On the Application of Trajectory Methods in Numerical Forecasting

By A. WIIN-NIELSEN,[1] International Meteorological Institute in Stockholm

(Manuscript received August 22, 1958)

Abstract

On the basis of a comparison between the results so far obtained by the usual numerical methods in physical weather prediction and the graphical methods it is likely that certain defects in the present forecast methods are due to the Eulerian technique. The possibilities for the use of a quasi-Lagrangian technique are investigated, and it is found that such a technique is possible and becomes especially simple when combined with a space-smoothing which eliminates the shorter fast moving waves in the spectrum. The Lagrangian method of computation is described in the barotropic case and also for a simplified two-parameter baroclinic model.

1. Introduction

The numerical method, which so far has been used in physical weather prediction, has been the Eulerian one. Only the graphical technique developed by FJØRTOFT (1952, 1955) is essentially Lagrangian. The author has together with HLYNUR SIGTRYGGSSON (1957) made a number of barotropic and semi-baroclinic forecasts based on the theories developed by Fjørtoft. With respect to the numerical procedure used in these forecasts we have, however, not deviated from the usual one, and the difference between our forecasts and the earlier time-integration of the simplified vorticity-equation consists mainly in the assumption that a field obtained through a space-smoothing of the initial height field has been assumed constant in time over several hours.

The results of the barotropic forecasts, which could be compared with barotropic forecasts computed without the smoothing technique, showed the result that the two methods were very closely of the same accuracy in the mean. An astonishing fact was, however, that even the more restricted method could in certain individual cases result in a better forecast as measured by correlation coefficients between observed and computed height changes and by the mean error over the forecast region. It is furthermore known that the models at present in use in numerical forecasting show a too large development in the sense that the forecast change in the mean exceeds the observed change (BOLIN, 1956 and DÖÖS, 1956). This erroneous development is visible in the forecasts essentially as an exaggeration of the gradient in the height field, or in other words: the Highs are too high and the Lows too low. A part of the fictitious development in the first barotropic forecast is due to the application of the geostrophic assumption as for instance shown by SHUMAN

[1] On leave from the Danish Meteorological Institute.

(1957). It may easily be shown that the mean production of vorticity in a non-divergent barotropic forecast over a closed region vanishes, while there in general exists a mean production, when the geostrophic assumption is applied. This latter production is due to the divergence of the geostrophic wind and is measured by the correlation between the (geostrophic) absolute vorticity and the meridional component of the horizontal wind. Consequently it is also found (BOLIN, 1956) that the development is somewhat but not completely surpressed by the use of the balance equation.

There may be found a number of reasons that the models at present in use contain characteristic defects. The most natural causes are probably found by pointing to a number of physical processes not yet included in the models. We should, however, also have in mind that the particular numerical methods used in the time-integration are not unimportant. Several experiments have shown this (compare for instance KNIGHTING et al., 1957). The content of the barotropic theorem is simply that the circulation per unit area is conserved following the area during its motion in the fluid. This formulation of the barotropic theory is a Lagrangian one. However, when we compute the forecasts we make no efforts to follow the small areas initially occupying the space and represented by the grid-points. The vorticity in a grid-point represents the circulation around a square with the side equal to Δs. when Δs is the gridsize. A number of investigations has shown that the region which initially is a square undergoes large deformations in the course of time (WELANDER, 1955). The greatest deformation can of course be expected in the regions of saddle points, but also along a jet-stream with a large horizontal shear we find large deformations. In these regions and to some extent also in other regions there is a tendency to deform the initial rather regular pattern into the structure of elongated bands. This holds not only for the initial squares, but is also true for the isolines for absolute vorticity.

When we make the barotropic and baroclinic forecasts by the use of the Eulerian technique it will be impossible to take the large deformations into account. The stretching is in some regions so large that the bands disappear between the grid points, and the distribution of vorticity or any other parameter which carry the history of the flow is not at all or only very approximative caught by the grid-point values as computed by finite differences. We neglect therefore to some extent the deformation properties of the flow when computing with the Eulerian method. Now, it has been argued that this is not so important for the following reason. Any time we construct or compute for instance the field of absolute vorticity we obtain a rather smoothed arrangement of the isolines. Consequently, it is argued, there is no tendency for these isolines to be arranged in bands, because we would then observe these bands on the weather maps. Answering this argument we have to bear in mind that the net work of observing stations on the Northern Hemisphere and to an even larger extent on the Southern has such a small density in space that we can not represent phenomena which are on a scale smaller than, say, 1,000 to 1,500 km. We are therefore bound to get a smooth picture. This does, however, not mean that we should neglect the deformation properties of the fields. It has been shown by FJÖRTOFT (1955) that the mutual deformation properties of the height and temperature field in the atmosphere are in an integrated sense responsible for a net flow of amplitude squared in one or another direction of the energy spectrum. The practical experience with the graphical prognostic technique which is essentially Lagrangian has also shown that the displacements of isolines, which are initially smoothed, ends up with "energy" on smaller scales. These smaller scale phenomena can in general also be verified.

Investigations made by VEDERMAN and HUBERT (1957) have further shown that the absolute vorticity is not conserved during a non-divergent barotropic forecast. They find changes in absolute vorticity during a forecast which amount to about 40 % of the initial values in extreme cases and a change amounting to 15—20 % is common. It is further observed that certain isolines completely disappear during the forecasts. This is not the case during a Lagrangian forecast. It seems therefore reasonable to try to develop a numerical technique which is closer to the Lagrangian, because the graphical technique has not shown the fictitious development. There may, however, be one reason which partly explains the less strong rate of development which is shown by the graphical technique. This is the method which is used for the solution of the Poisson or Helmholz equation, which must be solved in order to obtain the height field from the vorticity field. The approximate solution obtained with the aid of Fjørtoft's method will give a considerable underestimate of the amplitude of the large waves. This may at least act in the direction of decreasing the change on the longer scales (FJØRTOFT, 1952). This may, incidentally, be an advantage, because the recent experiments with barotropic forecasts on a hemispheric basis has shown that the planetary waves are not at all well forecasted by our present methods (MARTIN, 1958). It has in fact been shown by numerical experiments at the Joint Numerical Weather Prediction Unit in Washington that a large improvement in the hemispheric barotropic forecasts is obtained by keeping the first four wave-numbers constant through the forecasts. The space-smoothing technique as developed by Fjørtoft does essentially the same partly because of the technique used in the solution of the Poisson and Helmholz equations where a considerable underestimate of the change in amplitude on the planetary scale is made, and partly because of the assumed time independence of the smoothed field. The barotropic case will because of its simplicity be treated first.

2. The quasi-Lagrangian technique in the barotropic case

The barotropic vorticity equation applied to an isobaric surface expresses simply that the absolute vorticity is conserved along the path of a particle. This theorem is not explicitly used in the Eulerian integration method, where we always compute the local change of vorticity, in this case, only due to horizontal advection of absolute vorticity. As mentioned in the introduction we shall try to devise a method, where we explicitly make use of the essence of the basic barotropic vorticity equation, i.e. that the absolute vorticity is conserved during the trajectory of the particle.

When we want to follow a particle during its motion in the isobaric surface, it is not evident that a rectangular grid is the most convenient to use, because the successive positions of the particle in general will run between the grid points. The relaxation methods which are used to the solution of the finite difference form of the second order differential equations are, however, well suited for the rectangular grid. We shall in the first instant try to arrange the numerical procedure in such a way that we can still use the relaxation methods without major changes. This is obtained if we ask for the positions of the particles which after a certain time end up in the grid points of the rectangular grid.

Let the Lagrangian coordinate be denoted (x, y). Let further the Lagrangian coordinates coincide with the Cartesian coordinates for $t = 0$. These latter coordinates are denoted (a, b).

The trajectory is then expressed by the two equations:

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0 \tag{1}$$

We shall solve these equations by the Eulerian method, i.e.

$$\frac{\partial x}{\partial t} = -\mathbf{v} \cdot \nabla x, \quad \frac{\partial y}{\partial t} = -\mathbf{v} \cdot \nabla y \tag{2}$$

Provided we know $\mathbf{v} = \mathbf{v}(x, y, t)$ the solutions of these equations will give us the coordinates for the particle which to $t = T$ end up in a certain given point. The problems regarding the finite difference form of these equations and the boundary assumptions made for a and b shall not be considered here. A number of considerations in this respect have been made by DJURIC and WIIN-NIELSEN (1957). It suffices here to say that a number of experiments have resulted in the conclusion that the Eulerian solution of the equations will give the trajectory with a good accuracy, provided the time-integration not is extended too far. This is due to the fact that the initial grid is a rectangular grid, and that the truncation error in the computation of ∇x and ∇y in the beginning therefore is extremely small. Later, however, the initial rectangular grid is deformed to such an extent that the truncation error in the gradients of the coordinate field will play a larger and larger role besides the truncation error which always exists, when the wind vector is computed from a height-field or from a stream function. It is worth while to notice that the truncation errors play a smaller role in these equations because only first order differences are necessary, and also because the scale is large of all quantities which enter the computations. It is further an advantage that the truncation errors enter in the coordinate fields (i.e. the positions of the particles) and not in the physical parameter (here: the absolute vorticity) which determine the dynamic development of the flow.

By computing trajectories from the equations (2) in a stationary field it was found that the truncation errors are quite small. A particle, which initially has a position on a certain isoline, should in a stationary field stay there. The tests show that it is very nearly so.

Let us then to a time $t = T$ consider the particles which are in the grid points of a rectangular grid. From the solutions of the equations (2) we will know the coordinates of these particles to $t = 0$. The positions to $t = 0$ will in general not be in grid points. In the barotropic case we are then faced with the problem to determine the absolute vorticity for the particle which to $t = T$ arrives in a grid point. We shall first compute the relative vorticity.

Let now A, B, C and D to $t = 0$ be the positions of the particles which to $t = T$ occupy the grid points A_T, B_T, C_T and D_T (see fig. 1). The trajectories are denoted AA_T, BB_T etc. Any trajectory can as a curve be characterized by the three parameters a, b and t where a and b are Lagrangian coordinates. A parametric representation of a trajectory can therefore be given in the form:

$$x = x\ (a, b, t),\ y = y\ (a, b, t) \tag{3}$$

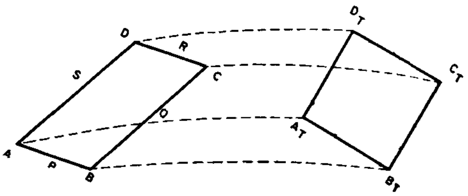

Fig. 1. Schematic picture showing the initial positions A, B, C, and D of the particles, which to the time $t = T$ arrive in the position A_T, B_T, C_T and D_T.

Let the coordinates for A_T, B_T, C_T and D_T be:

$$A_T\colon (a_T, b_T),\quad B_T\colon (a_T + \delta a, b_T),$$

$$C_T\colon (a_T + \delta a, b_T + \delta b),\quad D_T\colon (a_T, b_T + \delta b).$$

The particles which to $t = T$ are on the line through $A_T\, B_T$, are to $t = 0$ situated on the curve through AB. The parametric representation for this curve is:

$$x = x\,(a, b_T, 0),\ y = y\,(a, b_T, 0)$$

The coordinates for A and B are especially:

A: $(x\,(a_T, b_T, 0), y(a_T, b_T, 0))$

B: $(x\,(a_T + \delta a, b_T, 0), y(a_T + \delta a, b_T, 0)$

The infinitesimal vector AB is:

$$\vec{AB} = \left(\frac{\partial x}{\partial a}\delta a,\ \frac{\partial y}{\partial a}\delta a\right)$$

In a similar way we obtain:

$$\vec{BC} = \left(\frac{\partial x}{\partial a}\delta b,\ \frac{\partial y}{\partial b}\delta b\right)$$

$$\vec{CD} = -\left(\frac{\partial x}{\partial a}\delta a,\ \frac{\partial y}{\partial a}\delta a\right)$$

$$\vec{DA} = -\left(\frac{\partial x}{\partial b}\delta b,\ \frac{\partial y}{\partial b}\delta b\right)$$

The coordinates of a particle are:

$$(x\ (a, b, t), y\ (a, b, t))$$

The velocity for a particle is therefore:

$$\left(\frac{\partial x}{\partial t},\ \frac{\partial y}{\partial t}\right)$$

With these expressions we are now ready to compute the relative vorticity by the aid of the circulation thereon:

$$\zeta = \frac{1}{\delta A}\oint \mathbf{v}\cdot\delta\mathbf{r}$$

The circulation is:

$$\oint \mathbf{v}\cdot\delta\mathbf{r} = \left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial a}\right)_P \delta a + \left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial a}\right)_P \delta a$$

$$+ \left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial b}\right)_Q \delta b + \left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial b}\right)_Q \delta b$$

$$- \left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial a}\right)_R \delta a - \left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial a}\right)_R \delta a$$

$$- \left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial b}\right)_S \delta b - \left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial b}\right)_S \delta b$$

$$= \left\{\frac{\partial}{\partial a}\left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial b}\right) + \frac{\partial}{\partial a}\left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial b}\right)\right.$$

$$\left. - \frac{\partial}{\partial b}\left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial a}\right) - \frac{\partial}{\partial b}\left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial a}\right)\right\} \delta a \delta b$$

$$= \left\{J\left(\frac{\partial x}{\partial t}, x\right) + J\left(\frac{\partial y}{\partial t}, y\right)\right\} \delta a \delta b$$

If the flow is considered non-divergent, there will be no change in area, i.e.

$$\delta A = \delta a \;\; \delta b = \text{const.}$$

Consequently:

$$\zeta = J\left(\frac{\partial x}{\partial t}, x\right) + J\left(\frac{\partial y}{\partial t}, y\right) \tag{4}$$

It should be noted here that

$$J(\alpha, \beta) = \frac{\partial \alpha}{\partial a}\frac{\partial \beta}{\partial b} - \frac{\partial \alpha}{\partial b}\frac{\partial \beta}{\partial a}$$

The derivation has been given in details here, because the Lagrangian system seldom has been used in numerical forecasting. It has further been preferred to use a geometric representation instead of the formal derivation.

In actual computations it is necessary to use a certain wind-approximation. We shall in the following suppose that a stream function or a quasi-stream function has been computed from either the complete form or a simplified form of the balance equation. This stream function will be denoted $\psi = \psi(x, y, t)$.

3. On the application of space-smoothing in the barotropic case

As mentioned in the preceding section we want at first to keep the relaxation methods essentially unchanged. We must therefore find the initial positions of the particles, which after a certain fixed time T end up in the grid points of our rectangular grid. Further, in view of the results obtained earlier by the use of constant space-smoothed fields, and in order to keep the numerical computations rather simple, we shall first outline a method, where the trajectory method can be combined with the space-smoothing technique. The use of this procedure will simplify the computations and will probably not decrease the accuracy of the forecasts very much.

In the earlier experiments it was in the barotropic case considered necessary to neglect certain variations of the Coriolis parameter in order to obtain the most simple forecasting equation, containing only one Jacobian. We shall in this case define a smoothing operator which is slightly different, but avoids the difficulties with the variable Coriolis parameter.

The barotropic vorticity equation is in the non-divergent case:

$$\frac{\partial \zeta}{\partial t} = J(\eta, \psi), \quad \eta = \nabla^2 \psi + f \tag{5}$$

Let us define an operator:

$$\psi^{(N)} = \psi + \frac{1}{a_N}\eta \tag{6}$$

When (6) is introduced in (5) we obtain:

$$\frac{\partial \zeta}{\partial t} = J(\eta, \psi^{(N)}) \tag{7}$$

or

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v}^{(N)} \cdot \nabla \eta, \;\; \mathbf{v}^{(N)} = \mathbf{k} \times \nabla \psi^{(N)}$$

which is equivalent to:

$$\frac{d^{(N)}\eta}{dt} = 0, \quad \frac{d^{(N)}}{dt} = \frac{\partial}{\partial t} + \mathbf{v}^{(N)} \cdot \nabla \tag{8}$$

(8) says that the absolute vorticity is conserved in the velocity field determined by the smooth-

ed field. The smoothing operator (6) can in finite differences be written:

$$\psi^{(N)} = \psi + \frac{(\Delta s)^2}{\alpha_N}\left(\frac{1}{(\Delta s)^2}\nabla^2\psi + f\right)$$

or

$$\psi^{(N)} = \psi + \frac{1}{\alpha_N}\nabla^2\psi + \frac{(\Delta s)^2}{\alpha_N} f$$

where a_N has been written $\alpha_N/(\Delta s)^2$, α_N being a parameter, which determine the smoothing. The last term has a zonal distribution only depending on f and corresponds to the $J(\varphi)$-function used by FJØRTOFT (1952). The term is easily computed when f is known in the grid points.

It is probably worth while to point out that although small the effect from the last term should not be neglected, because it is systematic. The isolines for the term have a pure zonal distribution with a high value on the North Pole and zero at the Equator and represents therefore an anticyclonic circulation around the pole. The strength of this circulation is measured by the wind in this circumpolar anticyclonic vortex. The wind is

$$u_R = -\frac{\partial}{\partial y}\left(\frac{(\Delta s)^2}{\alpha_N} f\right) = -\frac{(\Delta s)^2 2\Omega}{R\cdot\alpha_N}\cos\varphi$$

For a value of $\Delta s = 600$ km and $\alpha_N = 4$ we obtain $u_R \sim 1.5$ m sec^{-1} in middle latitudes.

The author has earlier investigated a smoothing operator defined by the two first terms of the expression for $\psi^{(N)}$. A value $\alpha_N = 4$ is generally used and represents an optimum value in the sense that $\alpha_N < 4$ gives amplification on certain wavelength. For later use in this paper it is interesting to investigate the effect of different values of α_N. Assuming a Fourier expansion

$$\psi = \sum \psi_{mn} e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\nu_m}\right)}$$

we get for the reduction factor A for the different wavelengths:

$$A = \left(1 - \frac{4}{\alpha_N}\right) + \frac{4}{\alpha_N}\cos\frac{\Delta s}{\lambda}2\pi$$

where it has been assumed that $\lambda_m \sim \nu_n \sim \lambda$

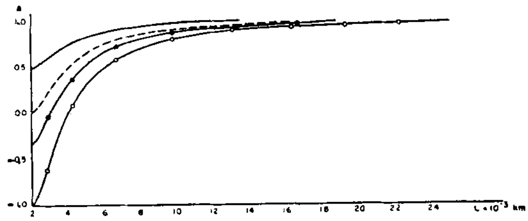

Fig. 2. Curves showing the degree of smoothing as a function of wavelength for different values of the constant α_N= in the smoothing operator. The full curve with open circles corresponds to $\alpha_N = 4$, with black circles to $\alpha_N = 6$. The dashed curve corresponds to $\alpha_N = 8$ and the full curve without marks to $\alpha_N = 16$.

Fig. 2 gives A as a function of wavelength for different values of α_N.

When $\psi^{(N)}$ now is assumed constant in time for some hours, we need not to recompute the stream function in every time step. We can then integrate the trajectory equations in the following form:

$$x^{\tau+1} = x^{\tau-1} + 2\Delta t J(x^\tau, \psi^{(N)})$$

$$y^{\tau+1} = y^{\tau-1} + 2\Delta t J(y^\tau, \psi^{(N)})$$

The computations can be performed using these equations until we have the positions of the particles to a time $t = T$ ($T \sim 12$ hours). We know then the initial positions of the particles which to $t = T$ occupy the grid points. Equation (4) is not directly suited for the computation of the vorticity, because $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ are not explicitly known. We get, however, an expression for the relative vorticity, when we eliminate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ by the aid of the equations:

$$\frac{\partial x}{\partial t} = J(x, \psi),\ \frac{\partial y}{\partial t} = J(y, \psi)$$

We obtain:

$$\zeta = J(J(x, \psi), x) + J(J(y, \psi), y)$$

When this expression is evaluated we obtain the following formula for ζ:

$$\zeta = -\left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right]\frac{\partial^2 \psi}{\partial a^2} + 2\left[\frac{\partial x}{\partial a}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\right]\frac{\partial^2 \psi}{\partial a \partial b} - \left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right]\frac{\partial^2 \psi}{\partial b^2} +$$

$$+\left[\frac{\partial^2 x}{\partial a^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial a^2}\frac{\partial y}{\partial b} - \frac{\partial^2 x}{\partial a \partial b}\frac{\partial x}{\partial a} - \frac{\partial^2 y}{\partial a \partial b}\frac{\partial y}{\partial a}\right]\frac{\partial \psi}{\partial b} + \left[\frac{\partial^2 x}{\partial b^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial b^2}\frac{\partial y}{\partial a} - \frac{\partial^2 x}{\partial a \partial b}\frac{\partial x}{\partial b} - \frac{\partial^2 y}{\partial a \partial b}\frac{\partial y}{\partial b}\right]\frac{\partial \psi}{\partial a} \quad (9)$$

This rather complicated expression is easily transformed into finite differences, if only the values of ψ are known in the initial positions of the particles. These positions will in general be between the grid points, but an interpolation between the known initial values in the grid points will probably be sufficient. This interpolation can be performed in the following way: Let us for simplicity denote the grid size by 1, and let $P=(p, q)$ (see fig. 3)

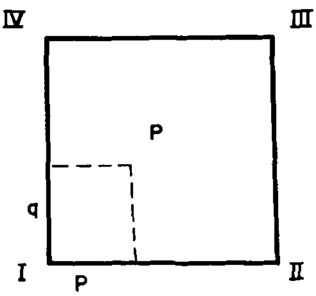

Fig. 3. The points and notations used in the interpolation of the streamfunction.

be the point in which we want an interpolated value of the stream function. One finds then easily that:

$$\psi_P = (1-p)(1-q)\psi_{\mathrm{I}} + p(1-q)\psi_{\mathrm{II}} + pq\psi_{\mathrm{III}} + (1-p)q\psi_{\mathrm{IV}} \quad (10)$$

The complicated expression (9) needs to be used only every time we are forced to solve for the stream function either because we no longer can assume that the smoothed value is quasi-constant in time, or because the deformation in the coordinate fields x (a, b, t) and y (a, b, t) has been so great that the finite difference form of (9) will give too great values for the truncation errors.

It is interesting to note that (9) is always an elliptic expression if ψ is considered as the unknown, because:

$$\left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right]\left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right] - \left[\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\right]^2 = \left[\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\right]^2 \geq 0$$

This property is unimportant in the present formulation of the problem, but will be important, if we want to follow the particles which initially occupy the grid points. In this case we will have to solve (9) for ψ with the left side equal to $\zeta_0 + f_0 - f$.

The computation of the relative vorticity from (9) may of course be replaced by a direct evaluation of $J\left(\frac{\partial x}{\partial t}, x\right) + J\left(\frac{\partial y}{\partial t}, y\right)$ if this is more convenient from the computational point of view.

The steps in a numerical integration of the system put forward in this paragraph would follow the scheme:

1. The stream function is found.
2. Smoothing of the stream function by the aid of (6).
3. Evaluation of x^τ and y^τ keeping $\psi^{(N)}$ constant in time. This process is continued up to approximately $T=12$ hours.
4. Interpolation, using expression (10).
5. Evaluation of the initial relative vorticity ζ_0 using (9)
6. Computation of $f_0 - f$
7. Solution of $\nabla^2\psi^T = \zeta_0 + f_0 - f$

Next, the steps are repeated from step 2.

4. Boundary conditions in the Lagrangian integrations

Special problems appear concerning the boundary conditions in the cases treated in section 3. A number of experiments regarding the boundary conditions in the trajectory problem has been performed by Djuric and the author. Other experiments are reported by Benton (1957). The best results were obtained by Benton in the cases when the boundary values were computed using an extrapolation from the values inside the region. If this system is adopted, one should have the same boundary conditions for ψ, x and y. If we, however, use the first system, in which ψ is kept constant on the boundary, we need boundary conditions for x and y, which are in consistence with the one for ψ. The most convenient conditions are probably those proposed by Benton.

5. The general barotropic case

In section 3 we have taken advantage of the assumed constancy of the space-smoothed field. By this assumption we could keep track of the individual particle which ended up in the grid points by computing "backwards" in the constant field and obtained in this way the knowledge of the relative vorticity after the time T in the points of the rectangular grid. This procedure cannot be directly transformed into the general barotropic case, when we do not use the space-smoothing. It should be noted that the method in section 3 is only quasi-Lagrangian in the time interval T. When a new period is started, we do not follow the same particles any longer, but ask then for the positions of the particle which after the next period arrive in the grid points.

A more general, but also more complicated numerical scheme can be arranged, when we give up the smoothing technique and the "backward" computation. We try in this case to follow the particles, which initially are situated in the grid points. Initially we can therefore compute the relative vorticity by the usual finite approximation formula. The different steps in the general time step may then be the following:

1. The displacement is computed from the following equations:

$$x^{\tau+1} = x^{\tau-1} + 2\Delta t\, J(\psi^\tau, x^\tau)$$
$$y^{\tau+1} = y^{\tau-1} + 2\Delta t\, J(\psi^\tau, y^\tau)$$

2. The Coriolis parameter $f^{\tau+1}$ in the point $(x^{\tau+1}, y^{\tau+1})$ is computed.

3. The relative vorticity in the new position is computed:

$$\zeta^{\tau+1} = \zeta^\tau + f^\tau - f^{\tau+1}$$

4. The field $\psi^{\tau+1}$ is computed by solving the equation:

$$\left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right]\frac{\partial^2\psi}{\partial a^2} - 2\left[\frac{\partial x}{\partial a}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\right]\frac{\partial^2\psi}{\partial a\partial b} + \left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right]\frac{\partial^2\psi}{\partial b^2} -$$
$$- \left[\frac{\partial^2 x}{\partial a^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial a^2}\frac{\partial y}{\partial b} - \frac{\partial^2 x}{\partial a\partial b}\frac{\partial x}{\partial a} - \frac{\partial^2 y}{\partial a\partial b}\frac{\partial y}{\partial a}\right]\frac{\partial\psi}{\partial a} - \left[\frac{\partial^2 x}{\partial b^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial b^2}\frac{\partial y}{\partial a} - \frac{\partial^2 x}{\partial a\partial b}\frac{\partial x}{\partial b} - \frac{\partial^2 y}{\partial a\partial b}\frac{\partial y}{\partial b}\right]\frac{\partial\psi}{\partial a} = \zeta^{\tau+1} \tag{11}$$

This equation is closely connected with (9), only are the signs of the different terms changed, because the formula now appears by developing the expression

$$J\left(\frac{\partial x}{\partial t}, x\right) + J\left(\frac{\partial y}{\partial t}, y\right)$$

and $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ has changed sign.

5. The next time step is then started from step 1.

It is obvious that this system will work rather slow because the solution of the equation in step 4 for ψ will be rather complicated. The equation is, however, as mentioned always elliptic, and a generalization of the usual relaxation method will most probably converge.

When the boundary conditions are treated in a way similar to the one discussed in the last section we will after a number of time steps end up with the information of the position of the particles and the corresponding values of the stream function. The stream function value must then be plotted in the point of position of the particle and the resulting map has to be analysed as a usual meteorological chart, when the points of position act as the sounding stations.

It is not necessary to recompute the height field, when the forecast is finished. The wind values in which we are mostly interested can be computed directly from the stream function. It is naturally important that the regions of interest are removed sufficiently from the initial boundaries in order to be sure that the final forecast will cover the region, and also such that the region of interest because of boundary errors are in some distance of the final boundaries.

6. The baroclinic case

It is essential for the application of the methods outlined in the preceding sections that a conservation theorem forms the basis for the time integration, and it has further been used that the horizontal flow can be assumed non-divergent. The baroclinic case may be treated in a similar way if we can formulate the prognostic equations as conservation theorems.

We shall in the following derive a two parameter model, which physically is equivalent to the greatest part of the models which have been used in numerical forecasting. The essence of this model was given by Fjørtoft on the conference on numerical forecasting in Stockholm, June 1957. The treatment given here will be somewhat more general in one respect. It was assumed by Fjørtoft that the same smoothing operator with the same value for the parameter α_N (eq. 7) could be used for the space-smoothing of the height field and the thermal field. It is, however, not a priori given that it is the same scale which we want to eliminate in the two fields. Unfortunately, there exist no detailed investigations of the spectral distribution of the kinetic energy on different pressure-levels in the atmosphere, and especially of the flow of energy from one scale to another. Kuo (1953) has from perturbation theory shown that the smaller scales will dominate at the lower levels, while the larger scales will occupy the higher levels in the troposphere. This distribution coincides with the general synoptic experience, although one should be careful with the conclusions, because our knowledge of the flow at higher levels is still limited because of the sparsity of data at these levels. The larger scale is obviously the one which is represented most accurately on our maps for 300 and 200 mb, while the density of the surface observations makes it possible to represent also the smaller scales. Until more information is obtained regarding this question it seems reasonable to keep the possibility open, that different scales should be removed from the height field for the mid-tropospheric level and the thermal field. If Kuo's results can be applied, the thermal field will contain the smaller scales, because it in the model is obtained as a difference between the height fields at a high and a low level.

It has been shown by Thompson (1956) and is inherent in the Sawyer-Bushby model (1953) that the vertical velocity has a zero point approximately at the level where the horizontal wind assumes its maximum value close to the tropopause. We shall adopt these results and assume as boundary conditions that $\omega \simeq 0$ for $p = 1{,}000$ mb and $p = 200$ mb, the 200 mb level being the level where the horizontal wind most likely has the maximum.

We shall denote the levels in the following way: The pressure levels 200, 400, 600, 800 and 1,000 mb will be denoted by subscripts 0, 1, 2, 3, and 4. The vertical boundary conditions are then:

$$\omega_0 \simeq 0 \qquad \text{and} \qquad \omega_4 \simeq 0$$

The vorticity equation shall be applied in the following simplified form:

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta = f \frac{\partial \omega}{\partial p} \tag{12}$$

where the vertical advection of vorticity and the "twisting" term has been neglected, and further the absolute vorticity has been approximated by the Coriolis parameter when it appears undifferentiated. These neglections and approximations are consistent with each other, when certain integral constraints have to be satisfied (Wiin-Nielsen, 1959).

We shall further apply the adiabatic equation:

$$\frac{d}{dt}\frac{\ln \Theta}{dt} = 0, \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega \frac{\partial}{\partial p} \tag{13}$$

and the hydrostatic equation:

$$\frac{\partial \phi}{\partial p} = -\alpha \tag{14}$$

The adiabatic equation (13) can by the aid of (14) be written in the following form:

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + \mathbf{v} \cdot \nabla \left(\frac{\partial \phi}{\partial p}\right) - \omega \alpha \frac{\partial \ln \Theta}{\partial p} = 0 \tag{15}$$

If we now further introduce a geostrophic stream function[1] ψ by the two relations:

$$\left.\begin{aligned} \nabla^2\psi &= \frac{1}{f}\,\nabla^2\phi - \frac{1}{f^2}\,\nabla\phi\cdot\nabla f \\ \frac{\partial\psi}{\partial p} &= \frac{1}{f}\,\frac{\partial\phi}{\partial p} \end{aligned}\right\} \qquad (16)$$

we can write (15) in the form:

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial p}\right) + \mathbf{v}\cdot\nabla\left(\frac{\partial\psi}{\partial p}\right) - \omega\frac{\alpha}{f}\frac{\partial\ln\Theta}{\partial p} = 0 \qquad (17)$$

which is the form in which we will use the adiabatic equation.

The vorticity equation (12) is applied at the levels 1 and 3 and vertical derivatives are approximated by finite differences:

$$\frac{\partial\eta_1}{\partial t} + \mathbf{v}_1\cdot\nabla\eta_1 = f\,\frac{\omega_2}{P} \qquad (18)$$

where $P = 400$ mb, and

$$\frac{\partial\eta_3}{\partial t} + \mathbf{v}_3\cdot\nabla\eta_3 = -f\,\frac{\omega_2}{P} \qquad (19)$$

The adiabatic equation is applied at level 2. We use here

$$\frac{\partial\psi}{\partial p} \sim \frac{\psi_3 - \psi_1}{P} = -\frac{\psi_T}{P}, \text{ where } \psi_T = \psi_1 - \psi_3 \qquad (20)$$

and obtain:

$$\frac{\partial\psi_T}{\partial t} + \mathbf{v}_2\cdot\nabla\psi_T = k\,\frac{f}{P}\,\omega_2 \qquad (21)$$

where

$$k = -\frac{P^2}{f^2}\,\alpha\,\frac{\partial\ln\Theta}{\partial p} \qquad (22)$$

The quantity k is closely related to the vertical stability and will in accordance with the approximations used in most baroclinic models be assumed quasi-constant.

The velocity $\mathbf{v}_2$ in the horizontal advection term in (21) can be replaced by either $\mathbf{v}_1$ or $\mathbf{v}_3$ if it in consistency with our evaluation of vertical derivatives is assumed that

$$\mathbf{v}_2 = \frac{1}{2}\,(\mathbf{v}_1 + \mathbf{v}_3) \qquad (23)$$

We get then:

$$\mathbf{v}_2\cdot\nabla\psi_T = \mathbf{v}_1\cdot\nabla\psi_T$$

and

$$\mathbf{v}_2\cdot\nabla\psi_T = \mathbf{v}_3\cdot\nabla\psi_T$$

We may therefore write equation (21) in the following form:

$$\frac{d_1}{dt}\left(\frac{\psi_T}{k}\right) = \frac{d_3}{dt}\left(\frac{\psi_T}{k}\right) = \frac{f}{P}\,\omega_2 \qquad (24)$$

where

$$\frac{d_1}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_1\cdot\nabla,\quad \frac{d_3}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_3\cdot\nabla$$

The equation (24) may now be combined with (18) and (19), respectively, and we get:

$$\frac{d_1}{dt}\left(\eta_1 - \frac{\psi_T}{k}\right) = 0,\quad \frac{d_3}{dt}\left(\eta_3 + \frac{\psi_T}{k}\right) = 0 \qquad (25)$$

Again, in accordance with our assumptions about the vertical variation of the horizontal wind we can write:

$$A = \eta_1 - \frac{\psi_T}{k} = \zeta_2 + f + \frac{1}{2}\,\zeta_T - \frac{\psi_T}{k} = \eta_2 + \left(\frac{1}{2}\,\xi_T - \frac{\psi_T}{k}\right)$$

$$B = \eta_3 + \frac{\psi_T}{k} = \zeta_2 + f - \frac{1}{2}\,\zeta_T + \frac{\psi_F}{k} = \eta_2 - \left(\frac{1}{2}\,\zeta_T - \frac{\psi_T}{k}\right)$$

The equations (25) are now the two conservation theorems which expresses that the quantities A and B as defined above are individually conserved in the horizontal flow at the levels 1 and 3, respectively. We may shortly write:

$$\frac{d_1 A}{dt} = \frac{\partial A}{\partial t} + \mathbf{v}_1\cdot\nabla A = 0 \qquad (26)$$

$$\frac{d_3 B}{dt} = \frac{\partial B}{\partial t} + \mathbf{v}_3\cdot\nabla B = 0 \qquad (27)$$

[1] The two relations (16) are not consistent with each other which can be seen by differentiating the first relation partially with respect to pressure and taking the horizontal Laplace-operator by the second. The inconsistency is not considered important.

The development at the two levels 1 and 3 is coupled through the close relationship between the two quantities A and B. It will in the numerical integration of the equations (26) and (27) be most convenient to use the flow at the level 2, determined through the stream function ψ_2 and the flow in the thermal field characterised through the stream function ψ_T as the two parameters, which carry the history of the flow.

7. On the application of smoothing in the baroclinic case

The introduction of smoothing may in the baroclinic case be performed in a way analogous to the barotropic case. As mentioned in section 6 we shall keep the possibility open to apply different smoothing parameters on the two fields ψ_2 and ψ_T.

Provided that the thermal field ψ_T contains more energy on a smaller scale than the field ψ_2 it is most convenient to define the smoothing operators by the following expressions:

$$\psi_2^{(N)} = \psi_2 + \frac{\eta_2}{a_N - c} \tag{28}$$

and

$$\psi_T^{(N)} = \psi_T + \frac{\zeta_T}{a_N} \tag{29}$$

α_N has the same meaning as before, while c is an arbitrary positive constant.

The introduction of the smoothing operators (28) and (29) into the forecasting equations (26) and (27) is demonstrated below for (26). We write (26) in the following form:

$$\frac{\partial A}{\partial t} = J\left(A, \psi_2 + \frac{1}{2}\psi_T\right) =$$

$$= J\left(A, \psi_2 + \frac{1}{2}\psi_T + \frac{t_A}{a_N}A\right)$$

when t_A is an arbitrary parameter to be determined later. When the expression for A is substituted into the last term of the second component of the Jacobian we obtain:

$$\frac{\partial A}{\partial t} = J\left(A, \psi_2^{(N)} + \left(\frac{1}{2} - \frac{t_A}{ka_N}\right)\psi_T^{(N)} + \right.$$

$$\left. + \left(\frac{t_A}{a_N} - \frac{1}{a_N - c}\right)\eta_2 + \left(\frac{t_A}{2a_N} - \frac{1}{2a_N} + \frac{t_A}{ka_N}\right)\zeta_T\right)$$

The parameter t_A is then determined in such a way that the coefficient to ζ_T becomes zero, i.e.

$$t_A = \frac{ka_N}{2 + ka_N}$$

The local change of A can with this value of t_A be written:

$$\frac{\partial A}{\partial t} = J\left(A, \psi_2^{(N)} + \left(\frac{1}{2} - \frac{1}{ka_N + 2}\right)\psi_T^{(N)} + \right.$$

$$\left. + \left(\frac{k}{ka_N + 2} - \frac{k}{ka_N - k_c}\right)\eta_2\right)$$

Denoting now:

$$q = \frac{1}{2} - \frac{1}{ka_N + 2}$$

and

$$2r = -a_N\left(\frac{k}{ka_N + 2} - \frac{k}{ka_N - k_c}\right)$$

we can then write the formula above as:

$$\frac{\partial A}{\partial t} = J\left(A, \psi_2^{(N)} + q\psi_T^{(N)} - 2r\frac{1}{a_N}\eta_2\right)$$

Noting now that

$$J(A, B) = J(A, A + B) = 2J(A, \eta_2)$$

we can also write:

$$\frac{\partial A}{\partial t} = J\left(A, \psi_2^{(N)} + q\psi_T^{(N)} - r\frac{1}{a_N}B\right) \tag{30}$$

(30) expresses that the quantity A is individually conserved in the field:

$$\psi_2^{(N)} + q\psi_T^{(N)} - r\frac{1}{a_N}B$$

With the usual assumption that $\psi_2^{(N)}$ and $\psi_T^{(N)}$ are varying so slowly in time that this variation can be neglected for some hours the only rapid varying term in the advection field is the last one.

The second forecasting equation (27) assumes the following form, when a derivation similar to the one demonstrated above is performed:

$$\frac{\partial B}{\partial t} = J\left(B, \psi_2^{(N)} - q\psi_T^{(N)} - r\frac{1}{a_N}A\right) \tag{31}$$

The equations (30) and (31) form the basis for the forecasts. The simplicity of the equations make them extremely convenient, if an Eulerian procedure can be used. In that case it is only necessary to solve the different equations once in the time-period T, in which $\psi_2^{(N)}$ and $\psi_T^{(N)}$ can be assumed constant. If, however, a quasi-Lagrangian system should be used the procedure becomes more complicated. A possible numerical scheme for the integration is given below.

1. The fields $\psi_2^{(N)}$ and $\psi_T^{(N)}$ are computed, and $\psi_2^{(N)}+q\psi_T^{(N)}$ and $\psi_2^{(N)}-q\psi_T^{(N)}$ are formed.
2. Initially A and B can be computed from the finite difference expressions and the two advection fields can be computed.
3. Two coordinate sets (x_1, y_1) and (x_3, y_3) are introduced. In the general time step the changes in the coordinates are computed from the formula:

$$x_1^{\tau+1}=x_1^{\tau-1}+2\Delta tJ\left(x_1^\tau,\ \psi_2^{(N)}+q\psi_T^{(N)}-r\frac{1}{a_N}B\right)$$

   and the formulas analogous to this one.
4. The next problem is to compute the values of $A^{\tau+1}$ and $B^{\tau+1}$ in the points $(x_1^{\tau+1}, y_1^{\tau+1})$ and $(x_3^{\tau+1}, y_3^{\tau+1})$, because these values have to be used to correct the two advection fields in the grid points. The fields A and B are, however, of such a large scale, because they contain the term $\frac{\psi_T}{k}$, that an interpolation directly between the grid points probably is sufficiently accurate. If this should not be the case, an interpolation in the fields $\psi_2+\frac{1}{2}\psi_T$ and $\psi_2-\frac{1}{2}\psi_T$ must be made, and a procedure similar to the one outlined in section 3 has to be used.
5. The advection fields are corrected by adding the term

$\frac{r}{a_N}(B^\tau-B^{\tau+1})$ to the upper field and the term

$\frac{r}{a_N}(A^\tau-A^{\tau+1})$ to the lower field

6. If the time T is not used, the computation is started from step 3 again and repeated up to the time T. When this time is used, we have the values of A and B in the grid points. In order to compute the corresponding stream functions we have to solve the two simultaneous equations:

$$\nabla^2\psi_2+f+\frac{1}{2}\nabla^2\psi_T-\frac{\psi_T}{k}=A$$

$$\nabla^2\psi_2+f-\frac{1}{2}\nabla^2\psi_T+\frac{\psi_T}{k}=B$$

Adding and subtracting the two equations, we get:

$$\nabla^2\psi_2=\frac{1}{2}(A+B)-f \tag{32}$$

$$\nabla^2\psi_T-\frac{2}{k}\psi_T=\frac{1}{2}(A-B) \tag{33}$$

These equations may then be solved by the usual relaxation methods.

It has been found convenient not at this stage to complicate the derivations by the additional problems which arise, when the formulae will be applied to actual computations. Finite differences have only been introduced at a few places, when necessary, and no attention has been paid to the map-magnification factor. It will, however, be illustrating to compute the numerical values in the baroclinic formulae in order to see that they have a convenient order of magnitude.

The vertical stability $\sigma=-\alpha\frac{\partial\ln\Theta}{\partial p}$ has in the standard atmosphere a value of approximately 1.88 m³ t⁻¹ cb⁻¹. With this value the parameter k takes the value $k=30{,}08\times10^{10}$ m². $a_N=\frac{\alpha_N}{(\Delta s)^2}=\frac{1}{6}\times10^{-10}$ m⁻², when $\Delta s\sim600$ km $=6\times10^5$ m. When we finally put $c=c_1/(\Delta s)^2$, $c_1=2$ we obtain for the two constants which determine the smoothed advection fields

$$q\sim0.36$$

and

$$r\sim0.40$$

$\alpha_N=6$ and $c_1=2$ mean that a wave with the wavelength $L=2{,}400$ km is smoothed completely in the field ψ_2, while the most effective smoothing in the thermal field appears on a wavelength $L=1{,}800$ km. The practical application of the quasi-Lagrangian method will give information about the most appropriate values for α_N and c_1.

The finite difference form of the quantities $\frac{1}{a_N}A$ and $\frac{1}{a_N}B$ are given by the following expressions, where again the map-magnification factor is neglected:

$$\frac{1}{a_N}A=\frac{1}{\alpha_N}\nabla^2\psi_2+\frac{(\Delta s)^2}{\alpha_N}f+$$

$$+\frac{1}{2\alpha_N}\nabla^2\psi_T-\frac{(\Delta s)^2}{\alpha_N k}\psi_T$$

$$\frac{1}{a_N}B=\frac{1}{\alpha_N}\nabla^2\psi_2+\frac{(\Delta s)^2}{\alpha_N}f-$$

$$-\frac{1}{2\alpha_N}\nabla^2\psi_T+\frac{(\Delta s)^2}{\alpha_N k}\psi_T$$

The numerical value of the coefficient $\frac{(\Delta s)^2}{\alpha_N k}$ is with the value above approximately $\frac{1}{5}$.

It is further of interest to find the numerical value of the coefficient to ψ_T in the Helmholz equation in (33), when this equation is brought on the finite difference form. We get then:

$$\nabla^2\psi_T-\frac{2(\Delta s)^2}{k}\psi_T=(\Delta s)^2\frac{A-B}{2}$$

With the values mentioned above we obtain $\frac{2(\Delta s)^2}{k}\sim 2.4$, a value which assures us that the convergence of the relaxation for equation (33) will be rather fast.

Finally, it should be mentioned that a computational scheme corresponding to the one in section 5 has not been constructed for the baroclinic case, when no smoothing is applied. The procedure becomes somewhat complicated in this case and it remains to be seen whether the technique will give any improvement in the barotropic case.

8. On the influence of the space smoothing on the phase speed of simple harmonic waves

The extensive use which has been made of the graphical technique in physical weather prediction in the last years makes it interesting to investigate in which way the speed of the atmospheric waves is influenced by the assumed constancy in time of the space-smoothed field. In the following we shall try to deal with this question in the barotropic case for certain simple flow-patterns.

In section 3 it was shown that the barotropic vorticity equation

$$\frac{\partial\zeta}{\partial t}=J(\eta,\psi),\quad \eta=\nabla^2\psi+f \tag{34}$$

exactly may be replaced by the equation

$$\frac{\partial\zeta}{\partial t}=J(\eta,\psi^{(N)},)\quad \eta^{(N)}=\psi+\frac{1}{a_N}\eta \tag{35}$$

(34) will have solutions of the form

$$\psi(x,y,t)=-Uy+A\sin k(x-ct) \tag{36}$$

when $c=U-\beta/k^2$, and $\beta=\partial f/\partial y$ is considered as a constant.

It is the purpose in the following to investigate the solutions to (35) and especially to see which difference there exists between those solutions and the solutions (36) for different values of the wave number k when the space-smoothed flow $\psi^{(N)}$ is kept constant over a certain time interval.

For this purpose we need first to compute the space-smoothed flow. We shall not at this moment be concerned with the differences which are introduced because of finite difference operations. With this in mind we obtain from (36)

$$\psi^{(N)}(x,y,t)=-\left(U-\frac{\beta}{a_N}\right)y+$$

$$+A\left(1-\frac{k^2}{a_N}\right)\sin k(x-ct)+\text{const.} \tag{37}$$

In the derivation of (37) we have treated the Coriolis parameter as $f=f^\star+\beta y$, where $f^\star$ and β are assumed to be constants.

Suppose further that $a_N=k_1^2$, where k_1 is a certain wave-number. The displacement field is then

$$\psi_0^{(N)}=\psi^{(N)}(x,y,0)=-\left(U-\frac{\beta}{k_1^2}\right)y+$$

$$+A\left(1-\frac{k^2}{k_1^2}\right)\sin kx \tag{38}$$

The procedure which will be used in the following is similar to the one which is used in the practical application of Fjørtoft's method. The displacement field $\psi_0^{(N)}$ is used to find the approximate trajectories. We know then the position (x, y) of the particles which to $t=0$ is situated in a certain point (x_0, y_0). In the next step we use the condition that the absolute vorticity is approximately conserved along the computed trajectory, and we know therefore also the distribution of absolute vorticity to this time. From this knowledge we will be able to compute the streamfunction by solving a Poisson equation.

The first step is to compute the trajectories in the field (38). The analytic expressions for the trajectories is obtained as solutions to the two equations:

$$\frac{dx}{dt} = u_0^{(N)} = U - \frac{\beta}{k_1^2} = c_1 \qquad (39)$$

$$\frac{dy}{dt} = V_0^{(N)} = kA\left(1 - \frac{k^2}{k_1^2}\right)\cos kx \qquad (40)$$

The solution to (39) is

$$x = x_0 + c_1 t \qquad (41)$$

while (40) has the solution:

$$\int_{y_0}^{y} dy = kA\left(1 - \frac{k^2}{k_1^2}\right)\int_0^t \cos k(x_0 + c_1 t)dt \qquad (42)$$

which leads to

$$y = y_0 + \frac{A}{c_1}\left(1 - \frac{k^2}{k_1^2}\right)\{\sin kx - \sin k(x - c_1 t)\} \qquad (43)$$

The next step is then to equate the absolute vorticity which to $t=0$ is situated in the point (x_0, y_0) to the absolute vorticity which to $t=t$ is in the point (x, y). This condition is expressed in the equation

$$\nabla^2\psi^\star + f = (\nabla^2\psi)_0 + f_0 \qquad (44)$$

which leads to

$$\nabla^2\psi^\star = -k^2 A \sin k(x - c_1 t) - \frac{A\beta}{c_1}\left(1 - \frac{k^2}{k_1^2}\right)\sin kx + \frac{A\beta}{c_1}\left(1 - \frac{k^2}{k_1^2}\right)\sin k(x - c_1 t) \qquad (45)$$

Apart from a possible linear function in y we can directly integrate (45) and obtain:

$$\psi^\star(x, y, t) = A\left(1 - \frac{\beta}{c_1}\left[\frac{1}{k^2} - \frac{1}{k_1^2}\right]\right)\sin k(x - c_1 t) + \frac{A\beta}{c_1}\left[\frac{1}{k^2} - \frac{1}{k_1^2}\right]\sin kx \qquad (46)$$

From the solution (46) we notice first that if $k = k_1$ we obtain the correct solution because in this case we have

$$\psi^\star = A \sin k(x - c_1 t) \qquad (47)$$

which coincide with the x-dependent part of (36) for $k = k_1$.

Next, we notice that the solution also becomes correct if the wave number coincide with the one corresponding to the stationary wave, i.e. $k = k_s$, when k_s satisfy the relation

$$k_s = (\beta/U)^{\frac{1}{2}} \qquad (48)$$

For this choice of k we obtain

$$\frac{\beta}{c_1}\left[\frac{1}{k_s^2} - \frac{1}{k_1^2}\right] = \frac{\beta}{c_1}\left[\frac{U}{\beta} - \frac{1}{k_1^2}\right] = \frac{\beta}{c_1}\left[\frac{c_1 + \frac{\beta}{k_1^2}}{\beta} - \frac{1}{k_1^2}\right] = 1$$

and (46) reduces to

$$\psi^\star = A \sin k_s x \qquad (49)$$

For all wave numbers different from k_1 and k_s we will not obtain a correct solution. In order to obtain measures of the error introduced by the assumption $\psi^{(N)}(x, y, t) = \psi^{(N)}(x, y, 0)$ we shall for convenience introduce the notation

$$a = \frac{\beta}{c_1 k^2}\left(1 - \frac{k^2}{k_1^2}\right) \qquad (50)$$

The expression (46) may then be written

$$\psi^\star = A(1 - a)\sin k(x - c_1 t) + Aa \sin kx \qquad (51)$$

By the aid of ordinary trigonometric formulae (51) may be written in the form

$$\psi^\star = R \cdot A \cdot \sin k(x - \mu t) \qquad (52)$$

Table I

L km	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000	12.000	14.000
R	0.92	0.88	0.89	0.93	1.00	1.07	1.15	1.24	1.43	1.55
m sec^{-1}	16.8	14.0	9.5	4.5	0.4	— 4.9	— 10.4	— 15.9	— 27.2	— 39.5
m sec^{-1}	16.4	13.6	10.0	5.6	0.4	— 5.6	— 12.4	— 20.0	— 37.6	— 58.4

The expressions for $R(t)$ and $\mu(t)$ are given by

$$R(t) = [(1 - 2a + 2a^2) + 2a(1-a)\cos kc_1 t]^{\frac{1}{2}} \quad (53)$$

and

$$\operatorname{tg}(\mu k_1 t) = \frac{(1-a)\sin kc_1 t}{a + (1-a)\cos kc_1 t} \quad (54)$$

To a certain fixed time, which in the following numerical computations is put equal to 12 hours, R measures the error in the amplitude, while μ, according to (52) gives the apparent phase-speed. A comparison between μ and c will show the error in phase-speed.

Table I contains numerical values for R, μ and c as a function of wave length. The values in the table are computed for $L_1 = 2{,}400$ km, $L_1 = 2\pi/k_1$, $U = 20$ m sec^{-1} and $\beta = 1.6 \times 10^{-11}$ m^{-1} sec^{-1}, corresponding to 45° N. μ and c are expressed in the unit: m sec^{-1}, while R is non-dimensional.

It is seen from table I that the error in amplitude is at most 12 % for $L = 4{,}000$ km for the waves with a wavelength between 2,400 km and the stationary wavelength. For these waves it is also seen that the difference between μ and c is numerically smaller than 0.5 m sec^{-1}. However, when we consider waves with a wavelength larger than the stationary one, we observe that the errors in amplitude may become quite large. With regard to the apparent phase speed we observe also quite large differences. The general tendency is, however, that the negative values of μ are numerically smaller than the values of c. This means that the retrogression of the longer waves will be smaller in Fjørtoft's method than the computed retrogression in a numerical integration of the barotropic vorticity equation.

The fast retrogression of the long waves predicted by the barotropic theory is, however, not observed in the atmosphere. We should therefore not be too concerned with the large values of R and the large difference between μ and c, which appears for $L > L_s$. Different "stop-gap" methods has been suggested to correct empirically for the large errors due to the retrogression of these planetary waves (WOLFF, 1958).

For values of $L < L_s$ we find that the errors due to the main assumption $\psi^{(N)} = \psi_0^{(N)}$ are rather small, especially with regard to the phase-speed. Although the present elementary computation does not allow nonlinear interactions, because U is constant, and because we only consider simple sinusoidal waves, it still shows that the results of integrations, performed with Fjørtoft's methods probably will be of about the same accuracy as a time-integration as they usually are performed on an electronic computer. Earlier comparisons (FJØRTOFT, 1956 and SIGTRYGGSSON and WIIN-NIELSEN, 1957) have in fact shown a close correspondence between such different integrations.

The considerations in the preceding sections have assumed that the smoothed field was computed using the differential form of the smoothing operator. In actual practice this is not the case. On the other hand, $\psi^{(N)}$ is computed using a formula of the form:

$$\psi^{(N)} = \psi + \frac{1}{\alpha_N}\ \nabla^2\psi + \frac{f(\Delta s)^2}{\alpha_N} \quad (55)$$

In order to investigate which effects this fact may introduce in the computations we shall consider a case where the assumption $\psi^{(N)} = \psi_0^{(N)}$ is exactly fulfilled. We consider a streamfunction

$$\psi(x, y, t) = -Uy +$$
$$+ A_1 \sin k_1 (x - c_1 t) + A_s \sin k_s x \quad (56)$$

with $k_s = (\beta/U)^{\frac{1}{2}}$.

For $\alpha_N = 2$ and $\Delta s = \frac{\pi}{2k_1} = \frac{L_1}{4}$ we find that the smoothed streamfunction becomes

$$\psi^{(N)} = -\left(U - \frac{(\Delta s)^2}{2}\beta\right)y + $$
$$+ A_s \cos k_s \Delta s \sin k_s x + \frac{f_0 (\Delta s)^2}{2} \qquad (56)$$

which is independent of time. Denoting $c^\star = U - \frac{(\Delta s)^2}{2}\beta$ we find as before the trajectories in the smoothed field from the equations

$$\frac{dx}{dt} = c^\star, \quad x = x_0 + c^\star t \qquad (57)$$

$$\frac{dy}{dt} = V_0^{(N)} = \frac{A_s}{2\Delta s} \sin(2k_s\Delta s) \cos k_s x \quad (58)$$

where $V_0^{(N)}$ also has been computed by finite differences in the usual way. The integration of (58) leads to

$$y = y_0 + \frac{A_s}{2k_s c^\star \Delta s} \sin(2k_s \Delta s) \cdot$$
$$\cdot [\sin k_s x - \sin k_s (x - c^\star \cdot t)] \qquad (59)$$

We may now again integrate the equation corresponding to (44), and we obtain in this case:

$$\nabla^2 \psi^\star = -k_1^2 A_1 \sin k_1 x_0 - k_s^2 A_s \sin k_s x_0 +$$
$$+ \beta(y_0 - y) \qquad (60)$$

This equation has a solution which apart from a linear function in y may be written:

$$\psi^\star = A_1 \sin k_1 (x - c^\star t) + \frac{\beta}{2k_s^3 c^\star \Delta s} \cdot$$
$$\cdot \sin(2k_s\Delta s) A_s \sin k_s x + A_s \left(1 - \frac{\beta}{2k_s^3 c^\star \Delta_s} \cdot \right.$$
$$\left. \cdot \sin(2k_s \Delta s)\right) \sin k_s (x - c^\star t) \qquad (61)$$

In the comparison between the solution (61) and the exact solution (56) we notice:

1. Because of the finite differences we get a wave-speed

$$c^\star = U - \frac{(\Delta s)^2}{2}\beta \qquad (62)$$

which shall be compared with

$$c_1 = U - \frac{\beta}{k_1^2} = U - \frac{4(\Delta s)^2}{\pi^2}\beta \qquad (63)$$

From (62) and (63) it is seen that

$$c^\star < c_1 \qquad (64)$$

indicating that the wave-speed is too small due to finite differences.

The percentual error for different values of U and for a value of β corresponding to 45° N is given in table II below.

Table II

U m sec^{-1}	10	20	30	40	50
$\frac{c_1 - c^\star}{c_1} \cdot 100$ %	7.5	3.3	2.1	1.5	1.2

2. From (61) we notice further that a wave with wave number k_s but moving with the speed $c^\star$ is introduced. The amplitude of this wave is, however, in most cases considerably smaller than the amplitude of the stationary wave. The ratio R_A between the two amplitudes, measured in per cent, is given in table III for different values of U.

Table III gives at the same time the amount with which the amplitude of the stationary wave is decreased in the forecasts.

Table III

U m sec^{-1}	10	20	30	40	50
R_A	8 %	5 %	3 %	2 %	2 %

The considerations mentioned above shows that only small errors are introduced because of the finite differences. We have therefore been justified in the application of the differential formulae in the first parts of this section.

9. Conclusions

It has been shown that a quasi-Lagrangian technique can be used in the models at present most commonly used in operational numerical forecasting. The procedure becomes extremely simple when combined with the space-smoothing technique.

An advantage of the method is that it assures us that the properties of the fluid which should be conservative also remain so during the forecast within the accuracy of the finite difference approximation used, and further the method assures that the deformation properties of the flow are taken intonsideration in a better way than in an Eulerian technique.

The possibilities of the technique need to be investigated by application to practical cases which also have been treated with the usual numerical technique. Numerical interations using a system which in many respects is similar to the one outlined here is being prepared by Mr. Økland at the Norwegian Weather Service, Oslo (oral communication).

10. Acknowledgements

The author wants to thank Dr. R. Fjørtoft, who first directed the author's attention towards the Lagrangian methods, for a number of interesting discussions during the preparation of this paper. Further acknowledgements are due to Dr. Welander, who kindly read the manuscript critically.

REFERENCES

BENTON, GEORGE S., 1958: Some comments on the numerical integration of the vorticity equation and related equations. *Tellus*, **10**, pp. 313 – 325.

BOLIN, B., 1956: An improved barotropic model and some aspects of using the balance equation for three-dimensional flow. *Tellus*, **8**, pp. 61 – 75.

DJURIC, D., and WIIN-NIELSEN, A., 1956: On the computation of meteorological trajectories with the aid of electronic computers, *Report, Dec. 1956*.

DÖÖS, BO R., 1956: Automation of 500 mb forecasts through successive numerical map analyses, *Tellus*, **8**, pp. 76 – 81.

FJØRTOFT, R., 1952: On a numerical method of integrating the barotropic vorticity equation, *Tellus*, **4**, pp. 179 – 194.

– 1955: On the use of space-smoothing in physical weather forecasting, *Tellus*, **7**, pp. 462 – 480.

– 1956: Letter to the Editor, *Tellus*, **8**, p. 115.

KNIGHTING, E., JONES, D. E., and HINDS, M. K., 1958: Numerical experiments in the integration of the meteorological equations of motion. *Ouart. Journ.* R.M.S., **84**, pp. 91 – 108.

KUO, H. L., 1953: The stability properties and the structure of disturbance in a baroclinic atmosphere, *Journ. of Met.*, **10**, pp. 235 – 243.

MARTIN, D. E., 1958: An investigation of systematic errors in the barotropic forecasts, *Tellus*, **10**, pp. 451 – 465.

SAWYER, J. S., and BUSHLY, F. H., 1953: A baroclinic model-atmosphere suitable for numerical integration, *Journ. of Met.*, **10**, pp. 54 – 59.

SHUMAN, F. G., 1957: Predictive consequences of certain physical inconsistencies in the geostrophic barotropic model, *Monthly Weather Review*, **85**, pp. 229 – 234.

SIGTRYGGSSON, H., and WIIN-NIELSEN, A., 1957: Experiments in numerical forecasting using space-smoothed fields, *Tellus*, **9**, pp. 296 – 313.

THOMPSON, P. D., 1956: A two-parameter non-geostrophic model suitable for routine numerical weather forecasting, *Tech. Mem.* No. **11**, *JNWP-Unit*, Washington.

VEDERMAN, J., and HUBERT, W. E., 1957: On the conservation of finite difference absolute vorticity in barotropic forecasts. *Monthly Weather Review*, June 1957, pp. 193 – 199.

WELANDER, P., 1955: Studies on the general development of motion in a twodimensional, ideal fluid. *Tellus*, **7**, pp. 141 – 156.

WIIN-NIELSEN, A., 1959: On certain integral constraints for the time-integration of barotropic and baroclinic models. *Tellus*, **11**, pp. 45 – 49.

WOLFF, P. H., 1958: The Error in Numerical Forecasts Due To Retrogression of Ultra-Long Waves. *Tech. Mem.* No. **13**, *Joint Numerical Weather Prediction Unit*.