A Theoretical Power-Law for the Size Distribution of Small Particles or Drops falling through the Atmosphere

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Abstract

A theoretical study is made of the size distribution of small particles or drops in the atmosphere as produced by vertical motion and coalescence. It is demonstrated that a power-law distribution is a possible solution both in the time-dependent case, when conditions are assumed to be spatially homogeneous, and in the steady state case. The theory is in special applied to cloud droplets falling and coalescing under the effect of gravity and Stokes friction. In this case it is found that the frequency distribution falls off as r^{-5} in the time-dependent case and as r^{-3} in the steady state case, r being the radius. A comparison with observed size distributions of droplets in the range 20—80 microns given by WEICKMANN and AUFM KAMPE indicate that, on an average, the r^{-5}-distribution and the r^{-3}-distribution are reasonable approximations for fair weather cumulus clouds and for cumulus congestus- and cumulonimbus clouds, respectively. In the time-dependent case the theory predicts that the concentration of droplets of a given size should change inversely to a linear function of time. After a definite time T, infinite concentrations are reached and the model breaks down. This characteristic growth time is inversely proportional to the initial concentration of particles. For the mean concentration of cloud droplets given by Weickmann and aufm Kampe, T takes on values of the order 150 seconds, assuming that only droplets which differ in radii less than 10 times coalesce.

1. The basic equation

Consider a distribution of particles or drops in the atmosphere having the frequency distribution $n(m, z, t)$. $n(m, z, t)\ \Delta m$ should then give the number of particles per unit volume having a mass between m and $m + \Delta m$, at a height z and at a time t. The distribution is assumed to change in time due to falling and coalescence. The total change of $n\Delta m$ is given by the equation

$$\frac{dn}{dt}\Delta m = \frac{\partial n}{\partial t}\Delta m + w(m)\frac{\partial n}{\partial z}\Delta m =$$

$$= \frac{1}{2}\iint\limits_{m \leq m_1 + m_2 \leq m + \Delta m} \pi\left[r(m_1) + r(m_2)\right]^2 E(m_1, m_2)\,|\overline{v_r}| \cdot (m_1, m_2)\, n(m_1)\, n(m_2)\, dm_1, dm_2 - \int\limits_{m_1 = 0}^{\infty} \pi\left[r(m_1) + r(m)\right]^2 E(m_1, m)\,|\overline{v_r}| \cdot (m_1, m)\, n(m_1)\, n(m)\, dm_1 \Delta m \qquad (1)$$

Here r is the radius of a particle, E is the collection efficiency (the fraction of particles on colliding paths that coalesce), v_r is the relative speed between two colliding particles, averaged over all occurring directions, and w is the fall velocity. A and w depend on the mass of the particle considered, E and v_r are assumed to depend on the masses of the two colliding particles.

Equation (1) is arrived at in the following way. Consider a particle of mass m_1 moving at a velocity v_1. In a time dt this particle sweeps a cylinder of volume $\pi r_1^2 v_1 dt$, and will then, on an average, coalesce with $E(m_1, m_2)\, \pi r_1^2 v_1 n(m_2)\, dm_2 dt$ particles in the range m_2 to $m_2 + dm_2$. The number of particles of the first kind $n(m_1)\, dm_1$ produce thus, in the time dt, $E(m_1, m_2)\, \pi r_1^2 v_1 n(m_1)\, n(m_2)\, dm_1\, dm_2 dt$ new particles in such collisions. The total positive gain of particles to the range m to $m + \Delta m$ is obtained by integrating the last expression over all collisions for which $m_1 + m_2$ lies in this range. The above reasoning gives the correct order of magnitude, but it is not quite exact. To be correct one should replace the factor $\pi r_1^2 v_1$ by $\pi(r_1 + r_2)^2 |\overline{v_r}|$, since a collision between two molecules of radii r_1 and r_2 occurs, in the present model, as soon as the distance between their centres is smaller than $r_1 + r_2$, and since it is the average relative velocity of the colliding molecules rather than the absolute velocity v_1 that determines the number of collisions. Finally, the integral should be divided by 2, since each collision situation is counted twice, as can be seen by interchanging m_1 and m_2.

The negative contribution of particles to the range m to $m + \Delta m$ is obtained by integrating a similar expression as derived above, considering the collisions between particles in the range m to $m + \Delta m$ and the other particles outside that range. The difference between the resulting two coalescence integrals constitutes the total change in $n(m)\, \Delta m$ as made up by local change $\frac{\partial n}{\partial t} \Delta m\, dt$ and fall-velocity advection $w \cdot \frac{\partial n}{\partial t} \Delta m\, dt$. Dividing this equation by dt gives finally the form (1).

One may transform equation (1) by putting $m_2 = m - m_1$ in the double integral and replacing the integration element $dm_1 dm_2$ by the element $dm_1 \Delta m$ (Fig. 1). After division by Δm one obtains then

$$\frac{\partial n}{\partial t} + w(m) \frac{\partial n}{\partial z} =$$

$$= \frac{1}{2} \int_{m_1=0}^{m} \pi [r(m_1) + r(m - m_1)]^2 E(m_1, m - m_1) \cdot |\overline{v_r}|(m_1, m - m_1)\, n(m_1)\, n(m - m_1)\, dm_1 -$$

$$- \int_{m_1=0}^{\infty} \pi [r(m_1) - r(m)]^2 E(m_1\, m)\, |\overline{v_r}|(m_1, m) \cdot \cdot n(m_1)\, n(m)\, dm_1 \qquad (1\,\mathrm{a})$$

This equation is of the same type as the coalescence equation studied earlier by MELZAK (1953) and HIRSCHFELD (1957).

In the following we make the specific assumption regarding the function $E(m_1, m_2)$

$$E(m_1, m_2) = \begin{cases} E_0 \text{ when } \varepsilon \le \dfrac{m_1}{m_2} \le \dfrac{1}{\varepsilon} \\ 0 \text{ when } \dfrac{m_1}{m_2} < \varepsilon \text{ or } > \dfrac{1}{\varepsilon} \end{cases} \qquad (2)$$

This means that all droplets on colliding paths coalesce unless the mass-ratio (smaller mass to larger) is less than ε, in which case no coalescences occur (one may assume that very small particles approaching a larger particle simply follow the air flow around the latter one and thus become deviated out from their collision paths). In reality conditions seem to be somewhat more complicated and there may be a decrease in E also for particles of almost equal size. However, for the purpose of the present investigation we will be satisfied to use (2).

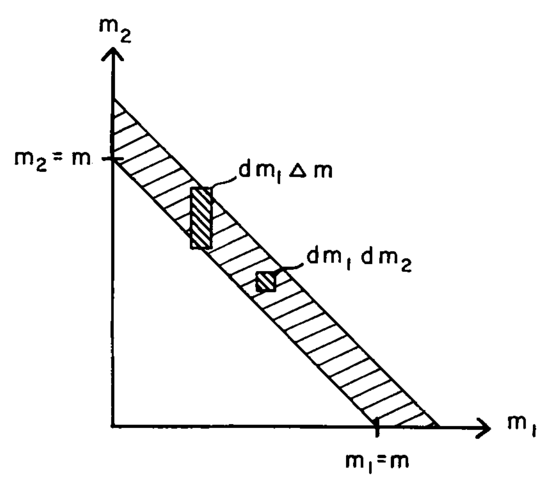

Fig. 1. Integration range in the coalescence integral.

For r, w and $|\overline{v_r}|$ we write

$$\left.\begin{aligned} r &= C_1 m^{\frac{1}{3}} \\ w &= -C_2 m^a \end{aligned}\right\} \qquad (3)$$

$$|\overline{v_r}|(m_1, m_2) = C_3 |m_1^b - m_2^b|$$

where C_1, C_2, C_3 are constant and a, b are pure numbers. The assumption that w and v_r vary with a power of the mass seems to apply to most cases of interest.

2. The time-dependent case

It is assumed that the conditions do not vary over the vertical so that $\frac{\partial n}{\partial z} = 0$ in equation (1a). Attempting a solution of the form

$$n = N(t)\, m^{\alpha} \qquad (4)$$

where α is a pure number, one finds after insertion of (2) and (3) and by introducing the new integration variable $x = \frac{m_1}{m}$ that the equation may be written on the form

$$\frac{dN}{dt} = \left\{m^{\alpha + b + \frac{5}{3}} \cdot E_0 C_1^2 C_3 I\right\} N^2 \qquad (5)$$

where

$$I = \frac{1}{2} \int_{\frac{\varepsilon}{1+\varepsilon}}^{\frac{1}{1+\varepsilon}} [x^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}]^2 |x^b - (1-x)^b| \cdot x^{\alpha}(1-x)^{\alpha} dx - \int_{\varepsilon}^{\frac{1}{\varepsilon}} (x^{\frac{1}{3}} - 1)^2 |x^b - 1| x^{\alpha} dx \qquad (6)$$

I has a value which only depends on the numbers ε, b and α.

One must now require that the expression in the right hand side of (5) is independent of m. This condition determines directly the exponent α:

$$\alpha = -\left(b + \frac{5}{3}\right) \qquad (7)$$

Equation (5) can now be solved to give N:

$$\frac{1}{N_0} - \frac{1}{N} = k(t - t_0) \qquad (8)$$

with

$$k = E_0 \pi C_1^2 C_3 I$$

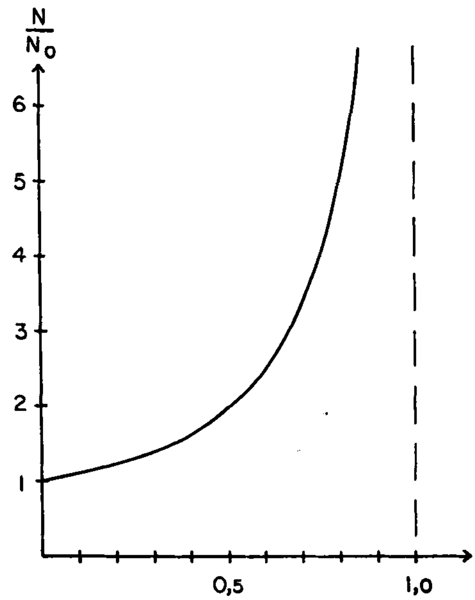

Fig. 2. Concentration of particles of given size as a function of time. The time unit is $\frac{1}{kN_0}$.

N is thus found to vary inversely linear with the time. To sustain the increase of N there occurs, of course, a mass flux through the spectrum from the small size end to the large size end. After a finite time interval

$$T = \frac{1}{kN_0} \qquad (9)$$

which value is inversely proportional to the initial concentration, the distribution will break down since N takes on infinite values (see Fig. 2). In reality the break-down cannot of course occur, since only a limited mass is available. It is seen, however, that the time T is only a little larger than the time needed to increase N by a factor of, say, 10 or 100, and so T may represent with good approximation a real growth-time. How much N actually increases in a real situation depends, of course, on the available mass of the smallest particles. It should perhaps be pointed out that the violent increase in the concentration when approaching the time $t = T$ is entirely due to including in the model a cascade coalescence process where all particles interact mutually. Such a result is not found in linear models, such as the one studied by TELFORD (1955).

3. The steady state

In the steady state case we put $\frac{\partial n}{\partial t} = 0$ in equation (1a). Inserting (2) and (3) and attempting a solution of the form

$$n = N(z)\, m^{\alpha} \tag{10}$$

gives this time

$$\frac{dN}{dz} = -\left\{ m^{\alpha - a + b + \frac{5}{3}} \cdot E_0 \frac{C_1^2 C_3}{C_2} I \right\} N^2 \tag{11}$$

where I is again given by (6).

In this case the condition that the expression on the right hand side should be independent of m leads to the exponent value

$$\alpha = a - b - \frac{5}{3} \tag{12}$$

The solution for N is then

$$\frac{1}{N_0} - \frac{1}{N} = -k_1(z - z_0) \tag{13}$$

with

$$k_1 = E_0 \pi \frac{C_1^2 C_3}{C_2} I$$

The concentration varies here inversely linear with the height, and increases downwards. In analogy with the previous case one may define a characteristic growth-distance

$$D = \frac{1}{k_1 N_0} \tag{14}$$

which varies inversely proportional to the initial concentration.

In applying the model it should be kept in mind.

4. Application to cloud droplets

As a specific application of the previous theory we study the case of a cloud droplet spectrum. It is generally assumed that the smallest droplets grow mainly by condensation. At a size of, say, 10 à 20 microns one may, however, expect that the coalescence process becomes important. Assuming that coalescence occurs only due to the variation of fall-velocity with the drop size we may put $|\overline{v_r}|\,(m_1, m_2) = |w(m_1) - w(m_2)|$. The fall-velocity w is determined by balancing the gravity force and Stokes friction:

$$6\pi\eta w r = mg \tag{15}$$

Here η is the coefficient of viscosity for air and g is the acceleration of gravity. This law could be applied to droplets of radii up to 60 à 80 microns without too large errors. Using the values given above one finds

$$\left.\begin{aligned} a &= b = \frac{2}{3} \\ C_1 &= \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \varrho^{-\frac{1}{3}} \\ C_2 &= C_3 = \frac{1}{6\pi}\left(\frac{4\pi}{4}\right)^{\frac{1}{3}} \frac{g\varrho^{\frac{1}{3}}}{\eta} \end{aligned}\right\} \tag{16}$$

where ϱ is the density of a droplet. Inserting the values $\varrho = 1$ g/cm^3, $g = 10^3$ cm/s^2 and $\eta = 1.7 \cdot 10^{-4}$ gcm/s one finds

$$C_1 \cong 0.62$$
$$C_2 = C_3 \cong 5.0 \cdot 10^5$$

With the values of a and b given by (16) one finds $\alpha = -\frac{7}{3}$ in the time-dependent case and $\alpha = -\frac{5}{3}$ in the steady state case. The corresponding power-laws for the frequency distribution of the radii are r^{-5} and r^{-3}, respectively.

This result may be compared with observed cloud droplets spectra in the range 20—80 microns, where coalescence processes are expected to be important. In Fig. 3 the theoretical power-laws have been compared with some mean droplet spectra reported by WEICKMANN and AUFM KAMPE (1953).

It is seen that the r^{-5}-law is a fairly good approximation to the mean droplet spectrum found in fair weather cumulus clouds, while the r^{-3}-law could fit the mean spectra found in cumulus congestus and cumulonimbus clouds. It seems, of course, reasonable that the transient model leading to the r^{-3}-law should apply best to the "young" clouds and the steady state model to the "older" clouds.

One must, however, be careful not to claim too much from this comparison. The spectra given by WEICKMANN and AUFM KAMPE are obtained by averaging both in space and time, and this averaging procedure may certainly

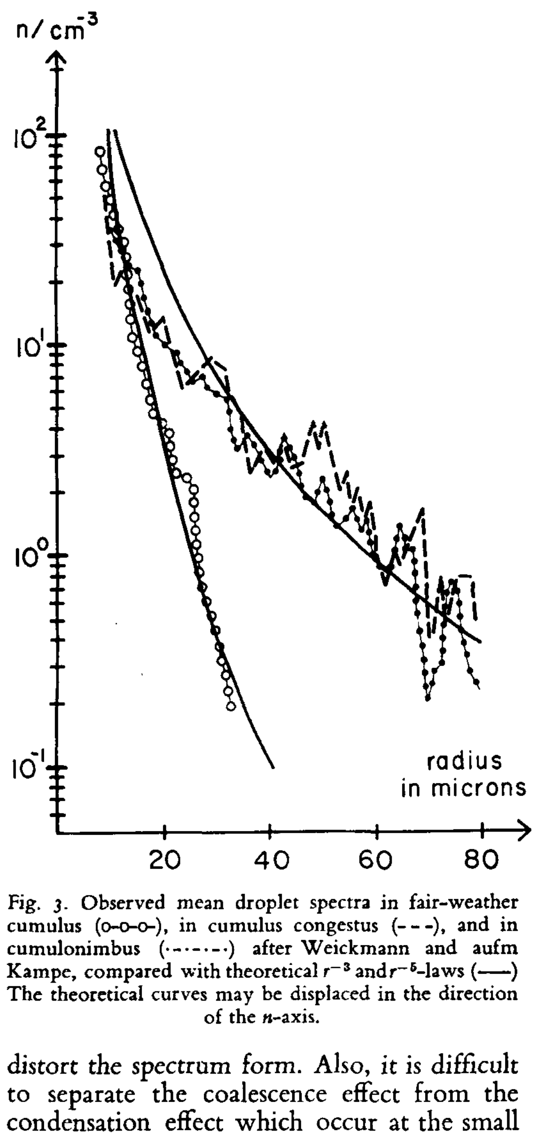

Fig. 3. Observed mean droplet spectra in fair-weather cumulus (o-o-o-), in cumulus congestus (- - -), and in cumulonimbus (·-·-·-·) after Weickmann and aufm Kampe, compared with theoretical r^{-3} and r^{-5}-laws (——) The theoretical curves may be displaced in the direction of the n-axis.

distort the spectrum form. Also, it is difficult to separate the coalescence effect from the condensation effect which occur at the small size end of the spectrum (10—30 microns).

To see whether the theory can give a coalescence growth of the droplets in a reasonable time a numerical computation of T may be of interest. We assume that the total number of droplets having a radius larger than 5 microns is 300 per cm^{-3}, this corresponds to the figure given by WEICKMANN and AUFM KAMPE for fair weather cumulus. Integrating (4) with $\alpha = \frac{7}{3}$ over all sizes larger than 5 microns one may determine N_0, its value becomes $1.6 \cdot 10^{-11}$ g$^{4/3}$ cm^{-3}. With regard to the collection efficiency we assume $\varepsilon = 10^{-3}$, $E_0 = 1$. This means that all droplets on colliding paths that differ in radii less than 10 times coalesce. With the values of b and α given, I depends only on ε, its value may be computed using the asymptotic formula

$$I(\varepsilon) \sim 7\varepsilon^{\frac{1}{3}} \qquad (17)$$

which is valid for sufficiently small ε. With the numerical values of E_0, C_1, C_3 and I the constant k in (8) takes on the value $4.2 \cdot 10^{7}$ $g^{-4/3}$ cm^{3} s^{-1}. The valuefor T is about 150 seconds. The value will not depend very critically on ε. Increasing ε to 10^{-2} gives a value of about 70 seconds, decreasing ε to 10^{-4} a value of about 300 seconds. As seen from (17) I goes to infinity when ε goes to zero, and so the growth time becomes zero. This result is of course unrealistic. In the model the power-law is assumed to hold even for the infinitely small droplets, which will take over the whole contribution when ε goes to zero. In reality this cannot occur since the distribution curve falls off again for the very small drops. When the mass-range over which the collection efficiency has finite values is of the same order or smaller than the mass-range over which a single power-law can be expected to give a good approximation the theory should anyway function all right. Luckily enough, it seems as if this case is also realized in practice.

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