# The Moving Flame Experiment

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## Abstract

When a flame is rotated around the outside bottom rim of a cylindrical pan of water initially at rest, D. Fultz has observed that the fluid acquires a net vertical component of angular momentum opposite to the rotation of the heat source. We have repeated this experiment in a cylindrical annulus in order to restrict the radial motions and have found that the same phenomenon occurs. Using a simple model based on the latter experiment we investigate the mechanism by which a fluid can acquire and maintain this momentum, considering that friction is the only force which can exert a net torque about the vertical.

The effects of the heat source are replaced by a forced temperature field which has sinusoidal variations in the circumferential direction and which rotates about the vertical axis at a constant rate. The correlation between vertical and horizontal velocity components in the equilibrium state has been computed from the linearized two-dimensional vorticity equation. It is shown that when the viscosity is small or when the speed of the heat source is large there is a convergence of horizontal momentum towards the center of the channel which is balanced by the viscous stress of the induced mean motion. The total momentum of the fluid is computed from the conservation principle and is found to vary inversely as the square root of the molecular viscosity and inversely as the 5/2 power of the speed of the heat source.


## 1. Introduction

In a recent unpublished report FULTZ (1956) describes an experiment in which a flame is rotated around the outside bottom rim of a cylindrical vessel filled with water. The experiment was stimulated by a question posed by Lord Kelvin in 1892 as to whether or not the fluid would acquire a net spin about a vertical axis as was originally suggested by HALLEY (1686). Fultz concludes that "... although no quantitative measurements have been made in the interior [of the fluid] it is practically certain that in the course of establishing the motion from rest the total absolute angular momentum of the fluid increases in the anti-flame sense. Thus the mechanisms involved must give rise to net torques during the establishment phase." We have reproduced this qualitative experiment using a central core to confine the motion to a cylindrical annulus whose radial width was small enough to be completely covered by the bunsen flame and large compared to the vertical depth (2 cm) of the water. In this manner we were able to reduce the radial convection which was present in the previously cited experiment and to determine whether the same phenomenon would occur in a flow which was approximately two-dimensional. When the flame was rotated at 4 r.p.m. at a mean radius of 15 cm we generally observed that the dust particles in the water performed small horizontal oscillations along a concentric circle in response to the alternating solenoidal field induced by the rotating heat source. However, the mean position of the

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particles changed slowly with a rotation which was opposite to that of the flame. Marked deviations from this flow pattern were also observed, from an impulsive motion of the particles, to a regular rotary motion with imperceptible oscillatory motion, to no net motion when the heating reached a stage where very intense vertical convection occurred. However, by observing the net displacement of paper markers placed on the surface and from permanganate tracers in the interior we have concluded that an average anti-flame rotation does exist in this geometry, whose order of magnitude was 0.1 % to 1.0 % of the rotation rate of the heat source, up to the point where boiling-like motions occur. This qualitative experiment encourages us to look at simple two-dimensional models in order to explain the following question. Considering that friction is the only force which can exert a net torque about the vertical axis, how can the fluid acquire and maintain a net angular momentum, and what determines its direction?

## II. The Theoretical Model

The rectangular channel shown in Fig. 1 represents an approximation to the geometry of a cylindrical annulus of large mean radius. The $x$-axis corresponds to the circumferential distance in the circular annulus, and by requiring that the motions be periodic in $x$ we shall achieve a correspondence in the lateral boundary conditions. The width of the channel in the $y$-direction, corresponding to the radial width of the annulus, is taken to be large compared with the vertical depth $H$. In addition we consider a distribution of heat sources and sinks which moves in the $-x$-direction with uniform speed $U$, and which forces a density

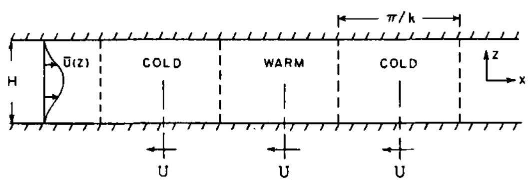


Fig. 1. Schematic diagram of the theoretical model. The hypothesized heat sources and sinks move with speed $U$ producing a neutrally stable temperature or density perturbation ($\varrho'$). The sinusoidal variations of density with $x$ and time induce a mean velocity in the fluid whose equilibrium profile $\bar{u}(z)$ is to be computed.

distribution which is a sinusoidal function of $x$ and time $t$, but which is independent of $y$ and $z$. The average horizontal momentum of the fluid in the equilibrium state will be determined, and this will correspond to the vertical component of angular momentum in the cylindrical geometry. Regarding the assumed forced density distribution it should be mentioned that in the experiments the temperature field does have variations of static stability and moreover cannot be divorced from the convective motions. However, the thermodynamics are secondary to the question at hand, since we should like to know how any thermal field can produce a net motion of the liquid perpendicular to the force of gravity. The model chosen is perhaps the simplest one to elucidate this phenomenon, and it does have a superficial resemblance to the differential temperature field produced by a moving flame.

If the Navier-Stokes equations are accepted, then the only way in which an average horizontal motion can be maintained in Fig. 1 is through a transport of momentum by the vertical ($w'$)and horizontal ($u'$) velocity perturbations which are produced by the alternating solenoidal field. In the equilibrium state, which will be considered first, the vertical flux of horizontal momentum

$$\nu \frac{d\bar{u}(z)}{dz} - \overline{w'u'} \tag{1}$$

must be constant, where $\nu$ is the viscosity and the single bar indicates an average value over a horizontal plane or over one wavelength. In this approximate statement of the momentum principle, as well as in what follows, we have used the Boussinesq approximation and neglected density variations except for the buoyancy force. Since $\bar{u}$ and $\overline{w'u'}$ are zero at the top and bottom of the channel, a vertical integration of (1), denoted by a second bar, gives the value of the constant as $-\overline{\overline{w'u'}}$. Consequently the total momentum of the fluid is proportional to

$$\bar{\bar{u}} = \frac{1}{H\nu}\int_{-H/2}^{+H/2} dz \int_{-H/2}^{z} \left(\overline{w'u'} - \overline{\overline{w'u'}}\right) dz_1 \tag{2}$$

## III. Computation of the Correlation

The correlation between $w'$ and $u'$ will be determined from the vorticity and continuity equations. Denoting the $-y$ component of vorticity by $\eta = \frac{\partial w'}{\partial x} - \frac{\partial u}{\partial z}$ these become:

$$\frac{d\eta}{dt} = -\frac{g}{\varrho_0}\frac{\partial \varrho'}{\partial x} + \nu\nabla^2\eta \qquad (3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Let the forced density perturbation $\varrho'$ be represented by the real part of $\Delta\varrho\, e^{ik(x+Ut)}$, where $\Delta\varrho$ is the amplitude and $k$ the wave number. Then the vertical velocity may be represented by $w'(x, z, t) = w(z)\, e^{ik(x+Ut)}$, if, we assume a linear response. Although the establishment of a net momentum is a non-linear effect, it is assumed that the correlation between the velocity components is insignificantly modified by the mean flow it induces, and that this quantity may be computed from the linearized hydrodynamic equations. This will be a valid approximation if, as was the case in the experiments, the induced mean motion and the velocity perturbations are small compared to $U$. The range of validity may be inferred from the final results. The horizontal velocity perturbation as computed from the continuity equation is

$$u'(x, z, t) = \frac{i}{k}\frac{dw(z)}{dz}\, e^{ik(x+Ut)} \qquad (4)$$

and consequently the vertical transport of momentum is

$$\overline{w'u'} = \frac{k}{2\pi}\int_0^{2\pi k} dx \quad Re\, w(z) e^{ik(x+Ut)}.$$

$$\cdot\; Re \frac{i}{k}\frac{dw}{dz}\, e^{ik(x+Ut)} \qquad (5)$$

or

$$\overline{w'u'} = \frac{1}{2k}\; Im w(z)\frac{dw^\star}{dz} \qquad (5)$$

where $Re$ denotes "real part", $Im$ denotes "imaginary part" and $w^\star(z)$ is the complex conjugate of $w(z)$.

When $\nabla^2 w' = \frac{\partial\eta}{\partial x}$ is substituted into the linearized vorticity equation the following expression for the vertical velocity is obtained:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w' = -\frac{g}{\varrho_0}\frac{\partial^2\varrho'}{\partial x^2} = \frac{g}{\varrho_0}k^2\Delta\,\varrho e^{ik(x+Ut)} \qquad (6)$$

If the depth of the fluid is small compared with the horizontal scale of the motion ($k/\pi \ll H^{-1}$) this reduces to the ordinary differential equation

$$\left(ikU - \nu\frac{d^2}{dz^2}\right)\frac{d^2}{dz^2}w(z) = \frac{g\Delta\varrho}{\varrho_0}k^2 \qquad (7)$$

and we shall confine the discussion to this limiting case. Introducing the abbreviations

$$z = H\zeta \quad R = kUH^2\nu^{-1} \quad \alpha = i^{1/2}R^{1/2}$$

$$w(z) = \frac{g\Delta\varrho}{\varrho_0}k^2H^4\nu^{-1}f(\zeta) \qquad (8)$$

into eq. (7) we obtain the following equation and boundary conditions for $f(\zeta)$:

$$\frac{d^2}{d\zeta^2}\left(iR - \frac{d^2}{d\zeta^2}\right)f(\zeta) = 1 \qquad (9)$$

$$f(-1/2) = f(1/2) = f'(-1/2) = f'(1/2) = 0 \qquad (10)$$

The solution of (9), (10) is

$$2iRf(\zeta) = \zeta^2 - 1/4 + \frac{\sinh\alpha}{\alpha(\cosh\alpha - 1)} - \frac{\cosh\alpha\zeta}{\alpha\sinh\alpha/2} \qquad (11)$$

Substituting this in (5) there results

$$\overline{w'u'} = \frac{1}{8R^2kH}\left(\frac{g\Delta\varrho}{\varrho_0}\,\frac{k^2H^4}{\nu}\right)^2 Im\phi(z) \qquad (12)$$

where

$$\phi(\zeta) = \left[\zeta^2 - 1/4 + \frac{\sinh\alpha}{\alpha(\cosh\alpha - 1)} - \frac{\cosh\alpha\zeta}{\alpha\sinh\alpha/2}\right]$$

$$\cdot\left[2\zeta - \frac{\sinh\alpha^\star\zeta}{\sinh\alpha^\star/2}\right] \qquad (13)$$

Since $\phi(-\zeta) = -\phi(\zeta)$ then $\overline{\overline{w'u'}} = 0$ and (2) becomes

$$\overline{\overline{u}} = \frac{H}{\nu} \int_{-1/2}^{+1/2} d\zeta \int_{-1/2}^{\zeta} \overline{w'u'} d\zeta_1$$

$$\overline{\overline{u}} = -\frac{2}{8R^2 k\nu} \left( \frac{g\Delta\varrho}{\varrho_0} \frac{k^2 H^4}{\nu} \right)^2 Im \int_0^{1/2} \zeta\phi(\zeta) d\zeta \tag{14}$$

It may be noted that the anti-symmetric form of $\phi(\zeta)$ implies that the net stress on the bottom and top surfaces are not only equal to each other but are also zero. The sign of the total momentum of the fluid depends upon the integral in (14), and although this may be evaluated exactly in terms of elementary functions it will be sufficient for our purposes to indicate the qualitative behavior and the quantitative results in two limiting cases. When the viscosity is very large, or when $R$ is small compared to unity, an expansion of $\phi(\zeta)$ in ascending powers of $\alpha$ leads to the result

$$Im \int_0^{1/2} \zeta\phi(\zeta) d\zeta = 2 \times 10^{-6} R^3 + \ldots.$$

The case which is of greater interest to us is when $R$ is much larger than unity. When we make the approximation $|e^{\alpha}| \gg 1$ the integral in (14) may be expanded in a power series in $\alpha^{-1}$, which leads to the following result:

$$Im \int_0^{1/2} \zeta\phi(\zeta) d\zeta = -\frac{1}{12\sqrt{2}} R^{-1/2} + \frac{3}{4} R^{-1} + \ldots \tag{15}$$

The significant point to note is the change in sign of the integral, and the consequent reversal of the total momentum (14) as $R$ increases from zero to values which are much greater than unity. By setting the right hand side of (15) equal to zero we estimate that this change occurs when $R = kH \dfrac{UH}{\nu}$ is of the order of $10^2$. For values of $R$ which are much larger than this (and when $kH \ll \pi$). The average $+x$ momentum of the fluid as obtained from (14) and (15) is

$$\frac{\overline{\overline{u}}}{U} = \frac{1}{48\sqrt{2}} R^{1/2} \left( \frac{gh}{U^2} \frac{\Delta\varrho}{\varrho_0} \right)^2 \tag{16}$$

## IV. Discussion

The quadratic dependency of the mean field (16) upon the buoyancy force is quite expected in view of the linear variation of the velocity perturbations with this force. However, the quantity $R$, which represents the ratio of inertial and viscous forces and may be thought of as the characteristic Reynolds number, is of primary importance for the qualitative features of the flow. When the relative velocity of the density field and the ambient fluid is large so that $R \gg 10^2$ the fluid acquires a net motion opposite to that of the density field. The form of the velocity profile for this regime may also be obtained from (13) and we find that $Im\,\phi(\zeta) \to -2^{-1/2} R^{-1/2} \zeta$ so that $\overline{w'u'}$ is proportional to $-\zeta$ for $|\zeta| < 1/2$. Therefore the mean velocity profile is parabolic except for the boundary regions where $\overline{u}'(\zeta) = 0$. Consequently in the equilibrium state both the top and bottom surfaces exert no net stress on the fluid, while in the interior there is an upward transport of momentum ($\overline{w'u'} > 0$) below the central level ($\zeta = 0$) and a downward transport of momentum above $\zeta = 0$, both of which are balanced by the viscous stress of the mean motion. It is noted that the transport is proportional to the square root of the viscosity while the total momentum varies as $\nu^{-1/2}$. In an ideal fluid without viscosity there would be no mean motion since there would be zero correlation between $w'$ and $u'$, whereas in a real fluid with very small viscosity the mean motion would be very large.

Although the transient process has not been studied quantitatively the mechanism by which the liquid acquires the net momentum seems reasonably clear. Suppose the thermal field is slowly imposed on a fluid initially at rest. We infer from the conclusions of the preceding paragraph that for large Reynolds numbers the slowly varying velocity perturbations will be such as to produce a convergence of horizontal momentum towards the center of the channel. Since there is initially no mean field to balance this, the $+x$ component of momentum will increase at the center of the channel at the expense of the fluid near the boundaries which will tend to move in the opposite direction. In so doing the fluid will exert a net stress on both boundaries in the

$-x$ direction, thereby increasing the total $+x$ momentum of the fluid as it comes to equilibrium.

In view of the fact that the phenomenon discussed here was used by Halley as an explanation of the atmosphere's general circulation, it is of some interest to estimate the magnitude of the angular momentum produced by the westward progression of the sun relative to the earth. Consider a 1 km depth which is alternatively heated and cooled by 1° C once a day, and assume that the induced horizontal motion is in a zonal direction on a latitude circle of radius $2 \times 10^8$ cm. Using the molecular viscosity of air we find that $R$ must be less than $3 \times 10^{+6}$ and that the angular momentum which is induced in the earth's atmosphere by the diurnal heating of the sun corresponds to an eastward moving current whose order of magnitude is less than one centimeter per second.

## REFERENCE

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