Note Concerning the Influence of Rotation on Convection

By VICTOR P. STARR, Massachusetts Institute of Technology

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My purpose in writing this note is not to present a model of an actual convective process, but instead to work with a particularly simple hypothetical scheme with the intent of illustrating the action of one single factor, namely, rotation. The material presented below may therefore be thought of as a scientific parable—a vehicle for conveying a thought. More detailed models embracing the effects of other pertinent factors in addition to rotation have been presented by many persons (see, e.g., SALTZMAN 1958 and references there given). There is much to be gotten, however, from analyses which focus one's view on individual items singly, to the extent that this can be done.

The work described herein was completed many years ago, during an early period of my career. Accordingly, the general outlines of it follow along the trend of my interests at the time, which ran to the study of the readjustment of simple fluid systems under the conservation of vorticity (see STARR 1939). In actuality, the treatment of the present topic depends upon three conservation principles—conservation of mass, conservation of vorticity, and conservation of energy.

We consider an ideal fluid layer of uniform depth D_0 and density ϱ constant throughout. This layer lies next to the plane ground surface beneath another layer of ideal fluid of great depth as compared with D_0 and of constant density $\varrho' > \varrho$. The presence of rotation is allowed for by assuming that the Coriolis parameter, f, is constant and uniform. We thus have an unstable discontinuity in density in the initial state which, moreover, is a state of rest with respect to the ground. The problem now is to say something concerning the shape and size of the finite disturbances which might develop at the internal boundary of the two fluids, if the latter is slightly perturbed. The entire point of this paper is contained in the suggestion that: *No given hypothetical disturbance can grow to finite size spontaneously from infinitesimal beginnings, if its growth requires the violation of one or more of the three conservation principles for mass, vorticity and energy.* Our technique is to assume that mass and vorticity conservation is fulfilled for each finite disturbance selected, and then to examine the fulfillment of the energy requirement. Other techniques could presumably result from other choices in this regard.

Let us suppose that there exists in some appropriate sense a size-parameter related to the horizontal dimensions of the disturbances, which is significant for our problem. Let it be supposed further that we do not run afoul of our energy criterion for a certain range of the size-parameter, but that beyond a given critical point energy considerations are violated. As this point is approached, presumably a state of indifference as to stability is likewise approached. Thus if some such behavior is present, then in seeking the critical point we are justified in using equilibrium concepts in regard to the hydrostatics of the system, and also in regard to the geostrophic balance as will be seen later.

As a simplest trial, we consider a finite disturbance of the form

$$D = D_0 + A \cos \frac{2\pi}{L} y \tag{1}$$

where use is made of a cartesian coordinate system with the x- and y-coordinates horizontal and z vertical, while D is the deformed depth of the lower layer, A is an amplitude factor and L is the wave length which now serves as the horizontal size-parameter. It is clear that (1) represents sinusoidal waves independent of the x-direction.

For a unit column it is easy to show that the change in potential energy occasioned by the presence of the disturbance is

$$\frac{1}{2} g (\varrho' - \varrho)(D^2 - D_0^2) \tag{2}$$

g being the acceleration of gravity considered uniform and constant. This is the amount of potential energy *released* by the disturbance. By use of (1) it is possible next to find the area average of this energy release, namely,

$$\Delta PE \equiv \frac{1}{L} \int_0^L \frac{1}{2} g (\varrho' - \varrho)(D^2 - D_0^2)\, dy =$$

$$= \frac{1}{4} g (\varrho' - \varrho) A^2 \tag{3}$$

We observe that due to the sinking of heavy fluid and the rising of an equal volume of light fluid, there is a lowering of the center of gravity, resulting in a decrease of potential energy, which is *independent of the wave length* L.

The equation of motion for the x-direction may in our case be simplified to the statement that

$$\frac{du}{dt} = fv \equiv f \frac{dy}{dt} \tag{4}$$

the pressure term being zero due to the uniformity of the system along x. Here u and v are the x- and y-components of particle velocity and t is time. An individual integration with respect to time between the undisturbed (initial) and disturbed (final) state may now be made so that

$$u - u_0 = f\,(y - y_0) \tag{5}$$

The significance of u_0 is the value of the x-velocity component in the initial state which in our model is zero. Likewise y_0 is the initial and y the final coordinate of the same individual particle. Equation (5) is in fact tantamount to a statement of the vorticity principle, namely that the potential vorticity remains unchanged. Its equivalence for our problem with this latter principle may easily be shown by differentiating it and combining it with the continuity equation in the form given later (see equation (14) below).

The equation of motion for the y-direction in the lower fluid is

$$\frac{dv}{dt} = -fu - \frac{1}{\varrho}\frac{\partial p}{\partial y} = -fu + g\,\frac{\varrho' - \varrho}{\varrho}\,\frac{\partial D}{\partial y} \tag{6}$$

where p is pressure as will be discussed presently, all motions and hence the disturbances of the pressure in the upper fluid are neglected for reasons of convenience. According to the borderline or marginal stability arguments already made, we are justified in replacing the pressure by its hydrostatic equivalent as shown in the second equality. It is to be noted that once this is done, the y-component of the pressure gradient is no longer dependent upon z. When this fact is combined with notion that in accordance with marginal stability concepts the left-hand side of (6), namely, dv/dt may be made as small as we please, if follows that u and therefore in its turn $y - y_0$ is independent of z, according to (5).

We may thus at once write the equation of continuity under the several conditions of our problem in the simple form

$$D = D_0 \frac{\partial y_0}{\partial y} \tag{7}$$

since columns initially vertical remain so in the disturbed state. It should be kept in mind that here again it is the initial and final states which are compared in quasi-Lagrangian fashion (see STARR 1946).

We may now proceed to examine the concepts concerning kinetic energy implied by the foregoing material. Although it is by no means a necessity but merely a convenience, we shall neglect the kinetic energy of the upper fluid *in toto*. This is justified as follows. Marginal stability considerations point to the smallness of the y- and z-components of velocity not only in the upper but also the lower fluid as well. As far as the x-component goes, the analogue of equation (5) for the upper layer would involve only small values of $y-y_0$ because the divergences due to the disturbance would be spread over a great depth. In the end it would result that the u kinetic energy is small.

Giving due consideration to what has been said, we now write the expression for the kinetic energy of the x-component of motion in the lower layer as follows, with the aid of (5).

$$\frac{1}{2}\varrho u^2 D = \frac{1}{2}\varrho f^2 D(y-y_0)^2 \tag{8}$$

Just as in the case of the potential energy change, this quantity is for a given unit column.

The continuity equation (7) becomes, upon elimination of D by (1), an ordinary differential equation for y_0 in terms of y, i.e.,

$$\frac{dy_0}{dy} = 1 + \frac{A}{D_0}\cos\frac{2\pi}{L}y \tag{9}$$

whose solution is

$$y - y_0 = -\frac{AL}{2\pi D_0}\sin\frac{2\pi}{L}y \tag{10}$$

where from symmetry considerations we drop the constant of integration.

We now insert the expression (1) for D and the expression (10) for $y-y_0$ into (8) and average over one wave length, so that

$$\Delta KE \equiv \frac{1}{L}\int_0^L \frac{1}{2}\varrho f^2 D(y-y_0)^2 = = \frac{\varrho f^2 A^2}{16\pi^2 D_0}L^2 \tag{11}$$

Upon appeal to the corresponding expression for ΔPE in (3), it is seen that whereas this latter quantity is independent of L, the expression just found shows that ΔKE increases as the square of L. Since, as one should expect, the amplitude A occurs squared in both quantities, and since all other factors are constants, the principle of conservation of energy carries the implication that values of L exceeding a certain unique limiting value, L_c, are not admissible. The critical wave length thus turns out to be

$$L_c = \frac{2\pi}{f}\sqrt{gD_0\frac{\varrho'-\varrho}{\varrho}} \tag{12}$$

If a diagram were to be made using energy per unit area as the ordinate and wave length as the abscissa, a graph of ΔPE would appear as a horizontal straight line, while that of ΔKE would constitute an arc of a parabola in the vicinity of L_c where the two curves intersect.

We have thus achieved a demonstration, without resort to linearization, of the fact that rotation produces an inhibition of large convective cell sizes in our system, and it is rather obvious why this is so. The kinetic energy developed is uniquely determined by the kinematics of the deformations. So also is the potential energy released. However, the kinetic energy increases rapidly with cell size, while the potential energy does not, depending upon the amplitude merely.

Actually, for cell sizes slightly smaller than L_c there is an excess of potential energy released, over and above what is required to set up the prescribed kinetic energy of the x-component of motion. It is to be presumed that this excess is used to generate significant amounts of kinetic energy of the y- and z-components of motion, thus implying the presence of stronger instability for smaller cell sizes. Probably the strongest instability would manifest itself for wave lengths tending toward zero, although in real fluids viscosity no doubt exerts strong damping effects in this vicinity.

Many variations of this demonstration can be, and in a number of particulars have been, made by myself and by my students. The kinetic energy of the upper fluid can be taken into account. The parallel rolls represented by (1) may be replaced by a square array. The concentration of the instability in a discontinuity may be replaced by a continuous gradation of density. As may easily be guessed, most of such alterations increase vastly the complexity of the manipulations, without adding much that is vital to the philosophical import of the results.

It is not without a certain utility to return to equation (6), and to verify in *a posteriori* fashion some of the assumptions made previously. Let us impress, in explicit fashion, now, the stipulation that the motion corresponding to the limiting case when L approches L_c, should be geostrophic, by dropping dv/dt. After differentiation we have

$$0 = -f\frac{\partial u}{\partial y} + g\frac{\varrho' - \varrho}{\varrho}\frac{\partial^2 D}{\partial y^2} \qquad (13)$$

Through differentiation of (5) and substitution from (7) it follows that

$$-\frac{\partial u}{\partial y} = f\frac{D - D_0}{D_0} \qquad (14)$$

Equation (13) may now be written in the form

$$\frac{gD_0}{f^2}\cdot\frac{\varrho' - \varrho}{\varrho}\cdot\frac{d^2 D}{dy^2} + D = D_0 \qquad (15)$$

whereupon it is seen that we may actually *derive* an expression for D of the form (1) provided, of course, that $L = L_c$.

From a more general viewpoint the material presented suggests a number of points for discussion. Some, at least, of these topics are the following ones.

1. The kind of size criterion developed here probably has little significance for cumulus-type convection in the atmosphere. Such convection represents much too small cell sizes to be influenced by the earth's rotation in the manner suggested above.

2. The larger convective units in the atmosphere (cyclones and the like) are more nearly of a scale for which the rotation is apt to be a governing factor from the energy standpoint. However, due to various altered and more complicated circumstances in the actual atmosphere, any computation of actual magnitudes from the theory here given, for purposes of comparison with observed data or with results of other theories which include other effects, is not justified and could be misleading.

3. Basically, it is the development of horizontal circulations due to divergence and convergence, in conformity with the Bjerknes circulation theorem which energywise brings about a profound difference between the rotating versus the nonrotating case of convection. These additional motions require energy fot their generation, which energy furthermore increases rapidly with cell size on an area-average basis, and leads to the results obtained. (Compare STARR 1955.)

4. A number of more sophisticated theories of convection deal with the steady motion of a fluid which is maintained against friction. In cases where rotation is present the horizontal circulations are again present and now it is their maintenance against friction which consumes an extra amount of energy.

5. The notion seems to persist in some quarters that convection necessarily depends upon fair-sized departures from hydrostatic equilibrium, and that therefore one is not at liberty to use the hydrostatic principle in studying convective processes. This is not a justifiable view. Actually any system in which the relative amount of kinetic energy of vertical motion developed is small, is sufficiently close to hydrostatic equilibrium to permit one to use this principle. In such cases when growth of kinetic energy occurs, it is the horizontal and not the vertical unbalance which plays an essential part. The large motions of the atmosphere possess this character.

6. The gross significance of what has been said does not reside cruicially in the precise fulfillment of the several approximations made for dealing with an idealized model. For conditions which depart somewhat from those postulated the corresponding expressions for energy, etc., would no longer possess the same simplicity, perhaps, but their fundamental nature would still remain about the same and lead to comparable physical conclusions.

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