs at 20, 80, and 136 minutes after dye was introduced into the source water, there being a source (S_0) through a slot in the eastern boundary near the rim and a sink (S_i) at the apex of the sector. As in the previous experiments, the radius was $a = 100$ cm, rotation rate $\Omega = 1.05$ sec^{-1}, the mean depth $\bar{h} = 8$ cm, and the ratio of the depth at the rim to that at the center h_a/h_0 was 2. The volume rate of flow was $S_0 = 110$ cm^3/min. As may be noted from the photographs there existed an obvious eastern boundary current transporting perhaps 10 % S_0, and repeated experiments showed this phenomenon to be reproducible. The hypothesis is advanced that a small portion of the fluid received negative absolute vorticity by means of the viscous forces at the edge of the source slot, and that this portion of the fluid could take part in an eastern boundary current.

Several theoretical models of boundary current circulations in the ocean have been summarized by STOMMEL (1958). It is clear from the various studies that for linear viscous models significant mass transports may occur only at western boundaries where it is possible to have a balance of changes of relative vorticity (ζ) due to latitudinal (or depth) variation with those caused by lateral or vertical viscous stresses. In addition, MORGAN (1956) has shown for the simplest non-linear models that only western boundary currents can be of significance. However, inasmuch as the possibility of negative absolute vorticity has not been considered and since complete non-linear solutions including viscosity have not been available for examination, there is no reason to rule out all possibilities of inertio-viscous eastern boundary currents.

The simplest physical arguments suggest that if the fluid next to the eastern boundary had negative absolute vorticity ($\zeta + 2\Omega < 0$) it would again be possible to balance vorticity changes due to the depth variation with those due to

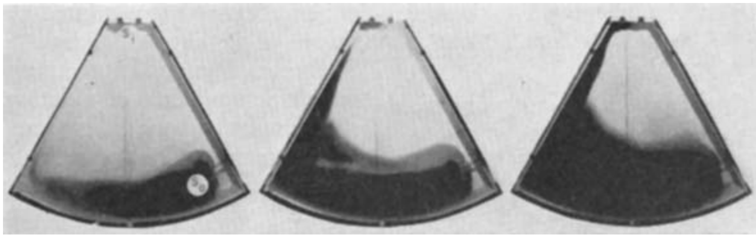

Fig. 5. Photographs at 10, 20, and 40 minutes illustrating the absence of an eastern boundary current when an interior source was used (compare with Figure 4) and the existence of strong "recirculation" from the western boundary.

lateral viscous stresses. However, in such an event many non-linear terms would become important and the simple arguments would be invalid. It is suggestive that for the experiment of Figure 4 6 % of the source fluid would enter with $\zeta + 2\Omega < 0$ given a slot width of 1/4 in., the flow rate of 110 cm^3/sec, and assuming a parabolic velocity profile for the flow through the slot. More critical experiments with various slot widths and flow rates are suggested as means of varying the percentage of fluid entering with negative absolute vorticity.

As a first test of the above hypothesis, the experiment illustrated in Figure 5 was conducted. Conditions were identical to those of Figure 4 except that the source water was introduced through a tube just away from the boundary so that there would be no opportunity for the fluid to obtain negative absolute vorticity. As may be seen, no eastern boundary current occurred.

6. Recirculation about an Interior Source or Sink

Introduction of the source water in the manner shown in Figure 5 eliminated the eastern boundary current but caused other interesting circulations. It is shown below that with an interior point source (distributed over a small area by turbulent mixing) an amplification of flow toward the western boundary is required. The sequence of Figure 5 shows a return flow from the western boundary current to the source, a circulation which is required by continuity in the vicinity of the source and the geostrophic constraints in the system with variable depth. To analyze this phenomenon we now consider the circulation about an interior source in the absence of bottom friction.

From equation (12) setting $\dfrac{\partial h}{\partial \Theta} = \dfrac{\partial h}{\partial t} = 0$ and $v = 0$ we obtain:

$$ru \frac{\partial h}{\partial r} - rQ = 0 \tag{24}$$

and by elimination of u with equation (7) we find:

$$\frac{\partial v}{\partial \Theta} = \frac{\dfrac{\partial^2 h}{\partial r^2}}{\left(\dfrac{\partial h}{\partial r}\right)^2} rQ - \frac{1}{\dfrac{\partial h}{\partial r}} \frac{\partial (rQ)}{\partial r}. \tag{25}$$

Integration of (25) from longitude Θ_0 to the eastern wall $\Theta = \Theta_E$ such that the entire source is included in this region gives:

$$v_{\Theta_0} = -\frac{\dfrac{\partial^2 h}{\partial r^2}}{\left(\dfrac{\partial h}{\partial r}\right)^2} Q_S + \frac{1}{\dfrac{\partial h}{\partial r}} \frac{\partial}{\partial r} Q_S \tag{26}$$

where we have taken $v_{\Theta_E} = 0$ (no zonal flow at the eastern boundary) and have written $Q_S = \int_{\Theta_0}^{\Theta_E} Qr d\Theta$, the zonally integrated source per unit of radius. Using the equation for the free surface (23) we obtain:

$$v_{\Theta_0} = \frac{a^2}{2h_0 l r^2} \frac{\partial}{\partial r}\left(\frac{Q_S}{r}\right). \tag{27}$$

Equation (27) shows that there should, in general, be both positive and negative values for v_{Θ_0} and that inflection points $v_{\Theta_0} = 0$ occur where $\frac{\partial}{\partial r}\left(\frac{Q_S}{r}\right) = 0$. For a smoothly varying source with a single maximum at r_0 we define the radius $r\ (v_{\Theta_0} = 0) = r'$ and write:

$$\underbrace{\int_0^{r'} hv\, dr}_{T(+)} + \underbrace{\int_{r'}^{a} hv\, dr}_{T(-)} = \int_0^a Q_S\, dr = S_0. \quad (28)$$

The integrals on the left of (28) represents transports toward and away from the source. We define $\frac{T(+)}{S_0}$ as the "recirculation" and find:

$$\frac{T(+)}{S_0} = \frac{1}{S_0}\int_0^{r'} \left(\frac{a^2 + lr^2}{2l}\right)\left[\frac{\partial}{\partial r}\left(\frac{Q_S}{r}\right)\right] dr. \quad (29)$$

Interpretation of (29) is facilitated by considering a specific functional form, say the Gaussian distribution $\frac{Q_S}{r} = \frac{S_0}{(2\pi)^{\frac{1}{2}} r_0 \sigma} e^{-\frac{(r-r_0)^2}{2\sigma^2}}$ so that $r' = r_0$, the radius of maximum source strength, and where σ is the radial spread of the source. Therefore for a source with small radial spread we may write approximately:

$$\frac{T(+)}{S_0} \approx \left(\frac{a^2 + lr^2}{2lS_0}\right)\int_0^{r_0} d\left(\frac{Q_S}{r}\right) =$$

$$= \left(\frac{a^2}{lr_0^2} + 1\right)\frac{r_0}{(2\pi)^{\frac{1}{2}} 2\sigma}. \quad (30)$$

We see that for small σ the recirculation is inversely proportional to σ. In any physical situation a lower limit on the practical size of σ probably occurs because of mixing associated with the source and with the increased circulation.

From Figure 5 we estimate $r_0 = 85$ cm and $\sigma = 5$ cm, and using the values $a = 100$ cm and $l = 1$, we obtain a theoretical recirculation $\frac{T(+)}{S_0} = 7.5$. This surprisingly large value was checked by evaluations from Figure 5. The total volume of dyed fluid which had been introduced by the time of the first picture was 1100 cm^3 (110 cm^3/min for 10 min). The volume of dyed water as evaluated from the photograph was 10,440 cm^3 or 9.5 $S_0 \Delta t$, which indicates that the source water was diluted by mixing at the source of 8.5 parts of recirculated clear water. This result is in good agreement with the theoretical recirculation considering the arbitrary specification of the form and width of the source region. Confirmation of these figures was obtained from the second photograph of Figure 5 which shows a volume of dyed fluid 10 times the total source input at 20 minutes. Comparison of Figures 4 and 5 shows the very large differences in the speeds of the zonal flow without recirculation (Figure 4) and with the additional flow produced by recirculation (Figure 5).

One may well inquire why the phenomenon of recirculation did not occur when the source water was injected through the slot in the wall. Formally the difference lies in the assumption following equation (26) where we have taken for the case of an interior source $v_{\Theta_E} = 0$, i.e., no geostrophic flow through the eastern wall and consequently no radial pressure gradient. For flow through the wall and with $Q = 0$ we would immediately obtain the solution to (25) $v_{\Theta_0} = v_{\Theta_E}$. Qualitatively we observe that for an interior source no isobars intersect the eastern wall but instead they form a region of high pressure extending to the western boundary with the source at the extreme eastern end. Having in mind this pattern the associated geostrophic recirculation and southward flow through the source region is obvious.

7. Partial Barrier Experiments

At the suggestion of Stommel, some experiments were performed to test the simple hypotheses proposed in (A) under somewhat more complicated conditions having geophysical interest. Figure 6a illustrates the anticipated circulation and Figure 6b the observed circulation in the 60 degree sector described in (A) but modified by a partial radial barrier 30 cm long extending inward from the midpoint of the rim. The barrier thus divided the sector into three portions referred to as the N, SW, and SE basins. Conditions of the experiment were $S_0 = 100$ cm^3/sec, $\bar{h} = 8$ cm, $h_0/h_a = 2$, $\Omega = 1$ sec^{-1} and since no water was removed from the tank the sink is represented by the uniform rise of the free surface.

The water budget of the predicted circulation is based upon the considerations developed in

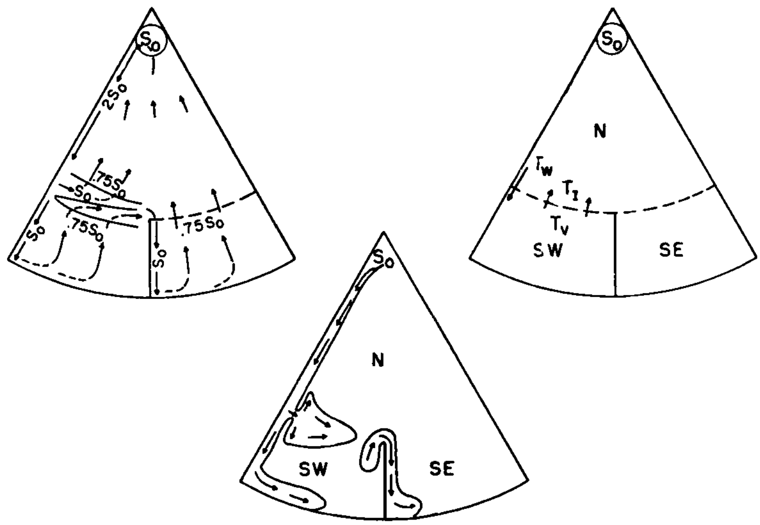

Fig. 6. The circulation obtained with a source at the apex of the sector and a uniform rise of the free surface in the presence of a partial radial barrier extending inward from the center of the rim. a) The theoretically predicted circulation; b) a sketch of the observed circulation; c) a schematic diagram for the definitions of N, SE, SW, T_W, T_I, T_V.

(A) as follows: we consider continuity in either the SW or the SE basin as indicated in Figure 6c where T_w is the transport of the western boundary current into the basin, T_I is the interior geostrophic transport out of the basin, and T_V is the upward flux due to the filling of the entire sector by the source. Continuity in SW requires $T_W = T_I + T_V$. $T_V = \frac{S_0}{2}\left(\frac{a^2 - r^2}{a^2}\right)$ is the fraction of S_0 used to fill the SE or the SW basin. T_I may be obtained from $T_I = \int_{\Theta} \int_{0}^{h+h'} fur\, d\Theta\, dz$ where h is the basic height at radius r and h' is the departure from h due to the steady geostrophic current. From equation (12) setting $v = Q = \frac{\partial h}{\partial \Theta} = 0$ we obtain $u = -\frac{\partial h/\partial t}{\partial h/\partial r}$ which permits evaluation of u. Using $\frac{\partial h}{\partial r} = \frac{2h_0 lr}{a^2}$ and $\frac{\partial h}{\partial t} = \frac{6S_0}{\pi a^2}$ we obtain $u = -\frac{3S_0}{\pi h_0 lr}$ and ignoring h' compared to h we find $T_I = \frac{S_0}{2l}\left(1 + \frac{lr^2}{a^2}\right)$. From the above expressions for T_I and T_V we obtain $T_W = \frac{S_0}{2}\left(1 + \frac{1}{l}\right)$.

Therefore for $l = 1$ we find $T_W = S_0$, and for $r = 70$ cm we obtain $T_V = +.25\, S_0$ and $T_I = +.75\, S_0$ for either the SW or the SE basin.

Of particular interest is the origin of the source water for the SE basin, for according to the simple theory 75 % of this water would be recirculated through the SW basin and 25 % would come zonally across from the western boundary current. This type of circulation was confirmed in the experiment by inserting crystals of dye in the SW basin to observe the flow around the end of the barrier as illustrated. Of further interest is the difference in history of the water in the western and eastern sides of the N basin; that on the west side having come directly from the western boundary current and 75 % of that on the eastern side having passed through both the SW and SE basins before flowing northward in the interior.

8. The Limiting Role of Bottom Friction

Figure 7a illustrates a partial barrier experiment similar to that of Figure 6 but with a localized sink near the rim as indicated and with the western and eastern walls coincident half way around the tank. The circulation performed in the expected fashion carrying the

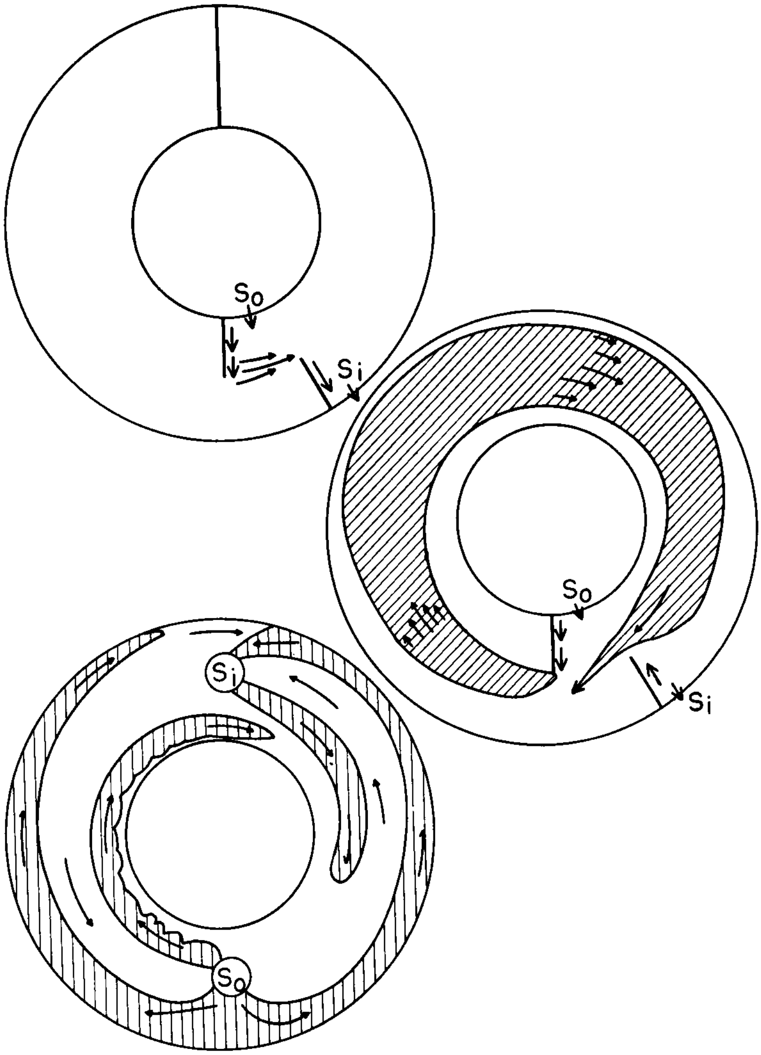

Fig 7. A series of experiments illustrating the progressive domination of bottom friction as radial barriers are removed. a) A partial radial barrier experiment similar to that of figure 6; b) the same experiment as in a) but with the complete radial barrier removed; c) a spiral circulation pattern required to transport mass from S_0 to S_i; with no radial barriers.

source water directly to the sink along the western boundaries.

Removal of the radial wall (Figure 7b) left an annular ring with partial barriers, a geometry somewhat similar to the geography of the antarctic circumpolar seas. For this case the boundary current from the inner barrier moved westward and proceeded around the tank at the same time spreading in accordance with the theory presented in section IV. After 1 circumference the dyed source water by-passed the outer barrier and the sink as indicated. Injection of dye beside the outer barrier clearly showed a boundary current flowing away from the sink. The interpretation is that the Ekman transport toward the rim was oversupplying the sink so that an inward flowing boundary current was required by continuity.

An extreme example of the domination of the flow by bottom friction in the absence of radial barriers is that illustrated in Figure 7c. The source and sink were at mid radii and 180 degrees apart. The complex spiraling pattern is evidence of the importance of the bottom boundary layer in the absence of radial barriers.

9. Applications to a Barotropic Ocean Circulation

The example of recirculation about an interior source is closely related to the "wind-spun vortex" suggested by MUNK (1950). In fact the theory of STOMMEL (1948) which includes a simple form of bottom friction and that of MUNK (1950) which includes lateral friction at the western boundary together contain nearly all of the essential elements of the theory and experiments presented above.

A striking feature of the recirculation about a source region is the extent of the large scale lateral mixing which may involve volumes of fluid several times the strength of the source itself. Also, the circulation of Figure 6 indicates significant mixing processes associated with the larger scale recirculations and the radial barriers. From this viewpoint it should be expected that the circulation of the deep ocean, which is now known to have some fairly strong currents (SWALLOW, personal communication), should be markedly affected by the large scale topography of the ocean floor. Accordingly, the distribution of properties may be significantly affected by localized mixing processes of large magnitude.

The amplitude of the recirculated zonal currents is of considerable interest especially in connection with the results recently published by ROBINSON and STOMMEL (1959). These authors described the localized amplification of the speed of barotropic currents of the oceans (VERONIS and STOMMEL, 1956) by relatively small scale irregularities in the ocean bottom. In a specific example, an amplification factor of 8 was suggested. Neither of the above named papers was concerned with nor implicitly included the recirculation phenomenon described here, and an additional amplification factor for barotropic currents should be considered. As in the model, the magnitude of the recirculation is primarily dependent upon the scale of the source-sink distribution as will be shown.

The barotropic theory for an ocean of constant depth D is readily obtained by conversion

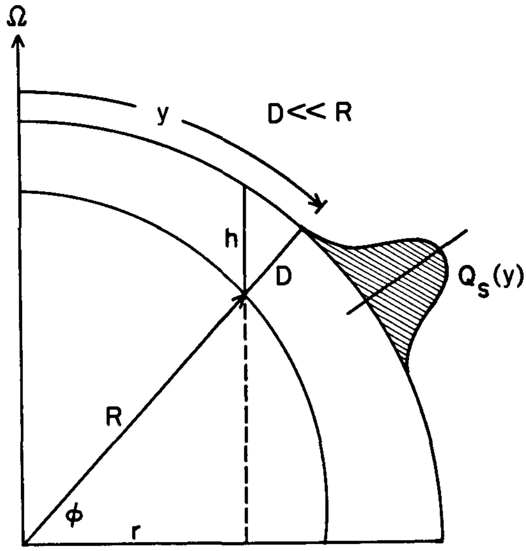

Fig 8. Conversion from polar to spherical geometry.

from polar coordinates according to the geometry of the sphere as in Figure 8. Into equations (19) and (20) we introduce the following relations for the sphere:

$$r = R\cos\varphi,\quad h = \frac{D}{\sin\varphi}$$

$$Q_S(r) = \frac{Q_S(y)}{\sin\varphi},\quad \frac{\partial}{\partial\varphi} = -R\frac{\partial}{\partial y}$$

We then obtain:

$$v_\Theta = -\frac{Q_S(y)}{D} + \frac{R}{D}\frac{\partial}{\partial y}(Q_S(y)\tan\varphi) \quad (31)$$

and

$$T(+) = -\int_0^{y'} Q_S(y)\,dy + R\int_0^{y'} d(Q_S(y)\tan\varphi) \quad (32)$$

which may be evaluated for any given latitudinal distribution of the source. For a source of small latitudinal extent and symmetrical with a maximum at the mean position y_0 we may approximate equation (32) by letting $\tan\varphi = \tan\varphi_0$ to obtain:

$$T(+) = -\frac{S_0}{2} + R\tan\varphi_0 Q_S(y_0) \quad (33)$$

where $Q_S(y_0)$ is the maximum value of $Q_S(y)$. As was done for the model we consider a normal distribution $Q_S = \frac{S_0}{(2\pi)^{\frac{1}{2}}\sigma} e^{-\frac{(y-y_0)^2}{2\sigma^2}}$ and obtain $Q_S(y_0) = \frac{S_0}{(2\pi)^{\frac{1}{2}}\sigma}$ so that the recirculation is:

$$\frac{T(+)}{S_0} = -\frac{1}{2} + \frac{R \tan \varphi_0}{(2\pi)^{\frac{1}{2}}\sigma}. \qquad (34)$$

$\frac{T(+)}{S_0}$ depends inversely upon the spread of the source region when $\sigma \ll R$. To take an extreme example we consider an idealized source due to precipitation or the wind stress convergence associated with a stationary hurricane having $\sigma \approx 20$ km at latitude $\varphi = 25$ degrees. The second term of (32) is dominant and gives a recirculation $\frac{T(+)}{S_0} = 59$! As another example, we consider a large-scale stationary system with $\sigma = 500$ km at $\varphi = 60$ degrees, and we obtain $\frac{T(+)}{S_0} = 10$ from the approximate formula (33). These figures suggest that the estimates of barotropic flow given by Veronis and Stommel (1956) and the partition of energy between barotropic and baroclinic currents (Veronis, 1956) should be reexamined in a framework which permits the recirculation phenomenon.

It is of some interest to estimate the spread of zonal barotropic flow due to bottom friction for the scale of an actual ocean. From section IV the formula for the standard deviation of the spread is $\sigma_x = (2Ay)^{\frac{1}{2}}$. For conversion to the spherical case we use

$$\sigma_y = \frac{\sigma_x}{\sin \varphi}, \; \frac{\partial h}{\partial r} = \frac{D \cos \varphi}{R \sin^3 \varphi}, \; \Omega \to \Omega \sin \varphi$$

in the Ekman equations, and $\nu \to \nu_e$ (an eddy coefficient of viscosity). We then obtain

$$A = -\frac{R \sin^3 \varphi}{\sqrt{2}\, D \cos \varphi} \left(\frac{\nu_e}{2\Omega \sin \varphi}\right)^{\frac{1}{2}}$$

For $\nu_e = 1$ cm^2/sec, $\Omega = .729 \times 10^{-4}$/sec^{-1}, $\varphi = 45$ degrees, $D = 5$ km, and $R = 6400$ km, we find $A = -45 \times 10^3$ cm, so that at a distance $x = 1000$ km from a point source $\sigma_y = 42$ km. We may take the total width of the current as $4\sigma_y = 168$ km. This width is latitude dependent, but it is reasonable to take 100 km as the minimum width of a barotropic zonal current except where local concentration due to bottom topography may be important. σ_y varies as $\nu_e^{\frac{1}{4}}$ so that only order of magnitude variations of ν_e can be of significance here.

10. Concluding Remarks

From time to time the uses and applications of model experiments are seriously questioned, particularly with respect to inability to improve our knowledge of real geophysical problems. The work described here contains a variety of experiments which suggest various possibilities and limitations for studies of the ocean circulation.

In the first place, the experiments have confirmed the qualitative predictions of the general ocean current theory suggested by STOMMEL (1957), and insofar as dimensional arguments can show that the same equations apply to the geophysical scale, the experiments represent confirmation of that theory.

Secondly, laboratory studies using a real fluid (as opposed to an idealized frictionless fluid) are sometimes capable of resolving incomplete or inconsistent theoretical analyses such as those proposed by HOUGH (1897) and by GOLDSBOROUGH (1933) each of whom considered source-sink circulations on a sphere. As has been pointed out by STOMMEL (1957) Hough had neither bottom friction nor meridional boundaries and could not obtain steady state solutions, whereas Goldsborough (having meridional boundaries) could obtain solutions only in certain restricted circumstances and did not conceive of boundary currents for the resolution of the general problem. The experiments described here and in (A) clearly resolve the difficulties of the above authors by means of experiments which would have been ideally suited to examination of their theoretical analyses.

Finally, laboratory experiments may suggest circulations of possible oceanographic significance which have heretofore been overlooked. Such is the case for the recirculation phenomenon and the spread of barotropic zonal flow each of which are contained within the elementary theory.

The eastern boundary current is an example of a circulation which may be of some geophysical interest but which is not contained within the elementary theories and deserves special attention. It is tempting to suggest on the basis of our experimental results that inertial eastern boundary currents may occur where there is a sufficiently strong discharge of water from a river or other source at an eastern boundary of an ocean. A geographical situation which resembles the experiment of Figure 4 is the saline discharge of approximately 1,680,000 m^3/sec (SVERDRUP, 1943) through the Straits of Gibraltar and into the Atlantic Ocean. It appears to be well established on the basis of observations of salinity (NIELSEN, 1912) that at least a small portion of the Mediterranean discharge moves in a narrow current northward along the coast of Spain, and a diagram of salinity would appear similar to the distribution of dyed source water in Figure 4.

Complete velocity profiles are not available, but direct observations of currents using Swallow floats at the salinity maximum (1,100 meters depth) off the south coast of Spain have shown on one occasion an average speed toward the West of 25 cm/sec at the closest observational distance 7.5 km from the coast (F. C. FUGLISTER, private communication). A shear of 25 cm/sec/3 km would equal the vertical component of the earth's vorticity at that latitude, therefore it seems probable that some small fraction of the Mediterranean discharge may have sufficient shear to realize negative absolute vorticity.

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