A Note on General Classification of Motions in a Baroclinic Atmosphere

By A. S. MONIN and A. M. OBUKHOV, Institute of Physics of the Atmosphere, Acad. Sci., U.S.S.R.

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Резюме

На основе решения задачи о малых колебаниях бароклинной атмосферы дается общая классификация основных типов динамических процессов в атмосфере (горизонтальные вихревые движения, гравитационные и акустические волны). Найден общий вид инварианта, с помощью которого может быть рассчитано при произвольных начальных данных предельное стационарное состояние атмосферы без анализа волновых процессов, обуславливающих перестройку полей. Прогностические уравнения могут быть выведены путем вычисления производной по времени от „инварианта" с учетом квадратичных членов. Выясняется „фильтрующая" роль квазистатического приближения: оно „отфильтровывает" внутренние акустические волны и несколько завышает частоты гравитационных волн. Показано, в частности, что время установления квазистатического равновесия в атмосфере составляет всего лишь несколько минут.

Abstract

In the present note we make an attempt to classify the basic types of dynamical processes in the atmosphere such as gravitational and acoustic waves and horizontal eddy motions by solving the problem of small oscillations in the baroclinic atmosphere.

We have found a certain invariant with the help of which one can estimate the final stationary state of the atmosphere under arbitrary conditions without the analysis of the wave processes which cause the field change. The most natural way of obtaining the special equations for the numerical forecasting is that of computing the first derivative in time of the invariant in the second approximation (quadratic terms being taken into account).

The filtrating role of the quasi-static approximation is being investigated in the note. Quasi-static approximation filtrates the internal acoustic waves and somewhat increases the frequencies of the gravitational waves.

It has, in particular, been shown that the time required for establishing the quasi-static balance in the atmosphere is equal to some minutes.

For hydrodynamical weather forecasting the simplified dynamical equations resulted from "quasi-static" and "quasigeostrophic" approximations are used (KIBEL, 1957). The equations are of the first order in time while the basic system of the hydrodynamical equations is of the fifth order. In the process of simplifying the basic equations some solutions, and namely those corresponding to the rapid wave processes, are excluded (or "filtered"). Charney who introduced the term "filtration" was the first to indicate this circumstance (CHARNEY, 1948).

To justify the above mentioned simplifications one needs to find out the nature of the rapid wave motions and show the mechanism of a field "adaptation". "Adaptation" is the process when the atmosphere changes for a short period from an arbitrary state characterized with five independent functions (a velocity vector, pressure and temperature) into quasibalanced state with certain fields agreeable to each other. The term "adaptation" has been introduced by OBUKHOV (1949) for consideration of dynamical processes in the two-dimensional model of the atmosphere. Similar consideration has been given to the three-dimensional atmosphere by Kibel and Monin in the works (MONIN, 1958) and (KIBEL, 1955), the hypothesis of "quasi-static character" being used ad hoc.

In the present note we make an attempt to classify the basic types of dynamical processes as horizontal vortices and gravitational and acoustic waves by solving the problem of small oscillations of the baroclinic atmosphere; the filtering role of the quasi-static approximation is being investigated.

Let us write the system of dynamical equations as follows:

$$\left.\begin{aligned} &\varrho\frac{du}{dt} = -\frac{\partial p}{\partial x} + l\varrho v; \quad \varrho\frac{dv}{dt} = -\frac{\partial p}{\partial y} - l\varrho u; \\ &\mu\varrho\frac{dw}{dt} = -\frac{\partial p}{\partial z} - g\varrho \\ &\frac{\partial \varrho}{\partial t} = -\left(\frac{\partial \varrho u}{\partial x} + \frac{\partial \varrho v}{\partial y} + \frac{\partial \varrho w}{\partial z}\right); \quad \frac{dp}{dt} = c^2\frac{d\varrho}{dt} \end{aligned}\right\} \qquad (1)$$

where x, y, z and t are Cartesian coordinates and time; u, v and w are the velocity components; p is the pressure; ϱ is the density, g is the acceleration of gravity; $l = 2\omega_z$ is the Coriolis parameter (a constant); $c^2 = \varkappa p/\varrho$ is the adiabatic velocity of sound squared; $\varkappa = \frac{c_p}{c_v}$ is the specific heat ratio. Into the third dynamical equation we introduce the auxiliary parameter the actual value of the parameter is equal to unit, but putting formally $\mu = 0$ we might obtain the equation system in a quasi-static approximation.

Let us take the relatively motionless state of the atmosphere with regard to the rotating coordinate system as the "main state" for which $\bar{u} = \bar{v} = \bar{w} = 0$, $\bar{p} = \bar{p}(z)$ and $\bar{\varrho} = \bar{\varrho}(z)$ are given functions of the height z connected by the static equation. Let us consider small deviations from this state which can be characterized with ψ, φ, π, χ, σ having the following sense:

$$\left.\begin{aligned} &\bar{\varrho}u = -\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x}; \quad \bar{\varrho}v = \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}; \\ &\bar{\varrho}w = \chi; \quad \pi = p - \bar{p}; \quad \sigma = \varrho - \bar{\varrho} \end{aligned}\right\} \qquad (2)$$

Using the linearization method for the dynamical equations (i. e. eliminating the quadratic terms from (1)) the below equation system can be obtained:

$$\left.\begin{aligned} &\frac{\partial \psi}{\partial t} = -l\varphi; \quad \frac{\partial \varphi}{\partial t} = l\psi - \pi; \\ &\frac{\partial \pi}{\partial t} = -c^2 \Delta_1 \varphi - \beta\chi - c^2\frac{\partial \chi}{\partial z} \\ &\mu\frac{\partial \chi}{\partial t} = -\left(\frac{\partial \pi}{\partial z} + g\sigma\right); \quad \frac{\partial \sigma}{\partial t} = -\Delta_1\varphi - \frac{\partial \chi}{\partial z} \end{aligned}\right\} \qquad (3)$$

where

$$\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \beta = (\varkappa - 1)g + \frac{dc^2}{dz}$$

Note that $\beta = \varkappa R\,(\gamma_a - \gamma)$ is the main characteristic of the stratification of the atmosphere $\left(\gamma_\alpha = \frac{\varkappa - 1}{\varkappa}\frac{g}{R}; \ \gamma = -\frac{\partial T}{\partial z}\right)$. The system (3) is of the fifth order in time as well as the basic system (1) is. It approximately describes the evolution of disturbances in time*). The boundary conditions according to the coordinate z can be given in the following form:

$$\chi = 0 \quad (z = 0); \qquad (4);$$

$$\chi \to 0 \quad (z \to \infty) \qquad (5)$$

(the conditions of the vanishing of the vertical velocity on the lower boundary and that of the mass flux at the infinity).

For solution of the Cauchy problem five initial conditions should be given if $t = 0$:

$$\psi = \psi_0(x, y, z); \quad \varphi = \varphi_0(x, y, z); \quad \chi = \chi_0(x, y, z);$$
$$\pi = \pi_0(x, y, z); \quad \sigma = \sigma_0(x, y, z) \qquad (6)$$

* The stationary values in the linear theory are actually quasistatic characteristics the changes of which in time are determined by the quadratic terms of the basic equations. In the forecast theory they play a very important role.

The system (3) admits a family of the stationary solutions dependent on one of the arbitrary functions of the coordinate $\psi_s(x, y, z)$. The relations given below are valid for each stationary solution:

$$\pi_s = l\psi_s; \quad \sigma = -\frac{l}{g}\frac{\partial \psi_s}{\partial z}; \quad \varphi_s = 0; \quad \chi_s = 0 \qquad (7)$$

In other words the stationary solutions are horizontal and nondivergent. The geostrophic wind equations and the static equations are true for them*):

$$u_s = -\frac{1}{l\varrho}\frac{\partial \pi_s}{\partial y}; \quad v_s = \frac{1}{lg}\frac{\partial \pi_s}{\partial x}; \quad \frac{\partial \pi_s}{\partial z} = -g\sigma_s \qquad (8)$$

For the system (3) there exists the invariant-function which is linearly expressed by means of the basic field characteristics and does not depend upon time:

$$\tilde{\Omega} = \Delta_1 \psi - l\left[\sigma - \frac{\partial}{\partial z}\left(\frac{\pi - c^2\sigma}{\beta}\right)\right] \qquad (9)$$

It is easy to verify that due to the equation (3) $\frac{\partial \tilde{\Omega}}{\partial t} = 0$. The invariant for the three-dimensional case found by us is a well known conception of a potential vorticity (Obukhov, 1949). It is also easy to verify the invariancy of S^* determined in the plane $z=0$ by using (3) and (4):

$$S^* = \pi^* - c^2\sigma^* \qquad (10)$$

(the asterisc denotes the values when $z=0$). Note that $\tilde{\Omega}$ and S^* are independent of the parameter μ and consequently, they keep on being invariant even when we come to quasi-static approximation ($\mu \to 0$).**

Let us call disturbances of a wave type those disturbances for which $\tilde{\Omega} = 0$ under an additional condition on the boundary $S^* = 0$.

* The solution of the equations (3) under the initial conditions (6) can be stationary only if these conditions are valid for the initial state:

$$\pi_0 = l\psi_0; \quad \sigma_0 = -\frac{l}{g}\frac{\partial \psi_0}{\partial z}; \quad \varphi_0 = 0; \quad \chi_0 = 0.$$

** The corresponding invariants of the quasi-static approximation have been found by A. S. Monin (1958).

The main characteristics of the disturbances of the wave type φ and χ satisfy the system of the fourth order:

$$\left.\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} &= c^2 \Delta_1 \varphi - l^2 \varphi + \beta\chi + c^2 \frac{\partial \chi}{\partial z} \\ \mu \frac{\partial^2 \chi}{\partial t^2} &= \frac{\partial}{\partial z}\left(\beta\chi + c^2 \frac{\partial \chi}{\partial z} + c^2 \Delta_1 \varphi\right) \\ &\quad + g\left(\frac{\partial \chi}{\partial z} + \Delta_1 \varphi\right) \end{aligned}\right\} \qquad (11)$$

It is easy to see that if $\mu = 0$ (a quasi-static hypothesis) the order of the system reduces by two and the number of independent field-characteristics correspondingly decreases. The system (11) is solved under the boundary conditions (4) and (5) and under the initial condition

$$\varphi = \varphi_0; \quad \frac{\partial \varphi}{\partial t} = l\psi_0 - \pi_0; \quad \chi = \chi_0;$$

$$\mu \frac{\partial \chi}{\partial t} = -\left(\frac{\partial \pi_0}{\partial z}\right) + g\sigma_0 \qquad (\text{if } t = 0)$$

The wave solutions have the property of disappearing without any "trace". It means that if at the initial moment the characteristics of the wave field are different from zero in a certain finite area then if $t \to \infty$ they will tend to zero.

Let us suppose that the initial field characteristics $\psi_0, \varphi_0, \pi_0, \chi_0, \sigma_0$ satisfy the conditions (7) everywhere except some finite area in which the field-characteristics mentioned above are given independently of each other. The initial field can be represented as the sum of the stationary and wave components. As time passes, the wave disturbance disappears and at any finite area of the field it approaches the stationary type corresponding to the stream function ψ_s and the pressure $\pi_s = l\psi_s$. That is the process of the field adaptation of the atmosphere.

In order to determine the function ψ_s corresponding to the final state the fact that the value of $\tilde{\Omega}$ for the initial state is equal to the one for the final state, can be used. Hence, taking (7) into account the equation of the elliptical type can be obtained:

$$\Delta_1 \psi_s + l^2 \frac{\partial}{\partial z}\left[\left(\frac{1}{\beta}+\frac{1}{g}\right)\psi_s + \frac{c^2}{\beta g}\frac{\partial \psi_s}{\partial z}\right] =$$
$$= \Delta\psi_0 - l\left[\sigma_0 - \frac{\partial}{\partial z}\left(\frac{\pi_0 - c^2\sigma_0}{\beta}\right)\right] \quad (12)$$

It is to be solved under the boundary condition resulting from the invariancy of S^*:

$$\psi_s + \frac{c^2}{g}\frac{\partial \psi_s}{\partial z} = \frac{1}{l}\left(\pi_0 - c^2\sigma_0\right) \quad (\text{if } z = 0) \quad (13)$$

The differential operator in the left part of (12) is known in the theory of a pressure field in the baroclinic atmosphere (KIBEL, 1955 and MONIN, 1958).

The estimation of the scale of time for the adaptation process requires a more detailed analysis of the wave disturbances in the atmosphere. The system (11) may be solved by means of separating the variables. For the case of isothermal atmosphere ($\overline{c}^2 = \text{const}$, $\beta = \text{const}$) the solution is obtained as the sum of the particular solutions of the form

$$\exp\left[-\frac{\beta+g}{2c^2}z + i\left(k_1 x + k_2 y + mz - \omega t\right)\right]$$

The frequency ω is determined for each three of the wave numbers (k_2, k_2, m) from the characteristic equation of the fourth order. For ω^2 we can obtain two values corresponding to the two types of the waves:

$$\left.\begin{matrix}\omega_a^2\\ \omega_g^2\end{matrix}\right\} = \frac{c^2}{2}\left[k^2 + \frac{l^2}{c^2} + \frac{m^2}{\mu} + \frac{1}{\mu}\left(\frac{\beta+g}{2c^2}\right)^2\right]\left\{1 \pm \right.$$
$$\left.\pm\sqrt{1 - \frac{4g\beta}{\mu c^4}\cdot\frac{k^2 + \left[m^2 + \left(\frac{\beta+g}{2c^2}\right)^2\right]\frac{c^2 l^2}{g\beta}}{\left[k^2 + \frac{l^2}{c^2} + \frac{m^2}{\mu} + \frac{1}{\mu}\left(\frac{\beta+g}{2c^2}\right)^2\right]^2}}\right\}$$

ω_a corresponds to the sign + and ω_g corresponds to −. It is natural to call the waves of large frequencies ω_a (with large propagation speed) acoustic waves nad the slower oscillations with frequencies ω_g the gravitational waves*.

$$\omega_j^2 = \frac{g}{H}\frac{K^2}{K^2 + m^2 + \frac{1}{4H^2}}; \quad H = \frac{p_0}{g\varrho_0}$$

Besides the solutions with $m \neq 0$ which can naturally be called internal waves there exist special solutions when $m = 0$ for which $\chi = 0$, $\varphi \neq 0$ with the frequency spectrum $\omega^2 = k^2c^2 + l^2$. These waves can be called two-dimensional. They are similar to the acoustic oscillations.

If μ tends to zero (quasi-static approximation) the internal acoustic waves disappear (i.e. their frequencies tend to infinity). The frequencies of the gravitational waves keep on being limited. Note that the change of the spectrum is not sufficient for the large scale disturbances (the horizontal scale is much higher than the height of the homogeneous atmosphere). So the quasi-static approximation is the method for filtration of the internal acoustic waves.

The duration of the adaptation process of the atmosphere to the quasi-static state is of the same order as that of the sound wave passing through the entire atmosphere**. It takes several minutes. After the static balance has been accomplished the adaptation process is progressing to the geostrophic balance. On average along the vertical such a state establishes after two-dimensional waves have disappeared. Somewhat later after the internal gravitational waves have disappeared the geostrophic balance is established at all heights. The second step of the adaptation process can be investigated with the help of approximate "quasi-static" equations that has been done by A. S. MONIN (1958).

* Acoustic oscillations are mainly dependent on the medium's compressibility (c^2) and to a less extent on the Arkhimed forces (β). For the gravitational oscillations the main factor is stratification although the compressibility of some importance too. For the case $l = 0$ if $\varkappa \to \infty$ that corresponds to the incompressible stratified medium we get the equation (PRANDTL, 1949):

** The sound wave passes through the layer of 20 km per min.

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