# The Diurnal Variation of the Wind

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(Manuscript received March 9, 1959)

## *Abstract*

The equations of motion are numerically integrated under assumptions which represent the diurnal variation of the wind. The coefficient of eddy viscosity varies periodically with time as well as nonlinearly with height. The time variation is represented by two trigonometric terms giving a 20-fold change between maximum and minimum eddy viscosity. Several different functions were used to depict the vertical variation of the eddy viscosity, among which were an increasing, but bounded, exponential function, and also an exponential function which first increased with height and then decreased to a residual value. The boundary conditions imposed were, firstly, that the wind vanishes (or approaches a constant value) at the lower boundary; and, secondly, that the wind becomes geostrophic at some arbitrarily selected upper level.

The solutions were in general agreement with observed data and showed the spiral characteristic of the planetary boundary layer. However, there is disagreement between theory and observation in some important features.

## 1. Introduction

The wind structure in the lowest several thousand feet is of interest in large-scale processes as well as in micrometeorology. Because of the difficulties involved in attempting to obtain analytical solutions to the general Reynolds' equations for turbulent motion, various simplifying assumptions have been devised. The introduction of a coefficient of eddy viscosity $K$ has provided a substantial simplification. Moreover, the layer of frictional influence has been subdivided into the laminar, surface, and spiral layers, each possessing certain characteristic properties. Steady state analytical solutions have been provided for the spiral layer, or the combined surface-spiral layers, for which $K$ is a constant or a simple function of height. More recently, BUAJITTI and BLACKADAR (1957) have provided numerical solutions for the case of a coefficient of eddy viscosity which varies periodically with time as well as with height. By expressing vertical variations of the variables as finite difference ratios, the partial differential equations are reduced to a system of ordinary differential equations, which, in turn, is solved by electrical analogue. The accuracy of such results will ordinarily increase with decreasing size of the increment of height over which the derivatives are approximated by finite differences. On the other hand, decreasing the height interval increases the number of equations necessary to represent a given layer, thus requiring a greater number of integrators and potentiometers in the analogue computer. Buajitti and Blackadar were able to consider 5 levels, the first and last of which were specified boundary conditions. A two-fold time variation of the coefficient of eddy viscosity was provided by a single cosine term, while the vertical variation consisted of a nonlinear decrease or increase with height. Empirical data (for example, see MILDNER 1932) indicates that the coefficient of eddy viscosity first increases with height, reaching a maximum near 250 meters; and then it decreases to a residual value. Observations of the coefficient of eddy conductivity (1956) indicate diurnal variations between 10 and 100-fold (depending on the height) which require two or more trigonometric terms for representation. In order to provide such variations in the eddy viscosity, a very large analogue computer would be needed. Buajitti and Blackadar essentially deal with the upper portion of the friction layer and show that better agreement is obtained between theory and observation when both the mean eddy viscosity and its diurnal variation decrease with height.

It was the purpose of this investigation to consider theoretical solutions for the diurnal variation of the wind which provide for a greater resolution and range in the vertical. Because of the complexity of the differential equations, numerical solutions were sought by means of CRC 102A electronic digital computer.

## 2. The Fundamental Equations

Since we are primarily concerned with the diurnal variation of the wind in the friction layer, horizontal differences are to be neglected and also the vertical variation of the geostrophic wind. Hence the equations of motion may be written in the form

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= fV + \frac{\partial}{\partial Z}\left(K\frac{\partial U}{\partial Z}\right) \\ \frac{\partial V}{\partial \tau} &= f(V_g - U) + \frac{\partial}{\partial Z}\left(K\frac{\partial V}{\partial Z}\right) \end{aligned} \tag{1}$$

where $U$ and $V$ are the horizontal velocity components; $f$, the Coriolis parameter; $K$, the kinematic coefficient of eddy viscosity; and $V_g$ is the geostrophic wind. For a simpler and broader interpretation of results, the following transformation of coordinates is made:

$$\begin{aligned} \tau &= \sigma t \qquad & Z &= 100qz \\ u &= U/V_g \qquad & v &= V/V_g \end{aligned} \tag{2}$$

The transformed equations are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sigma f v + \frac{10^{-4}\sigma}{q^2}\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) \\ \frac{\partial v}{\partial t} &= \sigma f(1-u) + \frac{10^{-4}\sigma}{q^2}\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) \end{aligned} \tag{3}$$

The boundary conditions imposed are that the wind vanishes (or is constant) at some reference level, $z=0$, presumably at or near the ground; and secondly, that the wind becomes geostrophic at an arbitrarily selected upper level. Thus

$$\begin{aligned} u &= 0, \quad v = 0, \quad \text{at } z = 0 \\ u &= 1, \quad v = 0, \quad \text{at } z = z_n \end{aligned} \tag{4}$$

The coefficient of eddy viscosity is assumed to have the form

$$K = g(t)\ k(z) \tag{5}$$

where

$$g(t) = 1 + b\ (\sin \omega t + c \sin 2\omega t) \tag{6}$$

Several different forms were tried for $k(z)$

$$k_1 = q^2 a_1[1 - a_2 \exp(-10^2\, a_3 q z)] \tag{7}$$

$$k_2 = q^2 a_1\{a_4 + a_5 \exp[-10^4\, a_6 q (z - z_m)^2]\} \tag{8}$$

With the exception of $a_2$, the $a_i$, $b$, $c$, and $\omega$ are constants chosen to give a variation of $K$ comparable to observed values. Two different choices were made for $a_2$; in Case 1, $a_2$ is constant; in Case 2, $a_2$ is of the form $a_2 = a_2' + a_2''$ $(\sin \omega t + c \sin 2\omega t)$. The latter choice gives a greater variation of $K$ with height during the time when $K$ is maximum. Estimations (STALEY, 1956) of the coefficient of eddy conductivity have indicated a 100-fold increase with height during the daytime, but only a 10-fold increase during night-time. A similar variation may apply to the eddy viscosity.

The second form $k_2(z)$ gives a $K$ which first increases with height, reaching a maximum at $z_m$, then decreases and becomes essentially constant at still greater heights. Such a distribution is similar to that found by MILDNER (1932).

For purposes of solving the equations numerically, the vertical derivatives of Equation (3) may be replaced by centered finite differences, giving the system of equations

$$\begin{aligned} \frac{du_i}{dt} &= \sigma f v_i + \frac{10^{-4}\sigma}{q^2}\left[\left(\frac{\partial K}{\partial z}\right)\frac{u_{i+1} - u_{i-1}}{2h} + \right. \\ &\qquad \left. + K_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right] \\ \frac{dv_i}{dt} &= \sigma f(1 - u_i) + \frac{10^{-4}\sigma}{q^2}\left[\left(\frac{\partial K}{\partial z}\right)_i \frac{v_{i+1} - v_{i-1}}{2h} + \right. \\ &\qquad \left. + K_i \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}\right] \end{aligned} \tag{9}$$

The levels $i=0$ and $i=n$ correspond to the lower and upper boundary conditions, respectively. Thus, (9) and (4) constitute a system of $2n$ equations. Now the friction layer may extend to several thousand meters. On the other hand, good resolution of the wind field, at least near the surface, probably requires a vertical increment of less than 10 meters, which leads to a system containing several hundred equations. In addition, a small time increment is required to maintain computational stability in the numerical solution; thus, the services of a large, high-speed computer are needed.

By transforming the vertical coordinate to $\ln z$, it is possible to obtain better resolution at low levels and still keep the number of equations to a more practicable value for the computer on hand. The transformation is straightforward (see HALTINER, 1958); and the resulting equations are similar in form to (9).

## 3. Constants

For convenience, consider the units of (1) to be of the c.g.s. system. Then a choice of $\sigma = 3{,}600$ makes the unit of time in (3) an hour. The parameter $q$ essentially disappears from the final equations; however, its value is related to the overall magnitude of the eddy viscosity by Eqs. (7) and (8) and the size of the vertical height increment. The parameter $q$ implies units of $q$-meters for $z$; i.e., $q=2$ gives 2-meter units for $z$, etc. Thus, as $q$ increases, the coefficient of eddy viscosity increases, but the vertical height increment over which the finite differences are taken also increases. It was necessary to resort to this scaling because typical values of the eddy viscosity required prohibitively small integration time increments for the capacity of the computer available. With $l$ representing the vertical increment of $\ln z$, a choice of $l = \ln 2$, together with $q=3$ would imply successive levels in a geometric series; i.e., 3, 6, 12, 24, ... meters.

The choices $c=0.3$ and $b=0.8$ give rise to a 20-fold diurnal variation of the eddy viscosity, with $\omega = 2\pi/24$. With regard to the vertical variation of $k$, the constant value $a_2 = 0.9$ gives a ten-fold variation of $K$ with height. On the other hand, if $a_2$ is allowed to vary with time as indicated in Section 2, a choice of $a_2' = 0.95$ and $a_2'' = .0352$ (Case 2) gives a 10-fold vertical variation of $K$ with its minimum diurnal value, and a 100-fold variation with height at the time of maximum $K$. The maximum value of $K$ obtained in this case is $1.13 \times 10^3\, q^2$ cm²/sec.

In connection with the function $k_2$ of Eq. (8), the values $a_4 = .01$ and $a_5 = .09$ give essentially a 10-fold variation with height first increasing to a maximum at $z = z_m$ and then decreasing as the altitude increases further. The choice of $z_m = 64$ $q$-meters will be referred to as Case 3, and $z_m = 16$ $q$-meters as Case 4. The maximum value of $K$ which was practicable on the available computer was $256\, q^2$ cm$^2$/sec in Case 3 and $64\, q^2$ cm$^2$/sec in Case 4. To obtain a magnitude of $K$ comparable to observed values of the order of $10^4$ to $10^5$ cm$^2$/sec, it is necessary to put $q$ in the range 10 to 20. For the former value and $l = \ln 2$, the vertical spacing will correspond to the sequence of heights 5, 10, 20, 40 ... etc. meters, while the latter gives the sequence ... 10, 20, 40, ... meters.

The initial point corresponds to the lower boundary at which the velocity is assumed to vanish. This point may be considered an arbitrary reference level from which the points above are measured. It is apparent that as $q$ increases, the vertical resolution decreases, particularly in the "surface layer" near the lower boundary. One may retain a greater degree of resolution at lower levels by decreasing the value of $l$; however, the vertical extent is correspondingly decreased, assuming no increase in the total number of equations. In all the cases studied the latitude was taken to be 45°.

## 4. Results and Conclusions

The velocity distributions obtained for Cases 1 through 4, representing the different forms of the coefficient of eddy viscosity, have very similar patterns. At any particular hour, the vertical variation of the wind shows the characteristic spiral; however, the shape varies significantly from the time of maximum to minimum $K$, as indicated by Figure 1. Note here that velocities are expressed relative to the geostrophic wind $V_g$ which is represented as 100 units in all figures. The time scale is arbitrary except that 0600 and 1400 correspond to the times of minimum and maximum $K$, respectively. In agreement with observation, the "night-time" cross-isobar angle exceeds the "daytime" case at low levels. Figures 2 and 3 show the diurnal variation of the wind at successive levels for Cases 1 and 3, respectively. The results are very similar in spite of the different vertical variation of the eddy viscosity. This is partly due to the fact that most of

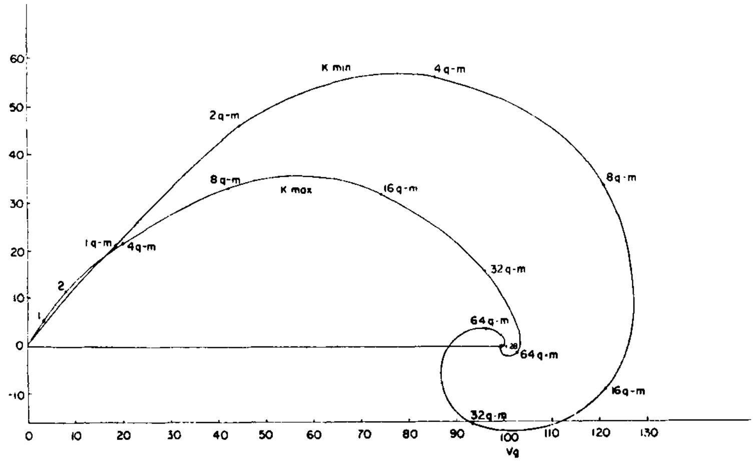


Fig. 1. Theoretical wind distributions as functions of height at the times of maximum and minimum eddy viscosity for Case 1.

the variation in the wind field takes place below the 64 $q$-meter level where $K$ is assumed to reach a maximum for Case 3.

Since the selection of $q$ is arbitrary, it is of interest to note the effect of varying this parameter. An increase in $q$ corresponds to an increase in the overall magnitude of the eddy viscosity and an increase in the elevations to which the ellipses of Figures 2, 3 and 4 apply. It is apparent that as the eddy viscosity increases, the height at which the wind becomes geostrophic also increases, as well as the height of the maximum diurnal variation of the wind. A comparison of Figures 2 and 3 shows that the diurnal variation of the wind is smaller in Case 1 than in Case 3, even though the overall magnitude of $K$ is less in the latter. However, the vertical extent to which deviations from the geostrophic are observed is greater in Case 1 than in Case 3. Figure 4 shows some observational data evaluated by Buajitti and Blackadar. Note that the shape and orientation of the ellipses of Case 3 from 8 $q$-m upward agree with the observed orientation from 15 ft. upward moderately well. However, the position of the wind vector at the time of maximum eddy viscosity (presumed to be near midday) is approximately opposite to the position of the observed wind vector at 1400. This disagreement is brought out in Figure 5 which shows the wind speed as a function of height for Case 3. Note that the low-level wind speed corresponding to the time of minimum eddy viscosity exceeds that corresponding to the maximum eddy viscosity. This condition is observed to hold above the "crossover", but not in the surface layer immediately adjacent to the ground. The zero momentum at the lower boundary tends to be propagated upward and the maximum penetration is observed at the time of maximum eddy diffusivity. This is similar to the diurnal temperature wave which is propagated upward from the lower boundary where heating takes place. Here there is in addition a source of momentum in the pressure and Coriolis forces. In passing, it may be noted in Figure 5 that at the time of minimum eddy viscosity the wind speed exceeds the geostrophic value by 25 % at one level and the strong shear occurs in the jet-type profile. Figure 6 shows the results for Case 4. The overall pattern is similar to Case 3 especially at and below 8 $q$-meters. A comparison of the 32 $q$-m ellipse of Case 3 and the 16 $q$-m ellipse of Case 4 shows a similarity in general shape; however, the time positions of the wind vectors have shifted clockwise on the ellipse about 60°. At 16 $q$-m in Case 4, the position of the wind vectors at the time of maximum and minimum eddy viscosity agree moderately well with the observed 1400 and 0600 winds at 3,650 feet; however, the orientation of the ellipse is only in fair agreement with observed orientation.

At the 32 $q$-m level there appears to be about $1^1/_4$ cycles to the ellipse during the 24 hr.

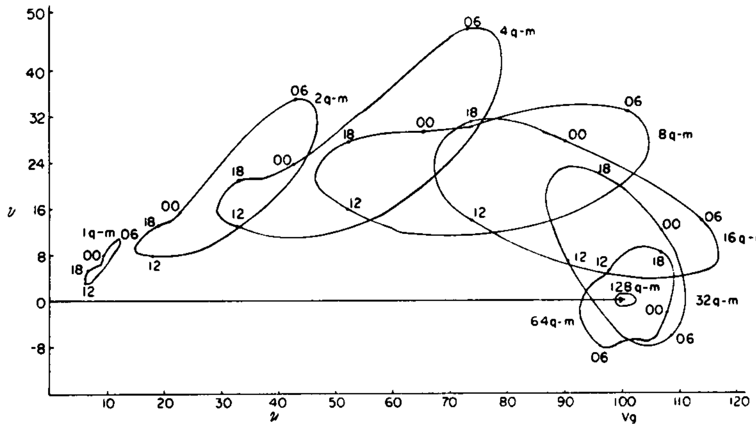


Fig. 2. The theoretical diurnal variation of the wind for Case 1.

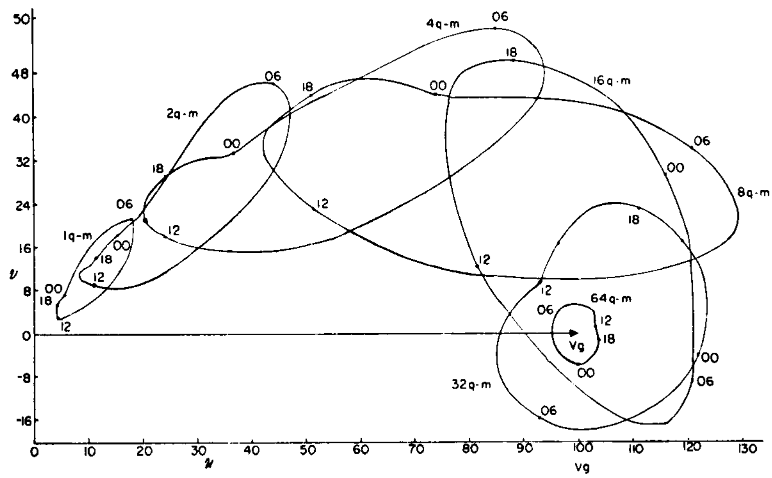


Fig. 3. The theoretical diurnal variation of the wind for Case 3.

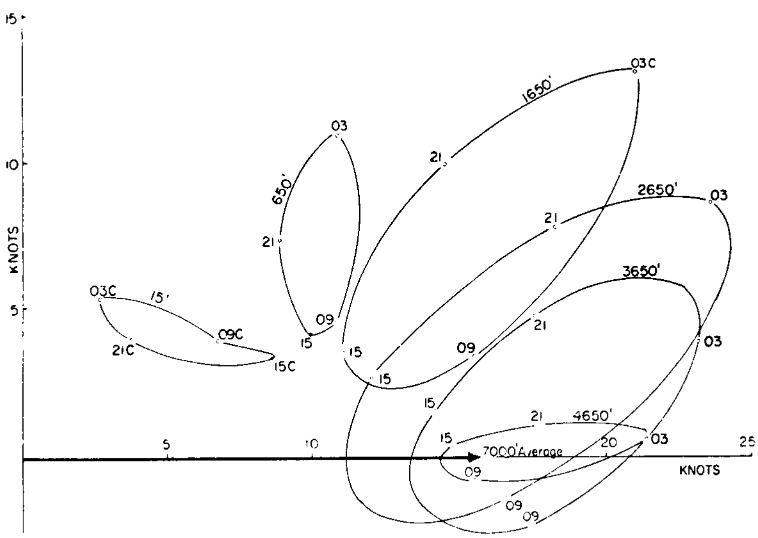


Fig. 4. The observed diurnal variation of the wind, after Buajitti and Blackadar (1957).

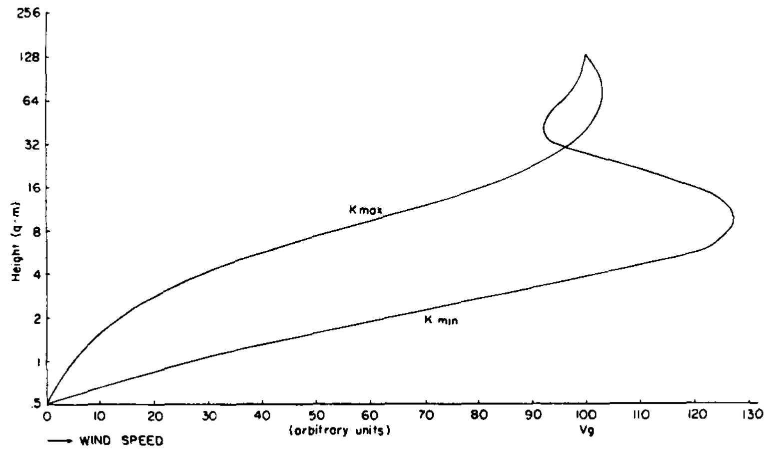


Fig. 5. The wind speed as a function of height at the times of maximum and minimum eddy viscosity in Case 3.

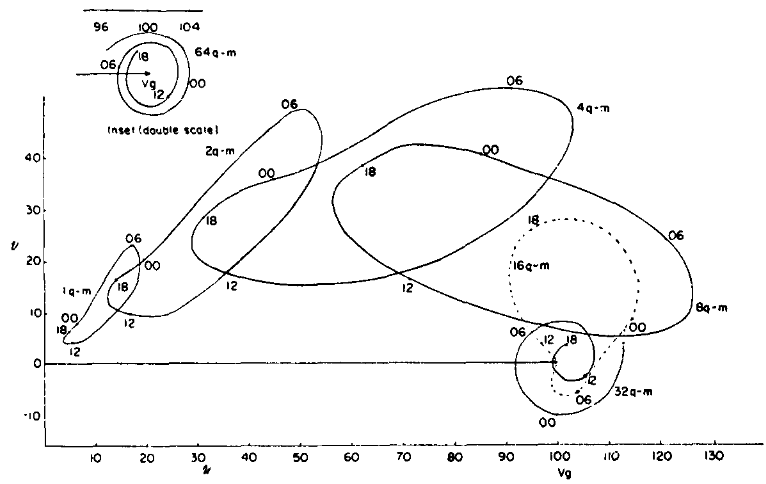


Fig. 6. The theoretical diurnal variation of the wind for Case 4.

period, while at 64 $q$-m the wind vector appears to pass through two complete cycles in a 24 hour period. These features did not occur in Case 3 even though the vertical variation of the eddy viscosity is similar. This phenomenon may be due to the initial conditions which were taken from Case 3 at 0600 hrs; however, this does not appear to be very likely. Time restrictions did not permit another 24 hour computation. Some computations were also made in which the geostrophic wind was allowed to vary linearly with height; however the resulting differences were minor and did not correct any of the significant dic-crepancies described above.

The results of this study show some agreement between observation and the theoretical models representing the diurnal variation of the wind. However, there is disagreement in some significant features of the wind field.

One may infer that coefficients of eddy viscosity of the type used here are inadequate to fully explain the diurnal variation of the wind. Perhaps some other type of function for $K$ will provide better agreement or a two-layer model may be necessary. A study of this problem is being conducted at present and will be reported on at a later date.

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