Integration of Four-Level Prognostic Equations over the Hemisphere

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Abstract

The three-dimensional quasi-nondivergent prognostic equation derived previously by Kuo is integrated over the hemisphere, the vertical distribution being represented by four levels. The effects of the mean radiational heating, friction and heating from the ground, and that of the turbulent diffusion have been included in this investigation. The initial disturbance considered is a random combination of ten wave components in x and seven wave components in y. The result of the integration is that at the end of the fourth day at 850 mb, three closed lows have been formed, all around latitude 60°, while the pole is occupied by a high. These lows have the dimensions of the semi-permanent lows in winter.

1. Introduction

This investigation is an attempt to integrate the quasi-nondivergent equations developed by KUO (1956, hereafter referred to as I) over the entire hemisphere by Green's method, so as to eliminate the errors introduced by placing arbitrary side boundaries. In this computation, the effects of the average radiational heating and cooling, the heat exchange between the atmosphere and its lower boundary (continent and ocean) and turbulent heat transfer, are all included.

The vertical distributions of the various quantities are represented by values at four levels, at 850 mb, 550 mb, 250 mb, and 50 mb. The 250 mb surface is taken as the mean level of the tropopause, so that the stratosphere is represented by one additional layer.

Because of the limitations of the electronic computer, it is impossible to include both hemispheres conveniently. Therefore we were compelled to take the vertical plane along the equator as a plane of symmetry and consider one hemisphere only. Furthermore, the nonlinear potential vorticity equation demands the stream function to be an odd function of y, which implies the vanishing of the north-south velocity along the equator. Thus any communication between the two hemispheres is excluded in the present computation. Such a condition would be a severe limitation for actual forecasting in low latitudes, and therefore only arbitrary artificial initial disturbances are considered in this investigation. Since a larger computer will be available in the near future, it will be possible to include both hemispheres in the computation and this limitation be removed.

2. Fundamental equations

As is well known, any horizontal velocity vector $\vec{v}$ can be separated into a nondivergent part $\vec{v}_1$ and an irrotational part $\vec{v}_2$ so that

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

$$\vec{v}_1 = \vec{k} \times \nabla \psi, \qquad \vec{v}_2 = \nabla X,$$

where ψ is the stream function for $\vec{v}_1$ and χ the potential function for $\vec{v}_2$, and $\vec{k}$ is the unit vector along the vertical. In I, Kuo discussed the class of motions in which $|\vec{v}_2|$ remains much less than $|\vec{v}_1|$ at all times, so that we have $|\vec{v}_1| \gg |\vec{v}_2|$ and $\left|\frac{\partial \vec{v}_1}{\partial t}\right| \gg \left|\frac{\partial \vec{v}_2}{\partial t}\right|$. This class of motions is called the "quasi-nondivergent" flow, and is considered to be representative of the large scale atmospheric motions. Since the vertical velocity is small and the horizontal dimension is large, the hydrostatic approximation holds at all times and the horizontal equation of motion may be written as

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial p}\right)\vec{v}_1 + (f+\zeta)\, \vec{k} \times \vec{v} = - \\ - \nabla\left(\phi + \frac{1}{2} v_1^2\right) + \vec{F} \qquad (1)$$

where the (λ, φ, p) system is used, $\phi = gz$ is the geopotential and $\omega = \frac{dp}{dt}$. Thus the dynamic equations do not contain $\frac{\partial \vec{v}_2}{\partial t}$ and $\frac{dw}{dt}$; these quantities can only be obtained indirectly from the $\vec{v}_1$ field.

Since $\vec{v}_1$ is determined completely by the stream function ψ, it is convenient to apply the curl operator to (1) and obtain the equation for the vorticity $\zeta = \nabla^2\psi$, given by

$$\frac{\partial \zeta}{\partial t} + \tau(\psi, \zeta + f) - (f+\zeta)^2 \frac{\partial}{\partial p}\left(\frac{\omega}{\zeta + f}\right) + \\ + \vec{k} \cdot \nabla \omega \times \frac{\partial \vec{v}_1}{\partial p} + \vec{v}_2 \cdot \nabla(\zeta + f) \\ - \vec{k} \cdot \nabla \times \vec{F} = 0 \qquad (2)$$

where $\tau(a, b) = \partial(a, b)/\partial(\lambda, \varphi)$.

In addition, we have the thermodynamic equation

$$\left\{\frac{\partial}{\partial t} + \left(\vec{v}_1 + \vec{v}_2\right) \cdot \nabla\right\}\frac{\partial \phi}{\partial p} - \frac{R\Gamma}{p}\omega = -\frac{R}{c_p}\frac{Q}{p} \qquad (3)$$

where $\Gamma = \frac{T}{\Theta}\frac{\partial \Theta}{\partial p}$, Θ being the potential temperature, and Q is the rate of the nonadiabatic addition of heat per unit mass. Thus (2) and (3) are the two fundamental equations. In these equations, the transport of heat and vorticity by $\vec{v}_2$ is of minor importance; therefore we may replace $\vec{v}_2$ by its first approximation.

Assuming that the motion $\vec{v}_1$ and the pressure distribution are always in a nearly balanced state, represented by the relation

$$(f+\zeta) \nabla \psi = \nabla\left(\phi + \frac{1}{2} v_1^2\right) \qquad (4)$$

we may then obtain the distribution of ψ or ϕ when one of the two is given, by integrating this equation. Furthermore, we may also obtain a rough approximation of $\vec{v}_2$ by substituting this relation in the equation of motion (1), to give

$$-\bar{f}\,\vec{v}_2 = \left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial p} - \frac{\partial}{\partial x_j} A_j \frac{\partial}{\partial x_j}\right) \nabla \psi \\ \approx \left(\frac{\partial}{\partial t} - A_j \cdot \frac{\partial^2}{\partial x_j^2}\right) \nabla \psi \qquad (5)$$

where $\bar{f}$ is a mean value of f which will be defined below, and A_j is the eddy viscosity coefficient in the direction of x_j and the summation convention over the repeated index j is being used. In this approximation, the first term represents the isallobaric wind while the second term represents the frictional wind, respectively. We note that in this form, both terms are expressed in terms of ψ alone. These quantities can be simplified further. For example, in the first term, we may replace $\partial\psi/\partial t$ by the past tendency, and in the second term, replace the sum $A_j \frac{\partial^2 \nabla \psi}{\partial x_j^2}$ by $A \frac{\partial^2 \nabla \psi}{\partial^2 z}$, or even by $-k_1 \nabla \psi$. It seems that this simple way of including the effect of $\vec{v}_2$ is a reasonable procedure and does not require much extra computation. On the other hand, we could also solve the equation (I 15) for ω and then obtain χ from the continuity equation $\nabla^2\chi = -\frac{\partial \omega}{\partial p}$. The divergence velocity $\vec{v}_2 = \nabla\chi$ can then be obtained. Since this procedure will increase the computation time enormously we shall not follow it in this computation.

We separate the temperature into two parts, a normal temperature T_0, taken as a function of p alone, and a departure T'. In this way, we may replace $\partial\phi/\partial p$ in equation (3) by $\frac{\partial \phi'}{\partial p} = -RT'/p$, which is assumed to be related to ψ by the integrated thermal wind relation

$$\frac{\partial \phi'}{\partial p} = \bar{f}\frac{\partial \psi}{\partial p} \tag{6}$$

where $\bar{f}$ is a certain mean value of f. By making use of the approximations (4), (5) and (6), the two equations (2) and (3) reduce to two equations in ψ and ω. Eliminating ω from these equations and neglecting some of the less important terms we then obtain

$$\left\{\frac{\partial^2}{dx^2} + \frac{\partial^2}{\partial y^2} - \frac{a^2 f\bar{f}\cos^2\varphi}{R}\frac{\partial}{\partial p}\frac{p}{\Gamma}\frac{\partial}{\partial p}\right\}\frac{\partial \psi}{\partial t} = -H_e(x, y, p) \tag{7}$$

where

$$H_e(x, y, p) \approx \tau_e(\psi, \zeta + f) - \frac{f}{R}\tau_e\left(\psi, \bar{f}\frac{\partial}{\partial p}\frac{p}{\Gamma}\frac{\partial \psi}{\partial p}\right) - \frac{fa^2\cos^2\varphi}{c_p}\frac{\partial}{\partial p}\left(\frac{Q}{\Gamma}\right) - a\cos\varphi\,\vec{k}\cdot\nabla\times\vec{F} - M_e \tag{8}$$

and $\tau_e(a, b) = \frac{\partial(a, b)}{\partial(x, y)}$ and $x = \lambda$, $y = \int_0^{\varphi}\frac{d\alpha}{\cos\alpha}$

are the horizontal coordinates in the Mercator projection. This projection is used in order to transform the Laplacian operator to the simple form so as to facilitate the finding of the Green's function, and also to facilitate the enumeration of the grid points in the coding. The subscript e in H and in the two Jacobians denotes that the horizontal differentiations are with respect to x and y. We note that the two Jacobians in H_e represent the effects of the transports of vorticity and heat by $\vec{v}_1$, and the last term M_e denotes the effect of transports by $\vec{v}_2$ and is given by

$$M_e \approx -\frac{2\bar{f}}{R\Gamma}\nabla_e\frac{\partial \psi}{\partial p}\cdot\nabla_e\frac{\partial}{\partial p}\left(\frac{\partial \psi}{\partial t} + k_1\psi\right) \tag{8a}$$

where k_1 is the coefficient of friction. For more details of the function H_e, see I.

The boundary conditions at earth's surface $p = p_0$, $\phi = \phi^\star$ is

$$\frac{\partial^2\psi}{\partial t\partial p} + s\frac{\partial \psi}{\partial t} + \tau\left(\psi, \frac{\partial \psi}{\partial p}\right) - \frac{s}{\bar{f}}\tau(\psi, \phi^\star) + \frac{RQ}{c_p p_0 \bar{f}} = 0 \tag{9a}$$

where $s = -\frac{1}{\Theta}\frac{\partial \Theta}{\partial p}$. For the top of the atmosphere $p = 0$ we have $\omega = 0$, we will also assume that the kinetic energy per unit volume vanishes there. A deduction hereof is

$$\lim_{p\to 0}\rho^{1/2}\frac{\partial \psi}{\partial t} = 0 \tag{9b}$$

which provides us with the upper boundary condition, ϱ being the density.

3. Green's Function for the Multi-layer System

In I the integration of the system (7) and (9a, b) by Green's method was discussed. To simplify the Green's function, a normal static stability is assumed and is given by

$$\Gamma_0 \equiv \frac{\partial T_0}{\partial p} - \frac{R}{c_p}\frac{T_0}{p} = -\frac{Cp}{(p + \alpha p_0)^2} \tag{10}$$

The value of C is taken to be 110° C and the value of α is $\alpha = 0.5$ for the troposphere and $\alpha = 0$ for the stratosphere, which corresponds to a nearly constant Γ_0 in the troposphere and an isothermal stratification in the stratosphere. The latter assumption is necessary in order to satisfy the upper boundary condition $\omega = 0$ at $p = 0$ if we assume T, V and Q to remain finite at the top of the atmosphere, as can be seen from the thermal energy equation (3). A very small but yet finite α may also be used for the stratosphere, representing a nearly isothermal condition in the lower stratosphere and $T \to 0$ as $p \to 0$.

Writing $\frac{\partial \psi}{\partial t}$ as the sum of two functions, $\frac{\partial \psi_1}{\partial t}$ and $\frac{\partial \psi_2}{\partial t}$ we then have

$$\frac{\partial \psi_1}{\partial t} = (p + \alpha p_0)^{-1/2} \iiint K(P, Q)\, H_s(Q) \frac{dp'}{(p' + \alpha p_0)^{1/2}} dA'_s \qquad (11)$$

where $dA'_s = a^2 \cos\varphi d\varphi d\lambda = a^2 \cos^2\varphi\, dx dy$, $H_s(Q) = H_e/a^2 \cos^2\varphi$, and $K(P, Q)$ is the Kern function and is given by

$$K(P, Q) = \frac{1}{4\pi l}\left\{\frac{e^{-\frac{lr}{2}}}{r} + \frac{e^{-\frac{lr'}{2}}}{r'} + 2hIe^{-h(q+q')}\right\}, \qquad (12)$$

$$r, r' = \left\{(x - x')^2 + (y - y')^2 + \frac{1}{l^2}(q \mp q')^2\right\}^{1/2},$$

$$r'(\beta) = \left\{(x - x')^2 + (y - y')^2 + \frac{1}{l^2}(q + \beta')^2\right\}^{1/2}$$

$$q = \ln\left[\frac{1 + \alpha}{p/p_0 + \alpha}\right], \; l = af\cos\varphi/\sqrt{RC}, \qquad (12a)$$

$$h = \frac{1}{2} + (1 + \alpha)p_0 \frac{\partial \ln \Theta}{\partial p},$$

$$I = \int_{q'}^{\infty} \operatorname{Exp}\left\{h(q + \beta) - \frac{lr'(\beta)}{2}\right\} \frac{d\beta}{r'(\beta)}.$$

Details of the K function are given in I. Since $K(P, Q)$ depends only upon the relative positions of the source point Q and the field point P under consideration, the integral in (11) may be evaluated in any coordinate system. However, both the derivation of K and the evaluation of ζ are greatly simplified by the use of the Mercator projection because of the simplification of the Laplacian operator ∇^2.

The second part of the solution $\frac{\partial \psi_2}{\partial t}$ which comes from the nonhomogeneous terms of the boundary condition (9a), is given by

$$\frac{\partial \psi_2}{\partial t} = (p/p_0 + \alpha)^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{A_{mn}}{h - \lambda_{mn}} \cdot e^{-\lambda_{mn} q} e^{i(mx+ny)} \qquad (13)$$

where

$$\lambda^2_{mn} = \frac{1}{4} + \frac{m^2 + n^2}{l^2}$$

$$A_{mn} = \frac{1}{4\pi^2} \iint B^\star(\xi, \eta) e^{-i(m\xi + n\eta)}\, d\xi d\eta$$

and

$$B^\star = (1 + \alpha)^{3/2} \left\{\tau_s\left(\psi, p\frac{\partial \psi}{\partial p}\right) - \frac{s_0 p_0}{f} \tau_s(\psi, \phi^\star) + \frac{RQ}{c_p f}\right\}_{p=p_0}$$

The subscript s in the Jacobians indicates that the differentiations refer to spherical coordinates. Thus $B^\star$ represents the effects of heat transfer and nonadiabatic heating at the ground level and the effect of topography.

To facilitate the computation, we shall also put $\frac{\partial \psi_2}{\partial t}$ into an integral form and try to combine it with $\frac{\partial \psi_1}{\partial t}$. Thus substituting A_{mn} into (13) we obtain

$$\frac{\partial \psi_2}{\partial t} = - \frac{1}{4\pi^2}(p/p_0 + \alpha)^{-\frac{1}{2}} \iint \sum_m \sum_n \frac{e^{-\lambda_{mn} q}}{\lambda_{mn} - h} \cdot B^\star(\xi, \eta) e^{im(x-\xi) + in(y-\eta)} d\xi d\eta = - \left(\frac{p}{p_0} + \alpha\right)^{-\frac{1}{2}} \iint K_2(x, y, q; \xi, \eta, 0) \cdot B^\star(\xi, \eta)\, d\xi\, d\eta \qquad (14)$$

where K_2 is another Green's function for the surface disturbance, and is given by

$$K_2 = \frac{1}{4\pi^2} \sum\sum \frac{1}{\lambda_{mn} - h} \operatorname{Exp}\left\{-\lambda_{mn} q + im(x - \xi) + in(y - \eta)\right\} \qquad (14a)$$

To evaluate K_2, let us put $m = k\cos\alpha$, $n = k \sin\alpha$, $-\pi < \alpha \le \pi$; $x - \xi = \varrho \cos\Theta$, $y - \eta = \varrho \sin\Theta$; we then have $\lambda^2_{mn} = \frac{1}{4} + \frac{k^2}{l^2}$ and

$$K_2(\varrho, q; \varrho', 0) = \frac{1}{4\pi^2} \sum\sum \frac{e^{-\lambda_k q}}{\lambda_k - h} \cdot e^{ik\varrho \cos(\alpha - \Theta)} \qquad (14b)$$

The limits of k and α are now $0 \le k \le \infty$ and $-\pi < \alpha \le \pi$. Changing the sums into integrals we then obtain

$$K_2 = \frac{1}{4\pi^2} \int_0^\infty \int_{\alpha_0}^{\alpha_1} \frac{e^{-\lambda_k q}}{\lambda_k - h} e^{ik\varrho \cos(\alpha - \Theta)}\, k\, dk\, d\alpha \quad (14c)$$

Putting $w = \alpha - \Theta$, we have

$$I_1 = \int_{\alpha_0}^{\alpha_1} e^{ik\varrho \cos(\alpha-\Theta)} d\alpha = \int_{\omega_0}^{\omega_1} e^{ik\varrho \cos \omega} d\omega = 2\pi J_0(k\varrho)$$

where $J_0(k\varrho)$ is the normalized Bessel function of order zero. This integral is made to be independent of Θ by allowing the imaginary part of w to go to infinity. Thus the Green's function K_2 is given by

$$K_2 = \frac{1}{2\pi} \int_0^\infty \frac{e^{-\lambda_k q}}{\lambda_k - h} J_0(k\varrho)\, k\, dk \qquad (15)$$

Note that both for large and for small values of k/l, we have

$$\lambda_k = \left(\frac{1}{4} + \frac{k^2}{l^2}\right)^{1/2} \approx \frac{k}{l} + \frac{1}{2} \approx \frac{k}{l} + h.$$

Therefore the integral above is given by

$$K_2 = \frac{e^{-q/2}}{2\pi} l \int_0^\infty e^{-kq/l} J_0(k\varrho)\, dk =$$

$$= \frac{l}{2\pi} \left(\frac{p/p_0 + \alpha}{1 + \alpha}\right)^{1/2} \frac{1}{r_0} \qquad (16)$$

where $r_0 = (\varrho^2 + q^2/l^2)^{1/2}$ is the effective distance from the source point Q_0 $(\xi, \eta, 0)$ on the earth's surface to the field point $P(x, y, q)$.

The complete solution $\frac{\partial \psi}{\partial t}$ is thus given by

$$\frac{\partial \psi}{\partial t} = \iiint G_1(P, Q) H_s(Q)\, d\zeta' dA_s' +$$

$$+ \iint G_2(P, Q_0)\, E_0\, dA_s' \qquad (17)$$

where $dA_s' = a^2 \cos^2 \varphi\, d\xi\, d\eta$ is the spherical area element and

$$G_1 = K_1(P, Q) / \sqrt{(\zeta + \alpha)(\zeta' + \alpha)}$$

$$G_2 = \frac{1}{(1 + \alpha)} \cdot \frac{1}{2\pi l r_0} \qquad (18)$$

and

$$E_0 = -(1 + \alpha)^{1/2} \frac{\bar{f}^2}{RC} B^\star =$$

$$= (1 + \alpha)^2 \left\{ \tau_s \left(\psi, \frac{\bar{f}^2}{RC} \frac{\partial \psi}{\partial \zeta} \right) + \right.$$

$$\left. + \frac{\bar{f} Q}{c_p C} \right\}_{\zeta = 1}$$

for a plane earth, and $\zeta = p/p_0$.

The Green's function G_2 for the surface effect is nearly equal to the value of $(G_1)_{\zeta'=1}$ for the near-by points, but decreases at a much slower rate with increasing distance, and becomes more than twice as large as $(G_1)_{\zeta'=1}$ at greater distances from the source point.

We note that the heating factor appears in the forcing function H_s in the differentiated form

$$H_2 = -\frac{\partial}{\partial \zeta} \frac{1}{p_0 \Gamma} \left\{ \tau_s \left(\psi, \frac{\bar{f}^2}{R} \zeta \frac{\partial \psi}{\partial \zeta} \right) + \right.$$

$$\left. + \frac{\bar{f} Q}{c_p} \right\},$$

and in undifferentiated form in the surface forcing function E_0. Putting $H_s = H_1 + H_2$, where H_1 denotes the other part of H_s given by (8), we may write the solution (17) in the following form

$$\frac{\partial \psi}{\partial t} = \iiint \left\{ G_1 H_1 + \frac{\partial G_1}{\partial \zeta'} E \right\} d\zeta' dA_s' +$$

$$+ \iint (G_2 - G_{1s})\, E_0\, dA_s' \qquad (17a)$$

where G_{1s} is the value of G_1 for $\zeta' = 1$ and

$$E = (\zeta + \alpha')^2 \left\{ \tau \left(\psi, \frac{\bar{f}^2}{RC} \frac{\partial \psi}{\partial \zeta'} \right) + \frac{\bar{f}}{c_p C} \frac{Q}{\zeta'} \right\} \quad (18a)$$

Here the heating effect appears in undifferentiated form in every integral. However, although equation (17a) gives a better physical insight into the heating effect, equation (17) is more convenient for computation since only one volume integral is involved.

In actual application, the atmosphere is divided into n layers of depth Δp. Similarly, the horizontal coordinates are replaced by the grid points $x = i\Delta x$, $y = j\Delta y$. The integrals in (17) are then replaced by summations of the products of the Green's function and the forcing function at the grid point, each result representing the mean value in the volume element $\Delta\zeta'\Delta\xi\Delta\eta$. For convenience sake, we shall put the thickness factor $\Delta\zeta'$ into the Green's function and obtain

$$\overline{G}_1(P, Q) = \frac{1}{(\zeta+\alpha)^{1/4}} \int_{\zeta'_Q - \frac{1}{2}\Delta\zeta'}^{\zeta'_Q + \frac{1}{2}\Delta\zeta'} \cdot$$

$$\cdot \int_{\xi_Q - \frac{1}{2}\Delta\xi}^{\xi_Q + \frac{1}{2}\Delta\xi} \int_{\eta_Q - \frac{1}{2}\Delta\eta}^{\eta_Q + \frac{1}{2}\Delta\eta} \frac{K_1(P, Q)}{(\zeta'+\alpha)^{1/4}} \frac{d\zeta' d\xi d\eta}{\Delta\xi\Delta\eta} \qquad (19)$$

Average values of the function G_2 may be obtained in a similar manner. Replacing the integrations by summations the solution (17) may then be written in the form

$$\frac{\partial\psi}{\partial t} = \sum_i \sum_j \sum_k \overline{G}_1(P, Q_k) \left\{ H_s(Q_k) + \right.$$

$$\left. + \frac{bE_0}{\Delta\zeta'_1} \right\} dA' \qquad (20)$$

where $b = \overline{G}_2 \Delta\zeta' / \overline{G}_1$, which is nearly equal to 1 for the near-by points and is larger than 1 for far away points. Since the surface effect is not very important, this ratio is replaced by 1 in the present preliminary study to avoid the use of two different Green's functions.

In the computation, the surface forcing function $E_0(Q_0)$ is calculated from an interpolation formula and the lower boundary condition. It is considered to be distributed in the lowest layer, and that $E_k = 0$ for all the other layers. Thus the object of the computation is simply to find the sum of the function H_s and E_0 at each "source point" Q, then to multiply by the influence function $\overline{G}_1(P, Q)$ and to add the resulting products together for all source points Q to obtain the rate of change of the stream function $\partial\psi/\partial t$ at the field point P. The value of ψ at time $t + \Delta t$ is then obtained by extrapolation, using a centered time difference to avoid computational instability.

It may be remarked that the value of the influence function $G_1(P, Q)$ decreases rapidly with the distance between P and Q, so that the contribution from the far away points is relatively small. We may therefore use a larger area element for the far away regions and a mean value of the influence function for this larger area element. In this way the computation time can be reduced greatly. It may be remarked that in the numerical form (19) in which the integrations have been replaced by summations, the function $\overline{G}_1$ (P, Q) is finite at every point, including the field point P itself. This is because $\overline{G}_1$ now represents the average value of G_1 for the whole volume element $\Delta\xi\Delta\eta\Delta\zeta'$ centered at P.

4. The Side Boundary Conditions

In the above only the boundary conditions at earth's surface ($p = p_0$) and at the top ($p = 0$) have been considered, and no side boundary has been assumed, and therefore the region of integration in (17) and (19) must include the whole atmosphere. Since in the actual atmosphere there is actually no side boundary, such an approach is really necessary in order to make forecasts for an extended time and area. Such a study has actually been attempted; the purpose was to take into consideration all the interactions between different parts of the atmosphere and to get away from the artificial restrictions implied by imposing side boundary conditions.

However, since we approximate the vertical structure of the atmosphere by dividing it into four layers, the inclusion of the entire atmosphere requires too many grid points and was difficult to handle in the computing machine available at the time. Therefore only one hemisphere will be included at the moment. This is physically possible if we consider the two hemispheres as being exactly similar, and taking the equator as a line of symmetry. Now the terms in the Jacobian $\tau(\psi, \nabla^2\psi)$ of eq. (2) always introduce odd y-functions because of the nonlinearity. We must therefore assume ψ to be an odd function of y if symmetry at the equator is supposed, implying that the north-south velocity is zero at the equator, so that there will be no communication between the two hemispheres. Since the rate of change of the vorticity ζ is then equal to zero, $\partial\psi/\partial t$ will also vanish at the equator. We shall therefore require

$$\frac{\partial\psi}{\partial t}=0 \quad \text{at } y=0. \tag{21}$$

Since eq. (7) is an elliptic equation in $\partial\psi/\partial t$, this condition together with the conditions (9a) and (9b) at $p=p_0$ and $p=0$ are sufficient for the determination of $\partial\psi/\partial t$ for the hemisphere.

The condition (21) is satisfied if we assume that for every disturbance H_s at the point Q (x, y, p) there is a disturbance equal to $-H_s$ at the point $Q'(x, -y, p)$ which is the image of Q with respect to the equator. Denoting the Green's function for the points P and Q' by $\overline{G}_1'$, which is obtained from $\overline{G}_1$, by replacing $y—\eta$ by $y+\eta$ in the distance $r=\{(x—\xi)^2+(y—\eta)^2+(q-q')^2/l^2\}^{1/2}$ and putting $\overline{F}=\overline{G}_1-\overline{G}_1'$, the solution that satisfies the boundary conditions (9a, b) and (21) is then given by

$$\frac{\partial\psi}{\partial t}=\sum\sum\sum \overline{F}(P, Q)\left\{H_s(Q_k)+\frac{E_0(Q_0)}{\Delta\zeta'}\right\}dA_s' \tag{22}$$

We note that in formula (22), both the quantities H_s and E_0 and the area element dA_s' refer to the spherical earth, and therefore it is possible to evaluate the integration in spherical coordinates.

It may be remarked that even though we have assumed that ψ does not change at the equator, the mean zonal velocity at the equator may still change with time, because of the turbulent friction and vertical motion, according to the equation:

$$\frac{\partial\bar{u}}{\partial t}=-\bar{\omega}\frac{\partial\bar{u}}{\partial p}-\left(\overline{\omega'\frac{\partial u'}{\partial p}}\right)+A_p\frac{\partial^2\bar{u}}{\partial p^2}+A_v\frac{\partial^2\bar{u}}{\partial y^2} \tag{23}$$

However, such changes must be exceedingly small because of the lack of disturbances in lower latitudes as a consequence of the side boundary condition (21).

5. Effect of friction

Frictional effects appear directly in the nonhomogeneous term H_e of (8) in the form of $\vec{k}\cdot\nabla\times\vec{F}$, and indirectly in the form of the frictionally driven cross isobar and isallobaric flows in the transport terms. We shall analyze the former first.

Since the atmospheric motion is always turbulent, the frictional effect in the dynamic equation may be written as

$$\vec{F}=A_v\nabla^2\vec{v}_1+\frac{1}{\varrho}\frac{\partial\vec{\tau}_z}{\partial z} \tag{24}$$

where A_v is the horizontal turbulent viscosity and $\vec{\tau}_z$ is the turbulent stress in the vertical planes, due to disturbances smaller than those under consideration. Since $\vec{\tau}_z=—g\varrho\mu\,\frac{\partial\vec{v}_1}{\partial p}$, we have

$$\frac{1}{\varrho}\frac{\partial\vec{\tau}_z}{\partial z}=-g\frac{\partial\vec{\tau}_z}{\partial p}=g^2\frac{\partial}{\partial p}\left(\mu\varrho\frac{\partial\vec{v}_1}{\partial p}\right)$$

Setting $\mu=K\varrho$, $K=K_0\left(\frac{T}{T_0}\right)^2$, and applying the operator $\vec{k}\cdot\nabla\times$ we obtain

$$\frac{1}{\varrho}\frac{\partial}{\partial z}\vec{k}\cdot\nabla\times\vec{\tau}_z=K_v\frac{\partial}{\partial p}\left(p^2\frac{\partial\zeta}{\partial p}\right) \tag{25}$$

where ζ is the relative vorticity, and

where $K_v=\frac{g^2K_0}{R^2T_0^2}\approx 1.6\times10^{-7}\,\text{sec}^{-1}$ if we choose $K_0=10^5$ cm^2 sec^{-1}. This corresponds to $\mu=100$ gm cm^{-1} sec^{-1}. The value of A_v is taken to be 10^5 m^2 sec^{-1}, after PHILLIPS (1956).

In addition, there exists the ground friction. For simplicity this frictional effect is assumed to be proportional to the wind speed at the ground level, instead of proportional to the kinetic energy. Thus we shall assume the surface stress is given by

$$\vec{\tau}_0=\varkappa\varrho\,|v_0|\,\vec{v}_0\approx k_g\varrho\,\vec{v}_0 \tag{26}$$

and take $k_g=3.0$ cm sec^{-1}. In the numerical integration the lowest level is taken at $p=p_0-\frac{1}{2}\Delta p$. Assuming that this frictional effect is uniformly distributed in this lowest layer, we obtain the coefficient of ground friction $k_0 = k_g/\Delta z$. For $\Delta p = 300$ mb, this gives $k_0 = 1.0 \times 10^{-5}$ sec^{-1}.

It may be remarked that if these frictional terms are evaluated at the centered time $t = n\Delta t$, they will produce computational instability; therefore we shall evaluate them at the previous time step $t = (n-1)\,\Delta t$ (personal communication from Dr. N. A. Phillips).

6. Nonadiabatic Heating

The nonadiabatic rate of heating Q occurs in the nonhomogeneous term H_s in the form $\frac{\partial}{\partial p}\left(\frac{Q}{\Gamma}\right)$, and in the boundary effect E_0 in the form of Q. The important physical processes determining Q are radiation, evaporation and condensation and conductive heat transfer, including that due to turbulent heat transfer produced by the disturbances which are smaller in scale than those under consideration. The heat transfer produced by ordinary thermal convection will also be included in this turbulence effect.

The atmosphere receives only a fraction of its heat supply through direct absorption of the solar radiation; the rest is received from the earth through long wave radiation and through evaporation and condensation. Since the atmosphere emits and absorbs the long wave radiation mainly through its water vapor content, and since the presence of clouds has a very large effect on the distribution of the radiative heat transfer, both the long wave radiation and the release of latent heat depend very much on the water vapor distribution and on vertical motion. Thus in order to take into consideration these processes it is necessary to add new dependent variables and new equations to our system. Such a system will become extremely complex. Since we are still at the very preliminary stage of the investigation, we shall include only the statistical effects of the radiational and evaporation-condensation processes so as to furnish the system with a thermal energy supply which is roughly comparable with that in the atmosphere.

Taking the total depth of the atmosphere as a whole, the air is gaining heat in lower latitudes and losing heat in higher latitudes through the processes of radiation and evaporation-condensation. This total gain or loss is measured by the difference between the effective incoming solar radiation E and the outgoing long wave radiation A. When the quantity $E - A$ is integrated over the area from the equator to the latitude φ we obtain the accumulation of heat T through radiation in this area. This also represents the amount of heat that must be transported across the corresponding latitude circles in order to maintain a steady mean temperature distribution.

Fig. 1 summarizes several computations of the required latitudinal energy transport T. The values represented by the circles are those of HOUGHTON (1954) and the crosses are those of ALBRECHT (1931). Values obtained by others all lie between these two limits.

In eq. (22), what is needed is not the total heating $E - A$ for the whole air column but the vertical distribution. At present no such vertical distribution has been computed. Therefore we must make a plausible assumption. It is assumed that the zonally averaged rate of heating Q, produced by the combined effect of radiation and evaporation-condensation decreases upward, and is directly proportional to the pressure p. The latitudinal variation is also represented by a simple function of φ, and is given by

$$Q_1 = 2\,B\,{}^{p}/_{p_0}\,(\cos^2\varphi - {}^2/_3) \qquad (27)$$

This represents a heating from the equator to 35° 15′ and a cooling from 35° 15′ to the pole, while the total heating over the hemisphere is zero. The constant B is determined by the mean rate of heating at the equator. Supposing this is equal to 0.05 cal cm^{-2} min^{-1}, which corresponds to a heating of 0.3° C per day at the equator, we then obtain $B = 2.4 \times$ $\times\, 10^{-6}$ cal gm^{-1} sec^{-1}.

Integrating (27) over the total pressure depth of the atmosphere and over the area from the equator to latitude φ we obtain the total gain of heat T per unit time. This is given by

$$T = \frac{\pi a^2}{3}\frac{p_0}{g}\,B \sin 2\,\varphi \cos\varphi \qquad (28)$$

where a is the radius of the earth. This function is represented in Fig. 1 by the curves T_a and

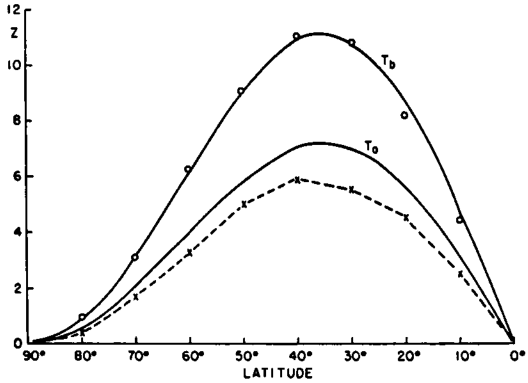

Fig. 1. Required poleward transport of heat across latitude circles to balance radiational excesses in the annual mean. Units are 10^{19} cal day^{-1}. Circles and crosses are values obtained by Houghton and Albrecht, respectively. Curves T_a and T_b are values computed from eq. (28).

T_b, corresponding to $B = 2.4 \times 10^{-6}$ cal gm^{-1} sec^{-1} and $B = 3.7 \times 10^{-6}$ cal gm^{-1} sec^{-1}, respectively. It is seen that the curve T_b fits the values of T obtained by Houghton very closely, while the curve T_a lies between Houghton's and Albrecht's values.

In the actual atmosphere, a large portion of the poleward transport of energy takes the form of latent heat. Since we are disregarding the water vapor content of the air, we may take a somewhat smaller value for the constant B so that less sensible heat need be transported poleward by the motion. The value of B used in the present computation is $B = 2.4 \times 10^{-6}$ cal gm^{-1} sec^{-1}, corresponding to the curve T_a in fig. 1.

The effect of lateral turbulent heat transfer on Q is given by

$$Q_2 = c_p A_T \nabla^2 T = -\frac{c_p}{R} \bar{f} A_T p \frac{\partial \zeta}{\partial p} \quad (29)$$

Assuming A_T is given by

$$A_T = A_0 \frac{p^2 (1 + \alpha)^2}{(p + \alpha p_0)^2}, \quad (30)$$

we then have

$$\frac{\bar{f}}{c_p} \frac{\partial}{\partial p} \left(\frac{Q_2}{\Gamma} \right) = \frac{\bar{f}^2 A_0}{RC} (1 + \alpha)^2 \frac{\partial}{\partial p} \left(p^2 \frac{\partial \zeta}{\partial p} \right) =$$

$$= K_T \frac{\partial}{\partial p} p^2 \frac{\partial \zeta}{\partial p} \quad (31)$$

This is of the same form as the vertical turbulent diffusion of vorticity given by eq. (25), and therefore, we may combine them into a single term. Taking $A_0 = 10^5$ m^2 sec$^{-1} = A_\nu$, $C = 110°\ A$, $\bar{f} = 10^{-4}$ sec^{-1}, $\alpha = 0.5$ we obtain $K_T = 0.7 \times 10^{-7}$ sec^{-1}. This is about one half of the value of K_ν in (25). Thus if we are to include the effect of the lateral diffusion of heat, we should also include the vertical diffusion of vorticity.

Another effect of the smaller scale turbulence is the vertical transfer of heat. Since the potential temperature is more conserved than the actual temperature, this heat transfer is expected to be proportional to the vertical gradient of the potential temperature. Since the mean potential temperature increases upward, the heat diffusion equation expressed in terms of the potential temperature will give a downward heat transfer. However, if the small scale turbulence is thermally driven, the heat transfer should be directed upward. Because of this uncertainty about its direction, we shall disregard this effect at the moment.

Still another heating factor is the heat exchange between the air and the ground. This effect can be represented by

$$Q_3 = c_p k_{0T} (T_g - T_0) \quad (32)$$

where T_g is the temperature of the ground and T_0 is the temperature of the air at ground level. This heating appears as a surface effect in the function E_0 in (22). The value of the coefficient k_{0T} may be assumed to be the same as the frictional coefficient k_0, which is taken as 10^{-5} sec^{-1}.

In considering the surface heating, we may assume T_g to be a known function of x and y at all times. On the other hand, the air temperature T_0 is unknown and is determined by the motion. Using the approximate relation (6), this effect is expressed in terms of the vertical gradient of ψ at the ground and Q_3 is given by

$$Q_3 = c_p k_{0T} \left(T_g + \frac{\bar{f}}{R} p_0 \left(\frac{\partial \psi}{\partial p} \right)_0 \right), \quad (33)$$

In this study, the ground temperature T_g is assumed to be given by

$$T_g = 245° + 48° \cos^2 \varphi + 14° \sin \frac{3\varphi}{2} \sin 2\lambda \quad (34)$$

in absolute degrees. The first two terms represent the zonal average and the third term the temperature departures due to two land and two ocean sectors, each 90° in extent.

7. The Grid Points

The size of the electronic computer and the available time determine the maximum number of grid points we can use to describe the motion of our atmosphere. The Computer used was the I.B.M. 704 at the General Electric Computation Laboratory, Lynn, Mass. It had an internal rapid-access memory of 8,192 36-binary digit words, together with 10 tape units and no drum. In order to make the time of computation for each time step of extrapolation not too long, and yet to have the vertical structure of the atmosphere described somewhat more accurately than by a two level model, we represented the atmosphere by four levels, namely 850, 550, 250, and 50 mb. Each of the first three levels represents the conditions in a layer of 300-mb thickness, while the 50 mb level represents the layer from zero to 100 mb. The reason for this division is that it was desired to have one level to represent the conditions in the more stable stratosphere, and another one for the tropopause level. These are assumed to be represented by the 50 mb level and the 250 mb level respectively, while the two levels at 850 and 550 mb represent the lower and middle tropospheric conditions.

Since one purpose of this study was to investigate the character of motion without introducing artificial boundaries, the spherical shape of the earth must be taken into consideration. We shall therefore use the conformal Mercator projection $x=\lambda$, $y=\int_0^{\varphi}\frac{d\alpha}{\cos\alpha}$. The advantage of this transformation is that it reduces the Laplacian operator to the simple form $(a\cos\varphi)^{-2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ and renders the finding of the Green's function more easy. However, it also has the disadvantage that by taking $\delta x\delta y$ constant the corresponding area on the earth becomes smaller and smaller as the latitude φ increases, and it requires too many grid points to represent the motion in higher latitudes. To avoid this difficulty we use one constant value for Δy in lower latitudes, from the equator to latitude 60°, and increase the values of Δx and Δy as φ increases. For the region between the equator and $\varphi=60°$, we take $\Delta x=\Delta y=0.1309$, so that there are 48 grid points around the latitude circle, the distance between two grid points being 7.5°. For middle latitudes this corresponds to a linear distance of about 500 km. For simplicity the value of Δx $(=\Delta y)$ is doubled at $j=11$ $(\varphi=60°)$, doubled again at $j=14$ and $j=15$, so that we have 48 points along the latitude circles for $0\le j\le 10$, 24 points for $j=11$, 12, 13, 12 points for $j=14$ and 6 points for $j=15$ $(\varphi=88.5°)$. Because of the cyclic requirement along the latitude circle, two auxiliary points are added for every j, one on each end, to facilitate the computation.

In order to compute the vorticity and the Jacobians for $j=15$, we need another row of six points at $j=16$. These points are very close to the pole indeed, although the pole itself is at infinity on the Mercator projection and therefore cannot be reached. Since we only want to use these six points along $j=16$ as auxiliary points, and since they are so close together on the actual map, we may take the average values of the stream function ψ and the temperature T of the six points along $j=15$ for the values at every one of these points. These mean values are also supposed to represent the values of the ψ and T at the pole. Thus we have 656 grid points for each level and 2,624 points altogether for the system.

8. The Initial State

For the study of the development of the general circulation of the atmosphere it may be desirable to take the atmosphere as being at relative rest and having a uniform temperature at the initial moment, and then begin the heating and investigate the development of the motion. However, such an approach is too time consuming and was beyond the monetary facilities of the project, we shall therefore start from a somewhat idealized state.

In this idealized initial state we assume that there already exists an average temperature gradient from north to south comparable to the mean state of the atmosphere, but no temperature contrast from west to east. The air temperature in contact with the ground is the same as the mean temperature of the ground and is represented by the first two terms of (34). The temperature decreases upwards with a lapse-rate given by (10). There also exists a mean zonal wind $\bar{u}$ at all levels. The vertical variation of $\bar{u}$ is in accordance with the thermal geostrophic wind relation in the troposphere, but with no vertical shear in the stratosphere (0mb < p < 250 mb). The variation of $\bar{u}$ with y is very small initially. The maximum zonal wind at the 250 mb level is 30 m sec^{-1}. Superimposed on this mean zonal flow is a somewhat random disturbance, represented by the stream function ψ^1. This ψ^1 is composed of nine wave components in the x-

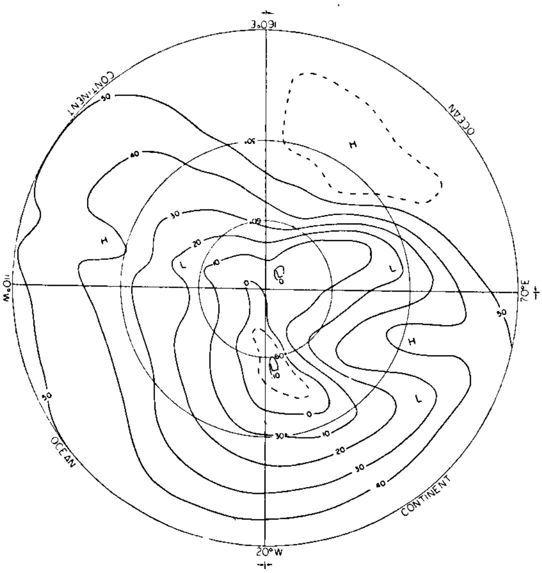

Fig. 2. Distribution of 850 mb stream function at time $t = 0$ at 10^6 m^2 sec^{-1} intervals.

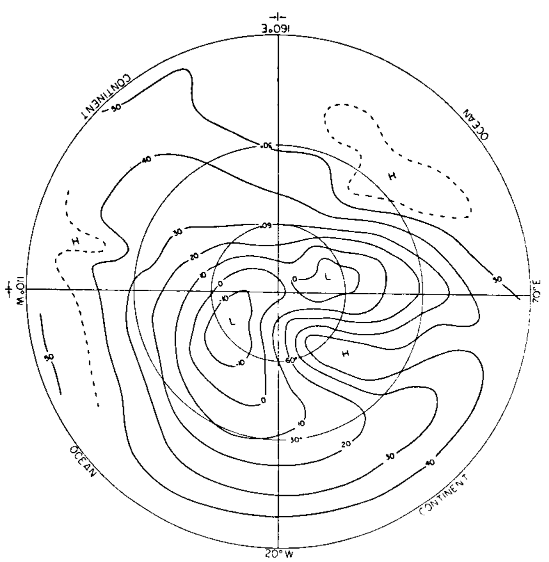

Fig. 3. 850 mb stream function distribution at $t = 24$ hours.

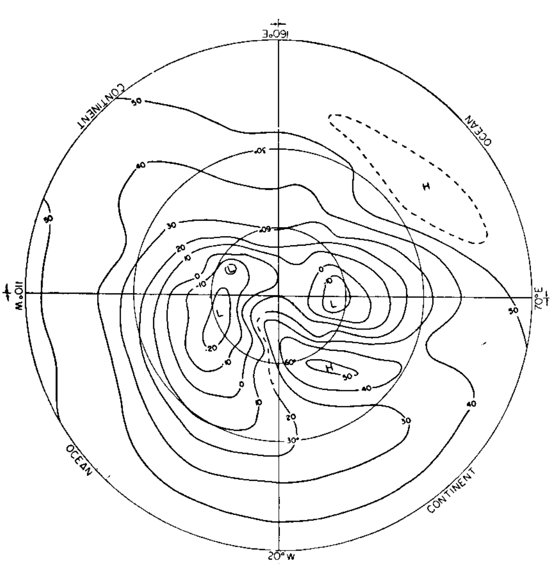

Fig. 4. 850 mb stream function distribution at $t = 48$ hours.

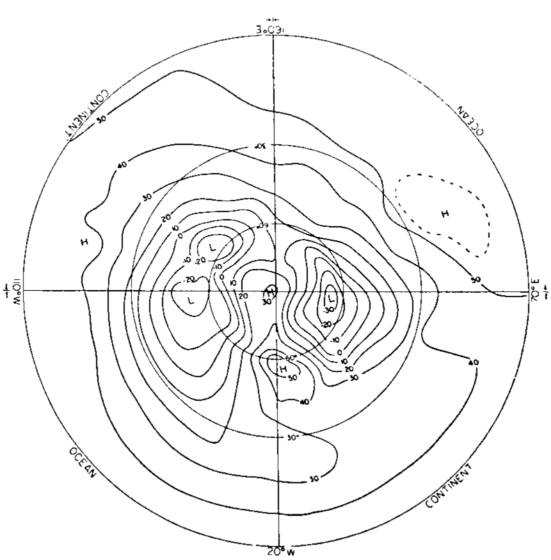

Fig. 5. 850 mb stream function distribution at $t = 72$ hours.

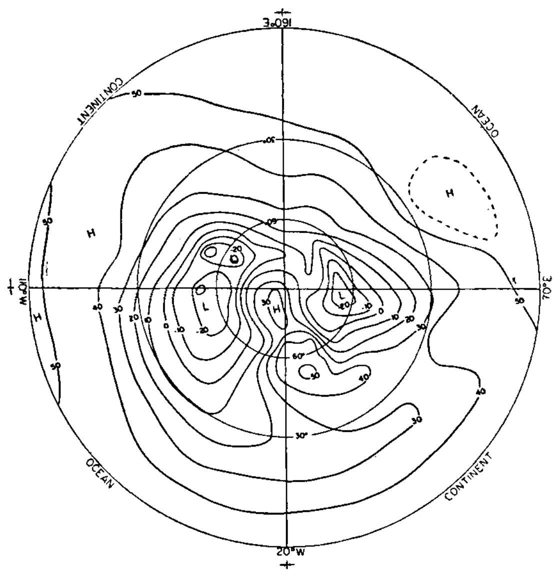

Fig. 6. 850 mb stream function distribution at $t = 96$ hours.

direction, the wave number m ranging from 1 to 9. The variation of ψ^1 with y is represented by sine and cosine function in y, the wave number ranging from 1 to 7. The amplitudes of the various wave components are determined so as to give the same amount of kinetic energy per unit mass for each wave component in x. The total kinetic energy of this disturbance corresponds to a mean perturbation velocity of 20 m sec^{-1}. The zero points of the various wave-components are shifted so as to give it a more random distribution. However, the disturbance so obtained is not very random, or shapeless, as we might like to assume. There is still large scale features present, as is indicated in figure 2 which gives the initial distribution of the total ψ at 850 mb. The prominent feature of this map are the two large troughs along the meridions 15° E and 75° E.

The disturbance represented by ψ^1 is assumed to be the same at all levels, so that the total ψ becomes more zonal at upper levels.

9. Development of the Flow

The rate of change of the composite stream function ψ is forecasted by eq. (22) through numerical integration. The time interval used in the integration is one hour and a half. The integration has been carried out for six days. The 850 mb flow pattern at 24, 48, 72, and 96 hours are represented in Figures 2—6. The following features of the developments may be noted:

a) The situation represented by Fig. 2, where the most prominent features are the two troughs situated along the longitudes 15° E

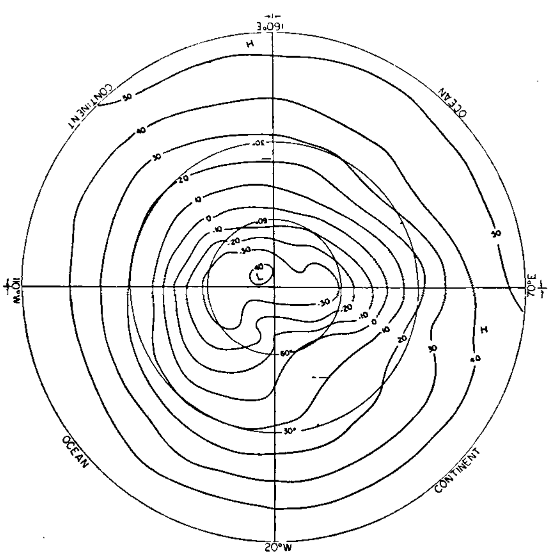

Fig. 7. 550 mb stream function distribution at $t = 48$ hours.

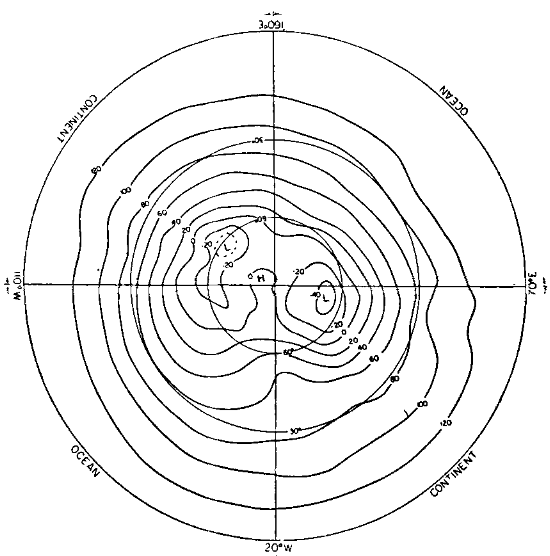

Fig. 8. 550 mb stream function distribution at $t = 96$ hours.

and 75° E[1], must be highly unbalanced because of the very short distance between them in one direction and very large distance in the other direction. Therefore it must undergo a readjustment of the distances. This adjustment takes the form of a rapid regression of the western trough.

b) The narrow ridge between the two troughs expanded into a fullsized high and moved toward the pole.
c) The two troughs developed very rapidly into two large centers, look more like the semi-permanent lows rather than the ordinary cyclones. Their latitudinal positions also correspond to the normal latitudinal position of the semi-permanent lows in winter.
d) On the third day (72 hours) the high has split up into two. The northern one moved over to the pole and stayed there.
e) Another closed low center is formed at the northern end of the weak trough along 155° W. Thus at the end of the fourth day there is a definite tendency to have three separate large lows developed.

Because of the much stronger zonal flow at the upper levels the changes of the flow patterns are not so pronounced as at 850 mb level. Figs. 7 and 8 represent the flow patterns at 550 mb after 48 and 96 hours, respectively. The general features of the developments are the same as that at the 850 mb level but become obscured because of the stronger zonal flow. The changes at 50 mb level are small during this period.

At the end of the 5th day, a certain computational instability has developed and it becomes not possible to carry the computation to a much longer period. Apparently this is connected with the change of grid distances at $j=11$ and $j=14$. It is hoped that this difficulty can be removed in the near future and we shall be able to extend this work further.

[1] The longitudes are for reference only and bears no fixed relation to the real earth.

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