A Numerical Investigation of Certain Features of the General Circulation[1]

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Abstract

The space variables of a two-layer, quasi-geostrophic model, which includes non-adiabatic terms and a variable static stability, are expanded in a few leading terms of a spherical harmonic series. The terms are chosen to specify a zonal current with a belt of easterlies and westerlies, and a single baroclinic wave. These expansions are used to transform the equations of the model into a set of thirteen ordinary differential equations. Numerical experiments have been carried out with this system for different wave lengths of disturbance and different levels of heating contrast. The solutions indicate that the system combines many effects noted in the "dishpan" experiments and theoretical stability studies of large-scale motions.

1. Introduction

In connection with numerical weather prediction, techniques have been developed for integrating the non-linear meteorological equations in simplified form. Simplification of the complete equations has been carried out in several ways. The general equations are modified so that motions corresponding to sound and gravity waves are filtered out. The vertical structure of the atmosphere is approximated by allowing the important meteorological variables to have only a small number of degrees of freedom in the vertical. In most cases all non-adiabatic terms are neglected, although recently attention has turned toward including approximations of the more important effects in connection with long range forecasting and numerical experiments investigating the maintenance of the general circulation (PHILLIPS, 1956).

For research purposes it may be desirable to simplify the meteorological equations still further while still retaining their non-linearity. One way in which this may be done is to expand the horizontal fields of the variables in a series, such as a double fourier series. The higher order terms in such a series would specify successively smaller and smaller scales of motion. When only the leading terms in such an expansion are retained, enough degrees of freedom remain to allow for important physical processes even though the specification may not be good enough for forecasting purposes. The "pre-filtered" set of equations obtained after transformation has the same energy invariants as the original model. This approach has been suggested by LORENZ (1958), who has illustrated the method in an investigation of a hypothesis for the index cycle.

In a study by Lorenz and Bryan three double fourier components were used to represent the stream function in each of two layers. In order to formulate a simple analogue to the general circulation non-adiabatic terms were intro-

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duced in an approximate way. Instead of arbitrarily choosing a constant static stability, it was introduced as a seventh variable in the system. This permitted the investigation of the relationship between circulation intensity and static stability. The investigation to be reported on here is a similar study. It makes use of the same physical model, but spherical geometry has been taken into account by expanding the stream function in a few leading terms of a spherical harmonic series. The components representing the non-zonal disturbance were taken to be complex with respect to longitude. This enabled disturbances to move without change in shape, a feature missing in the seven-variable system. The final system contained thirteen independent parameters.

Computations carried out with the system were designed to test its non-linear response to three different levels of forced heating introduced through the non-adiabatic terms. The single baroclinic disturbance was varied over a range of intermediate wave lengths. The results of these computations are discussed under the heading of experiment I. Experiment II is a single integration of the system extending over the equivalent of a long time span. The forced temperature contrast is slowly changed to simulate seasonal variations.

2. The Physical Model

Let

$$\mathbf{v} = \nabla \times \mathbf{k}\psi + \nabla \frac{\partial \chi}{\partial p}$$

where the first term on the right represents the non-divergent part of the horizontal wind and the second term the divergent part. This particular notation is used for later convenience. Neglecting non-adiabatic terms for the moment, the continuous equations on which this study is based are the following:

$$\frac{\partial \zeta}{\partial t} = J(\zeta + f, \psi) - \nabla \cdot \left(f \nabla \frac{\partial \chi}{\partial p} \right) \tag{1}$$

$$\frac{\partial \Theta}{\partial t} = J(\Theta, \psi) - \nabla \frac{\partial \chi}{\partial p} \cdot \nabla \Theta - \omega \frac{\partial \Theta}{\partial p} \tag{2}$$

$$0 = \frac{c_p}{p_0^{\varkappa}} \nabla^2 \Theta + \nabla \cdot \left(f \nabla \frac{\partial \psi}{\partial p^{\varkappa}} \right) \tag{3}$$

$$0 = \frac{\partial \omega}{\partial p} + \nabla^2 \frac{\partial \chi}{\partial p} \tag{4}$$

In equation (3) $\varkappa$ is equal to R/c_p. The remaining symbols have their conventional meaning. Written with pressure as the vertical coordinate, equations (1) through (4) are respectively the vorticity equation, the thermal equation, the thermal wind relation corresponding to geostrophic balance, and the continuity equation. Note that these equations are similar to those frequently used in connection with numerical weather prediction with the exception of two extra terms. In equation (1) the advection of the absolute vorticity by the divergent part of the wind is included. The advection of temperature by the divergent part of the wind is included in (2). As shown by LORENZ (1957) these terms must be included if a quasi-geostrophic model is to have the proper energy invariants involving kinetic energy, available potential energy and the vertical temperature structure.

For convenience in writing down the two-layer specialization of these equations the following new variables are introduced:

$$\overline{\Theta} = \frac{1}{2}(\Theta_1 + \Theta_3)$$

$$s = \frac{1}{2}(\Theta_1 - \Theta_3)$$

$$\overline{\psi} = \frac{1}{2}(\psi_1 + \psi_3)$$

$$\tau = \frac{1}{2}(\psi_1 - \psi_3)$$

The subscripts 1 and 3 refer to the upper and lower layer respectively. The subscript 2 will refer to the interface. As a further simplification, it is assumed that s is a variable with respect to time only. Since s is a measure of the static stability, for simplicity it will be referred to as such in the following discussion.

Very simplified approximations are made of the effects of turbulent viscosity and non-adiabatic heating. It is assumed that radiation and smaller scale dynamic processes not included explicitly in the system would maintain a certain temperature distribution given by the constants, $\overline{\Theta}^*(\varphi)$ and s^*. Non-adiabatic heating in the system is taken to be proportional to the difference between the ambient temperature distribution given by Θ and s and the "forced" distribution. The constant of proportionality, h, has the dimensions t^{-1}. Similarly the effects of friction at the lower boundary and the internal eddy viscosity coupling the layers are introduced by the relations,

$$\left(\nabla^2 \frac{\partial \psi_3}{\partial t}\right) \text{surface fric.} = -k\nabla^2\psi_3$$

and

$$\left(\nabla^2 \frac{\partial \tau}{\partial t}\right) \text{int. eddy vis.} = -l\,\nabla^2\tau$$

Numerical values of h, k and l are given later in connection with the actual experiments. The viscous term involving lateral shear is not represented in the model.

In the case of just two layers the continuity equation (4) reduces to

$$\omega_2 = -\nabla^2\chi_2$$

This equation may be used to eliminate all reference to ω. With this substitution and the addition of the approximation above for the non-adiabatic effects, the equation for the two-layer system based on (1) through (3) may be written as:

$$\nabla^2 \frac{\partial \bar{\psi}}{\partial t} + J(\bar{\psi}, \nabla^2\bar{\psi} + f) + J(\tau, \nabla^2\tau) = \frac{k}{2}\nabla^2(\tau - \bar{\psi}) \tag{5}$$

$$\nabla^2 \frac{\partial \tau}{\partial t} + J(\bar{\psi}, \nabla^2\tau) + J(\tau, \nabla^2\bar{\psi} + f) - \nabla\cdot(f\nabla\chi_2) = \frac{k}{2}\nabla^2(\bar{\psi} - \tau) - l\,\nabla^2\tau \tag{6}$$

$$\frac{\partial \bar{\Theta}}{\partial t} + J(\bar{\psi}, \bar{\Theta}) + \frac{2s\nabla^2\chi_2}{p_0} = h(\bar{\Theta}^* - \bar{\Theta}) \tag{7}$$

$$\dot{s} + \frac{1}{\sigma}\int_\sigma \frac{2s\nabla^2\chi_2}{p_0}\bar{\Theta}\,d\sigma = h(s^* - s) \tag{8}$$

$$c_1 c_p \nabla^2\Theta = \nabla\cdot f\nabla\tau \tag{9}$$

The constant, c_1, in equation (9) is equal to $\frac{1}{2}\left[\left(\frac{3}{4}\right)^{\varkappa} - \left(\frac{1}{4}\right)^{\varkappa}\right]$. σ is the total area of the system. Excluding the non-adiabatic terms on the right hand sides, equations (5) through (9) represent a special case of a more general formulation of energy consistent meteorological equations suggested by LORENZ (1957). Note that these particular equations are very similar to the so-called two-parameter model with the exception of equation (8). In most numerical weather prediction models the static stability is treated as a constant. Since s is assumed to vary with respect to time only, equation (8) is an ordinary differential equation. The second term on the left hand side represents the covariance over the entire system of temperature and vertical motion. Generally this term is almost exactly balanced by the non-adiabatic term, which would otherwise tend to reduce the lapse rate to a value s^*.

In describing the solutions, frequent reference will be made to different forms of energy. The kinetic energy of the non-divergent flow and the available potential energy of the entire system are:

$$K = -\frac{p_0}{2g}\int_\sigma (\bar{\psi}\nabla^2\bar{\psi} + \tau\nabla^2\tau)\,d\sigma$$

$$A = \frac{c_1 c_p p_0}{g}\int_\sigma [(\bar{\Theta}^2 + s^2)^{\frac{1}{2}} - s]\,d\sigma$$

These two expressions have been derived previously by PHILLIPS (1954) and LORENZ (1957), respectively. Available potential energy has been discussed in some detail in connection with the general circulation by LORENZ (1955) and VAN MIEGHAN (1957). The simplified expression above may be arrived at by considering the expression for the total potential and internal energy of the system.

$$\text{Pot. Energy} + \text{Int. Energy} = \frac{c_p p_0}{g}\int_\sigma (c_2\bar{\Theta} - c_1 s)\,d\sigma$$

The constant, c_2, is equal to $\frac{1}{2}\left[\left(\frac{3}{4}\right)^{\varkappa} + \left(\frac{1}{4}\right)^{\varkappa}\right]$

The mean temperature of the entire system cannot change by adiabatic processes. On the other hand, since the mean square potential temperature is conserved under adiabatic flow,

$$s_{\max}^2 = s^2 + \bar{\Theta}^2$$

Here $s_{\max}$ is the maximum static stability that could be attained by an adiabatic redistribution of the mass field. Thus

$$A = \frac{p_0 c_p}{g} \int_\sigma c_1 (s_{\max} - s)\, d\sigma$$

3. Constrained Harmonic Form of the Equations of the Model

As outlined in the introduction only a small number of terms of the complete harmonic expansions of the variables are retained in the final formulation. Using the correct variation with latitude of the coriolis parameter, a problem arises in applying the geostrophic thermal wind relation at the equator where f is zero. This may be avoided by adopting a boundary condition of symmetry in the flow across the equator. In this case the temperature gradient is zero there so that both terms in the thermal wind relation vanish identically. This symmetry condition is satisfied by choosing only even-valued harmonics across the equator to specify the temperature field, and only odd-valued harmonics to specify the stream function field.

Let Y_n^m represent a normalized spherical harmonic of degree n and order m. The order indicates the number of waves around latitude circles. The number of zeros between poles is given by the difference, $n-m$. Therefore when $n-m$ is odd, the function is odd-valued across the equator. In the very minimal specification used in this study only the two lowest degree harmonics, odd-valued across the equator, were retained to represent the zonal flow and the disturbance of wave number m.

$$\nabla^2 \bar{\psi} = A_1^0 Y_1^0 + A_3^0 Y_3^0 + A_{m+1}^{\pm m} Y_{m+1}^{\pm m} + \\ + A_{m+3}^{\pm m} Y_{m+3}^{\pm m} \qquad (10)$$

$$\nabla^2 \tau = B_1^0 Y_1^0 + B_3^0 Y_3^0 + B_{m+1}^{\pm m} Y_{m+1}^{\pm m} + \\ + B_{m+3}^{\pm m} Y_{m+3}^{\pm m} \qquad (11)$$

The harmonic form of (9), the thermal wind relationship for the two-layer system, is:

$$\frac{-c_1 c_p}{f r^2} \frac{L_n^m}{N_n^m} = \\ \frac{(n-m)}{(2n+1)n^2} \frac{B_{n-1}^m}{N_{n-1}^m} + \frac{(n+m+1)}{(2n+3)(n+1)^2} \frac{B_{n+1}^m}{N_{n+1}^m} \qquad (12)$$

In this relation L_n^m is the complex amplitude of a harmonic specifying $\bar{\Theta}$, and N_n^m is the normalization coefficient for a spherical harmonic of order m, and degree n. This equation brings out a basic difficulty in using a small number of harmonics to specify a two-layer system. With the condition of symmetry across the equator, every adjacent pair of odd-valued harmonics, specifying τ, is related through (12) to an intermediate, even-valued function across the equator, specifying the temperature field. Thus from this point of view either the temperature must be under-specified with respect to the shear stream function, τ, or vice-versa. The first alternative was followed in this study.

$$\Theta = L_2^0 Y_2^0 + L_{m+2}^{\pm m} Y_{m\pm 2}^{\pm m} \qquad (13)$$

$$\frac{2\nabla^2 \chi_2}{p_0} = W_2^0 Y_2^0 + W_{m+2}^{\pm m} Y_{m+2}^{\pm m} \qquad (14)$$

It is important to note that Y_2^0 has just a single zero between equator and pole at approximately 40° of latitude. Thus the specification of the vertical motion permits a special type of Hadley circulation, but the three-cell, meridional circulation of the type proposed by Rossby is eliminated *a priori*.

SILBERMAN (1954) has described the transformation of the barotropic vorticity equation by means of spherical harmonics. If the limited expansions, (10) and (11) are substituted into the vorticity equation, (5), and an integration is made over the entire sphere, the orthogonal properties of the harmonics lead to four ordinary differential equations.

$$\dot{A}_1^0 = F_1 \qquad (15)$$

$$\dot{A}_3^0 = F_2 \qquad (16)$$

$$\dot{A}_{m+1}^m = F_3 \qquad (17)$$

$$\dot{A}_{m+3}^m = F_4 \qquad (18)$$

Since

$$A_n^{-m} = (-1)^m \overline{A_n^m}$$

where $\overline{A_n^m}$ is the complex conjugate of A_n^m, equations involving $\dot{A}_{m+1}^{-m}$ and $\dot{A}_{m+3}^{-m}$ are redundant, and are therefore omitted. $F_1 \ldots F_4$ are simple, but cumbersome expressions in terms of the amplitudes of τ and $\bar{\psi}$. For conciseness they have not been included. Note that the last two equations are complex. In the same way equation (6) becomes:

$$\dot{B}_1^0 + b_1 W_2^0 = F_5 \tag{19}$$

$$\dot{B}_3^0 + b_2 W_2^0 = F_6 \tag{20}$$

$$\dot{B}_{m+1}^m + b_4 W_{m+2}^m = F_7 \tag{21}$$

$$\dot{B}_{m+3}^m + b_4 W_{m+2}^m = F_8 \tag{22}$$

where b_1, b_2, b_3 and b_4 are constants determined by m and n, and the coriolis parameter. The specialized harmonic form of the thermal equation is given by only two ordinary differential equations.

$$\dot{L}_2^0 + sW_2^0 = F_9 \tag{23}$$

$$\dot{L}_{m+2}^m + sW_{m+2}^m = F_{10} \tag{24}$$

Equation (8) becomes simply

$$\dot{s} - L_2^0 W_2^0 - \frac{1}{2}\left(L_{m+2}^m \overline{W_{m+2}^m} + \overline{L_{m+2}^m} W_{m+2}^m\right) = F_{11} \tag{25}$$

Equations (15) through (25) were solved through numerical integration. At each time step $F_1 \ldots\ldots\ldots F_{11}$ were known quantities and the terms on the left the unknowns to be solved for. The first four equations, (15) through (19), contain only one unknown and therefore could be solved directly. The remaining equations, with the exception of (25), were solved by combining them with the aid of the thermal wind relation. Specializing equation (12), and differentiating with respect to time,

$$\dot{L}_2^0 + a_1 \dot{B}_1^0 + a_2 \dot{B}_3^0 = 0 \tag{26}$$

$$\dot{L}_{m+3}^m + a_3 \dot{B}_{m+1}^m + a_4 \dot{B}_{m+3}^m = 0 \tag{27}$$

where $a_1 \ldots\ldots a_4$ are constants.

If equation (26) is used to eliminate $\dot{L}_2^0$ from (23), and (23) is combined with (19) and (20),

$$\begin{bmatrix} a_1 & a_2 & s \\ 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \end{bmatrix} \begin{bmatrix} \dot{B}_1^0 \\ \dot{B}_3^0 \\ W_2^0 \end{bmatrix} = \begin{bmatrix} F_4 \\ F_6 \\ F_9 \end{bmatrix} \tag{28}$$

Similarly (27), (24), (21) and (22) may be combined to form:

$$\begin{bmatrix} a_3 & a_4 & s \\ 1 & 0 & -b_3 \\ 0 & 1 & -b_4 \end{bmatrix} \begin{bmatrix} \dot{B}_{m+1}^m \\ \dot{B}_{m+3}^m \\ W_{m+2}^m \end{bmatrix} = \begin{bmatrix} F_7 \\ F_8 \\ F_{10} \end{bmatrix} \tag{29}$$

In (29) the coefficient matrix is real, but the column of unknowns and the non-homogeneous part are complex. Separating the real and imaginary parts, the resulting set of linear non-homogeneous equations may be solved by inverting the coefficient matrix. If (28) and (29) are solved first, the values of W_2^0 and W_{m+2}^m obtained at each time step may be substituted in the second and third terms of (25), the equation for the static stability. When this is done, (25) may be solved without further modification.

4. The Harmonic Form of the Expressions for Energy

The expressions for the kinetic energy and the available potential energy of the system, equations (6) and (7) respectively, may also be written in specialized harmonic form:

$$\begin{aligned} K = -\frac{\sigma p_0 r^2}{2g}\Bigg[&\frac{1}{2}\left([A_1^0]^2 + [B_1^0]^2\right) \\ &+ \frac{1}{12}\left([A_3^0]^2 + [B_3^0]^2\right) \\ &+ \frac{1}{(m+1)(m+2)}\left(A_{m+1}^m \overline{A_{m+1}^m} + B_{m+1}^m \overline{B_{m+1}^m}\right) \\ &+ \frac{1}{(m+3)(m+4)}\left(A_{m+3}^m \overline{A_{m+3}^m} + B_{m+3}^m \overline{B_{m+3}^m}\right)\Bigg] \end{aligned} \tag{30}$$

$$A = \frac{\sigma c_1 c_p p_0}{g}\left[\left(s^2 + [L_2^0]^2 + L_{m+2}^m \overline{L_{m+2}^m}\right)^{\frac{1}{2}} - s\right] \tag{31}$$

In the expression for K, the first four terms within the brackets, involving zero order harmonics, represent the kinetic energy of the zonally averaged flow. The other terms represent the "eddy" kinetic energy of the constrained system.

Recently there has developed a great deal of interest of the rates of transformation and destruction of various forms of energy in the general circulation. In the discussion of the solutions reference will be made to: (a) the transformation of zonal potential energy into zonal kinetic energy, (b) the transformation of eddy potential energy into eddy kinetic energy, and (c) the transformation of eddy kinetic into zonal kinetic energy. These three transformations will be denoted respectively by $\{\overline{A}\cdot\overline{K}\}$, $\{A'\cdot K'\}$ and $\{K'\cdot\overline{K}\}$. Expressions for these transformations may be derived in terms of the special harmonic expansions of the variables used in this investigation by dividing (30) and (31) into their zonally averaged and eddy components and differentiating each component with respect to time (PHILLIPS, 1954). All the tendency terms in the resulting equations may be replaced by the specialized harmonic form of the vorticity, thermal vorticity, and thermal equations. The quantity $\{\overline{A}\cdot\overline{K}\}$, for example, may then be identified as the common term in the expression for $\frac{d\overline{A}}{dt}$ and $\frac{d\overline{K}}{dt}$. The other two expressions may be found in a similar way. Writing down the result directly:

$$\{\overline{A}\cdot\overline{K}\} = \frac{\sigma p_0 c_1 c_p}{g} W_2^0 L_2^0 \tag{32}$$

$$\{A'\cdot K'\} = \frac{\sigma p_0 c_1 c_p}{2g}\left(W^m_{m+2}\overline{L^m_{m+2}} + \overline{W^m_{m+2}}L^m_{m+2}\right) \tag{33}$$

$$\begin{aligned}\{K'\cdot\overline{K}\} = {} & \frac{\sigma p_0 r^2\, im}{24g} \\ & \cdot\left[A_3^0\left(A^m_{m+1}\overline{A^m_{m+3}} + B^m_{m+1}\overline{B^m_{m+3}}\right)\right. \\ & \left. + B_3^0\left(A^m_{m+1}\overline{B^m_{m+3}} + A^m_{m+3}\overline{B^m_{m+1}}\right)\right] \\ & \cdot\left[\frac{1}{(m+1)(m+2)} - \frac{1}{(m+3)(m+4)}\right]\xi\end{aligned} \tag{34}$$

where

$$\xi = \int_{-1}^{+1} P_3^0\left(P^m_{m+1}\frac{dP^m_{m+3}}{d\mu} + P^m_{m+3}\frac{dP^m_{m+1}}{d\mu}\right)d\mu$$

and

$$\mu = \sin\phi$$

5. Numerical Method

The numerical integration of an equation of the type,

$$\frac{\partial\zeta}{\partial t} = J(\zeta,\psi) - k\zeta$$

involves certain difficulties which occur regardless of whether the stream function is represented by values at discreet grid points or by the amplitudes of orthogonal harmonics. The non-linear and the linear term on the right hand side of the equation have different stability properties for ordinary finite difference schemes. PHILLIPS (1956) has discussed this problem, and presents a method which is very efficient when the linear term is small relative to the other term. With the rather high values of the friction and heating coeffients used in this study, the method was nof practical. Instead the Runge-Kutte method ot integration was used, taking second order accuracy. This method was stable in the ordinary sense, if

$$\Delta t < \frac{t_1}{2\pi}$$

where Δt is the time step, and t_1 the period of the most rapidly fluctuating amplitude. This method had the disadvantage of introducing a slight, but steady amplification through the non-linear terms, which increased with the fourth power of the time step. This resulted in a small fictitious source of energy. The effect would be very serious in an adiabatic system, but in this case it could be balanced by a slightly higher rate of dissipation with only a slight modification of the solutions.

6. Experiment I

An important non-dimensional quantity introduced in connection with the "dishpan" experiments is the forced thermal Rossby number,

$$R_{o_T^*} = \frac{U_T^*}{\Omega r}$$

U_T^* is the zonal thermal wind implied by the latitudinal temperature gradient forced by external heating, and the denominator is the velocity of solid rotation. Thus the forced thermal Rossby number is determined solely by externally imposed conditions and not by the flow itself. In the constrained system the forced temperature distribution (see equation 7) was specified by a single zonal harmonic,

$$\bar{\Theta}^*(\phi) = -L_2^{0*} Y_2^0$$

The profile of Y_2^0 is proportional to the cosine of the latitude plus a constant. This forced temperature distribution implies a given forced zonal wind shear between the upper and lower layer provided some assumption is made about the ratio of the amplitudes, B_1^0 and B_3^0, in (26). In order to conveniently define Ro_t^*, U_T^* is taken to be twice the difference in velocity between the layers at latitude 45°, provided that the two amplitudes specifying the zonal vertical shear are equal.

In all twelve integrations included in this experiment the other constants were taken to be:

$$s^* = 0$$

$$h, k = \frac{1}{2} f$$

$$l = \frac{1}{8} f$$

The value of s^* may be interpreted as meaning that the effect of radiation and small scale effects was to drive the lapse rate toward a neutral value. The constants h, k and l, involved in the heating and friction terms reflect the physical character of the fluid, but are necessarily very abstract quantities since they represent effects due to all scales less than the very largest ones included explicitly in the constrained system. If any attempt was made to model the earth's atmosphere more closely, which is not justified in this simple system, smaller values of these constants would be more appropriate. In particular the heating coefficient h should be several orders of magnitude less than k. It should be noted that solutions of quite a different character might have been obtained with other values of these constants.

In some preliminary experiments it was found that the same equilibrium solutions were converged on rather rapidly by the system no matter what the initial conditions were. Therefore initial conditions were chosen more or less arbitrarily except that the disturbance was given a small initial amplitude in every case. Runs were made for three different levels of the forced thermal Rossby number with the disturbance taken to be successively wave number three through six. In some cases the perturbation damped out completely, but in most cases a constant amplitude baroclinic wave appeared.

a. Properties Associated with Baroclinic Instability

The twelve different equilibrium solutions may be compared in terms of the levels of different forms of energy and the rates of conversion from one form into another. These quantities are given in tables I and III (see p. 174). Note that the disturbance disappears altogether when the disturbance was taken to be wave number six, and Ro_t^* was 0.1 and 0.3. For these cases the equilibrium solution was as pecial type of pure Hadley regime in which the entire release of potential energy was through rising motion in low latitudes and sinking in high latitudes.

If the expression for the rate of conversion of available potential into kinetic energy is substituted into equation (25),

$$\dot{s} = \frac{g}{c_1 c_p p_0 \sigma} \left[\{\bar{A} \cdot \bar{K}\} + \{A' \cdot K'\} \right] - hs$$

Since $\dot{s}$ is equal to zero in the equilibrium solutions, the static stability itself is directly proportional to the rate of conversion of available potential to kinetic energy. The values of s in the equilibrium solutions, shown in table II (see p. 174), indicate that the maximum intensity of the energy cycle was associated with lower and lower wave numbers as the level of forced thermal Rossby number was increased. The available potential energy, on the other hand, was a minimum where the intensity was greatest.

In figure 1 the ratio of the rate of conversion of eddy available potential energy into eddy kinetic energy to the total conversion of potential to kinetic energy is plotted. The diagram indicates that, while a pure Hadley regime appeared in only two cases, a remnant symmetric circulation is present in all the other cases as well. It is least important, however, when the energy cycle as measured by the static stability had its maximum for any given

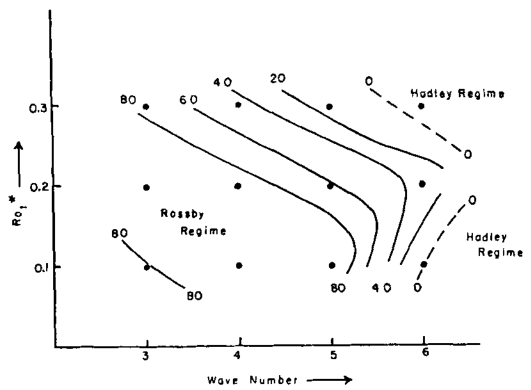

Fig. 1. The percentage of the total release of potential energy accomplished by the disturbances of different wave number. The plot is based on the solutions obtained at points indicated by dots. Note that the disturbances actually may have only integral wave numbers. A continuous plot is used for more graphic representation.

value of Ro_t^*. It may appear surprising that the meridional circulation could persist at all in these cases. An explanation may lie in the fact that the constrained system lacks any mechanism by which the disturbance itself could have induced a three-cell circulation (PHILLIPS, 1954, KUO, 1956). If this had been present the effect of the reverse cell in mid-latitudes might have more than canceled the energy released by the single Hadley cell.

Many of the features of figure 1 could have been predicted from the many stability studies of planetary baroclinic currents initiated by CHARNEY (1947). In these investigations it has been found that for a given value of the static stability there is a critical wave length of maximum instability. As the static stability is increased this wave length becomes longer, and the minimum shear necessary to induce growth of perturbations on the zonal current becomes greater. The relationship between this theory and the intensity of the energy cycle was first pointed out by LORENZ (1954) in connection with the "dishpan" experiments. The mechanism is as follows: an increase in circulation intensity causes the static stability to rise as the more rapid vertical transport of heat cannot be fully offset by radiational cooling at the top of the system. This in turn means that both the critical wave length and the critical shear for baroclinic instability increase. For the most intense circulations all wave lengths would be baroclinically stable and the zonally symmetric Hadley cell would be the only possible mechanism for the release of potential energy on a large scale.

The available potential energy (table I a) of the system is a rough measure of the threshold baroclinicity required to sustain the release of potential energy. In terms of the level of forced temperature contrast the most intense circulations are those associated with the minimum available potential energy. In a system with more degrees of freedom this wave number would dominate. The shift in the maximum intensity from shorter to longer wave lengths with increasing forced thermal Rossby number reflects the mechanism outlined in the preceding paragraph. Apparently in the case of wave number six the necessary threshold baroclinicity needed to sustain the disturbance could not be maintained. It is difficult to draw definite conclusions about the disappearance of wave number six for Ro_t^* equal to 0.1. An examination of the model indicates that dissipation is much less for the Hadley regime than for the wave regime. This is principally due to the fact that there are no geostrophic winds in the lower layer in the case of the pure symmetric regime. This feature may strongly influence the system when the thermal driving is weak.

b. Properties Related to Barotropic Stability

A comparison of $\{\bar{A} \cdot \bar{K}\}$ and $\{A' \cdot K'\}$ to $\{K' \cdot \bar{K}\}$ in table III indicates that the exchange of kinetic energy between the disturbance and the mean flow plays a comparatively insignificant role quantitatively. Since this exchange is so important in the atmosphere, however, its behavior in these solutions is of interest.

For all twelve cases the winds in the upper layer were strong westerlies with a maximum at mid-latitudes. In the lower layer the winds were much weaker and the winds in mid-latitudes were of opposite sign to those in low latitudes. With steady winds in a zone at the surface the effect of friction is to create a source or sink of angular momentum there for the system. The areas directly over surface sources or sinks of angular momentum must be regions of divergence or convergence of angular momentum respectively. Studies of the general circulation have shown that a positive covariance over the whole mass of the atmosphere between westerly flow and the convergence of westerly angular momentum indicates a positive flow of kinetic energy from the disturbances to the zonal flow (Kuo, 1950).

In investigations of the stability of disturbances in a barotropic current (Kuo, 1949, 1951) it was found that for average zonal wind distributions in the atmosphere both very long and very short waves were stable. Waves of intermediate synoptic scale showed a tendency to abstract kinetic energy from the zonal flow and amplify. For very sharp, jet-like zonal wind distributions all wave lengths showed a tendency to amplify and thereby act as a brake on the zonal current. The primary physical factors were found to be the variation of the coriolis parameter and the latitudinal profile of the zonal current. Recent observational studies of general circulation have shown that in the exchange of kinetic energy with the zonal flow there are important differences of this type between scales of disturbances (Saltzman, 1958, Eliasen, 1958).

The disturbances in the constrained system were specified by two harmonics of the same wave number around the sphere, but different latitudinal profiles. In the absence of non-linear and possibly baroclinic effects there would be a continual dispersion since the two harmonics would not have exactly the same phase speed. In the actual solutions a difference in phase angle between the two harmonics arose which allowed the non-linear and/or baroclinic effects to exactly compensate any further tendency to disperse. For those cases in which the "beta" effect was predominant in determining the phase velocity, the distortion of the disturbance was such that the trough lines tilted SW—NE, and there was a positive flow of kinetic energy from the disturbance to the zonal flow. As indicated in table II these were the cases in which the wave length of the disturbance was long and the zonal flow relatively weak.

7. Experiment II

In experiment II the effect of a slow variation of the forced temperature contrast was studied. The single baroclinic disturbance was taken to be wave number three and the forced temperature contrast was varied sinusoidally over two periods of 108 rotations (days) each. Maximum and minimum values of Ro_t^* were identical with the highest and lowest values used in experiment I. The friction and heating constants were one half those of experiment I.

$$h, k = \frac{1}{4} f$$

$$l = \frac{1}{16} f$$

$$s^* = \frac{1}{10} \Delta_\varphi \overline{\Theta}^*$$

The value of s^* (see equation 8) may be interpreted physically as an assumption that a slightly stable lapse rate would be established even without the presence of large scale motions. The ratio of this "forced" lapse rate and the horizontal forced temperature contrast between the equator and pole was taken to be one tenth.

Contrary to what was expected, the fairly rapid change of the forced thermal Rossby number did not set up oscillations in the levels of various forms of energy in the system. Except for a slight lag, the system remained fairly near to equilibrium with the changing forced temperature contrast. The single disturbance slowly changed in amplitude so that the release of potential energy and frictional dissipation were closely balanced. As a result the solution reflected many of the features of the equilibrium states of experiment I.

The speed of the zonal flow was closely tied to variations of Ro_t^*. As a result when Ro_t^* was low, the effect of beta dominated, and the phase velocity of the disturbance was negative. For higher values of the forced temperature contrast the disturbance tended to be carried along by the zonal current. The transition from a positive to a negative value of $\{K' \cdot \overline{K}\}$ coincided quite closely with the change from a negative to a positive phase velocity. One of the most interesting aspects of the solution was due to damping of the disturbance brought about by the increase of static stability during the period of maximum forced temperature contrast. When the contrast was low, the rate of release of potential energy by the Hadley cell was insignificant. As the contrast was increased, however, the disturbance first became more intense, but finally was stabilized

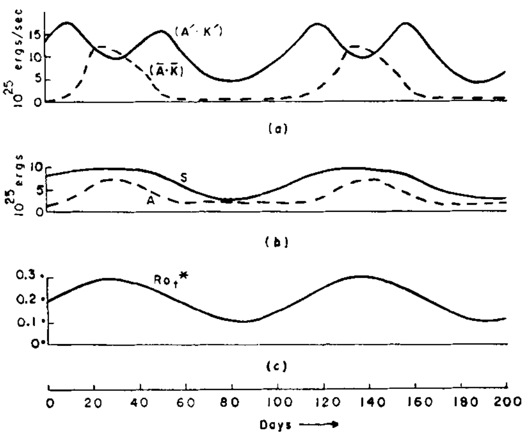

Fig. 2. Results of experiment II. a) the rate of transformation of zonal available potential energy to zonal kinetic energy and the rate of transformation of eddy available potential energy to eddy kinetic energy. b) the gross static stability and the available potential energy of the system. c) the amplitude of the harmonic representing the forced temperature distribution. The dimensions of the system are taken to be those of the earth's atmosphere.

by the great increase in the vertical temperature gradient. At this point the Hadley type of circulation became the dominant mode for the release of potential energy. Note in figure 2 that the Hadley mode was much less efficient in terms of forced temperature contrast. As a result the combined release of potential energy tended to flatten out rather than rise to a peak when Ro_t^* was at a maximum.

The curve for s in figure 2 reflects this leveling off in the intensity of the energy cycle. The behavior of the total available potential energy was also interesting. While the wave regime was dominant, the rising static stability meant that the threshold baroclinicity also had to rise. In this particular case the rise of the baroclinicity and the static stability compensated each other in such a way that the available potential energy remained nearly constant. When the disturbance was damped, however, the baroclinicity jumped to a much higher value to permit a greater release of potential energy by the symmetric mode. This sudden rise is also reflected in A, since the static stability, s, changes rather slowly.

8. Conclusions

In spite of its extreme simplicity many more numerical experiments would be required to completely explore the properties of the constrained system. In particular, solutions for a whole range of values of the heating and friction constants should be investigated. The most significant results of experiment I and II centered on two non-linear effects. First there was the distortion of the disturbance from symmetry along a north-south axis, which meant that it could exchange kinetic energy with the zonal flow. The distortion was such that non-linear terms and/or baroclinic effects could exactly balance any tendency for the harmonic components of the disturbance to disperse. For weak zonal currents and low wave numbers the distortion was such that the trough lines had a SW—NE tilt, and there was a positive transfer of kinetic energy from the disturbance to the zonal flow. Secondly, the inclusion of a variable static stability led to a special pattern of response (see figure 1). For slow variations of the forced heating the static stability was approximately proportional to the intensity of the release of potential energy. The static stability therefore increased as the forced temperature contrast was increased, which in turn caused a shift of the most "baroclinically" unstable wave number to lower values. The extended numerical integration of experiment II illustrated how this mechanism brings about a change in the dominant mode of potential energy release. In this case the change over is accompanied by a sudden rise of the available potential energy which was necessary to provide the increased zonal shear to sustain the Hadley regime.

In both experiments the tendency was very marked for the system to rapidly come into equilibrium with the forced temperature contrast, and for fluctuations in the levels of various forms of energy to be rapidly damped. This is surprising in the light of results obtained by LORENZ (1958) for a simplified barotropic model. In an adiabatic system it seems likely that both the non-linear processes described above could under certain circumstances lead to important feed-back oscillations. Experiments must be carried out with lower values of the heating and friction coefficients to be sure that such oscillations have not been excluded by the dominating effect of the linear non-adiabatic terms.

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Table I. Levels of different forms of energy in the equilibrium solutions of experiment I. The units are 10^{27} ergs and the dimensions of the system are taken to be that of the earth's atmosphere

Ro_t^*	3	4	5	6
	Wave number			
0.3	638	986	1 352	1 485
0.2	441	404	739	890
0.1	442	272	240	569
	(a) *Total Available Potential Energy*			
0.3	282	507	637	648
0.2	108	137	235	286
0.1	58	36	40	81
	(b) *Total Zonal Kinetic Energy*			
0.3	1 222	803	235	0
0.2	592	839	525	251
0.1	117	234	252	0
	(c) *Total Eddy Kinetic Energy*			

Table II. Values of s in degrees centigrade assuming that p_0 is 1000 mb

Ro_t^*	3	4	5	6
	Wave Number			
0.3	125	110	69	52
0.2	71	93	62	44
0.1	18	36	39	9

Table III. Rates of energy conversion in the equilibrium solution of experiment I. (a) conversion of zonal available potential energy into zonal kinetic energy. (b) conversion of eddy available potential energy into eddy kinetic energy. (c) conversion of eddy kinetic energy into zonal kinetic energy. The units are 10^{24} ergs per second

Ro_t^*	3	4	5	6
	Wave Number			
0.3	148	301	374	283
0.2	40	73	133	171
0.1	18	13	14	32
	(a) $\{\overline{A} \cdot \overline{K}\}$			
0.3	531	333	32	0
0.2	337	440	231	79
0.1	76	179	189	0
	(b) $\{'A \cdot K'\}$			
0.3	−27	−52	−51	0
0.2	1	−41	−21	−35
0.1	4	11	0	0
	(c) $\{K' \cdot \overline{K}\}$			

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