On Density and Isotropy of an Ionized Gas in a Magnetic Field

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Abstract

The relation between the density on the surface $|v|$ = constant in the velocity space and the density in the space is treated. The relationship is given by an integral equation. First the three-dimensional case for homogeneous magnetic field is treated, then a two-dimensional case in a magnetic field the intensity of which varies linearly with the space coordinate to show the dependence on the inhomogeneity in the field.

1. Introduction

In a paper DATTNER and VENKATESAN (1958) have discussed the anisotropy of the cosmic radiation. In the present paper we shall make some calculations which may be of some interest in this connection. We shall try to get a relation between the particle density and the isotropy in the velocity distribution, when the gradient in absolute value of the magnetic field is not zero. To simplify we assume that the magnetic field has only one component in a Cartesian frame of reference and that the density gradient and the field gradient are parallel to each other but perpendicular to the magnetic field.

The problem we are going to discuss is related to the discussions of COWLING (1932) and later SPITZER (1952). The difference is that we are interested in the properties of a region much smaller than the gyration radius but in the referred papers the mean values over regions large compared to the gyration radius are discussed. The results of these two treatments must be related, but are not identical.

2. A homogeneous magnetic field

Assume $N(x_1, y_1)$ particles with velocity v have their guiding centres at the point (x_1, y_1). The velocity distribution is assumed to be isotropic. The probability that the velocity vector falls within the solid angle $d\Omega$ is therefore $d\Omega/4\pi$. Hence the number of particles passing a surface dS situated at the point (x, y) within the solid angle $d\Omega$ round the normal to the surface is

$$I\,dS\,d\Omega =$$

$$= \frac{v}{4\pi}\,dS\,d\Omega\,N(x + R\cdot\sin\varphi,\ y - R\cdot\cos\varphi) \quad (1)$$

if φ is the angle between the projection of the velocity vector on the xy-plane and the x-axis. The magnetic field is in the z-direction.

The particles move in helical orbits. The projection on a surface perpendicular to the magnetic field is a circle with radius $R = \frac{p}{eB}\cdot$ $\cdot\sin\Theta$, where Θ is the angle between the velocity vector and the magnetic field lines.

Since we then define N at the centre of the circular motion but I at the point where the particles are actually situated we get the argument given in (1).

The density at a point (x, y) is

$$n(x, y) = \int \frac{I(x, y) \cdot d\Omega}{v} \tag{2}$$

and the integration has to be extended over all particles which pass the point (x, y).

Let us simply assume $N = e^{-\alpha x}$. Then we find

$$I = \frac{v}{4\pi} e^{-\alpha x} e^{-\alpha R \sin \varphi} \tag{3}$$

and from (2)

$$n = e^{-\alpha x} \frac{sh\, \alpha R}{\alpha R} \tag{4}$$

Let us expand I in a Fourier series in φ. We obtain

$$I(x, \varphi) = \frac{v}{4\pi} e^{-\alpha x} I_0(\alpha R) + + 2 \sum_{\nu=1}^{\infty} I_\nu(\alpha R) \cos \nu \left(\varphi - \frac{\pi}{2}\right) \tag{5}$$

Since I is symmetric relative to the point $\varphi = \frac{\pi}{2}$, we have used $\varphi - \frac{\pi}{2}$ as argument instead of φ. The quantities $I_\nu(r)$ are modified Bessel functions of the first kind and order ν. The relation between n and I in the case of homogeneous magnetic field is given by the integral equation (2). There exists an infinity of linearly independent solutions, I, even if $n = 0$, and hence there does not exist a single-valued relation between n and I. If we require that all points shall be on the same footing as far as possible, we should get a single-valued relation. Since we have at present no reason for making any other assumption we have here discussed this simple case.

3. A certain type of inhomogeneous magnetic field

In the case when there is a gradient in the absolute value of the magnetic field strength we shall make the simplifying assumption that the velocity component in the direction of the magnetic field vanishes. Thus we determine the number of particles entering a plane angle instead of a solid angle.

We assume as above that the magnetic field lines are directed along the z-axis and that the gradient in the magnetic field intensity is in the x-direction. For convenience in writing we give the magnetic field in the form

$$\frac{eB}{m} = 2\alpha x \tag{6}$$

where e and m are the charge and rest mass respectively of the particle under study. We restrict the study to the case of a linear gradient in the field, and to variation in one Cartesian coordinate only.

The equation of motion is

$$m\dot{\mathbf{v}} = e\mathbf{v} \times \mathbf{B} \tag{7}$$

from which we get

$$\ddot{y} + \dot{x} \frac{eB}{m} = 0 \tag{8}$$

A dot means derivation with respect to the 'eigen time' and v is defined so that mv is the momentum. If we introduce (6) in (8) we can integrate and the result is

$$\dot{y} + \alpha x^2 = C \tag{9}$$

C is a constant of integration. It is a measure of the mean distance of the orbit from the y-axis. Since B is independent of time and the electric field vanishes identically we have

$$\dot{x}^2 + \dot{y}^2 = v_0^2 \tag{10}$$

where v_0, the velocity of the particles, is a constant of motion. Equation (9) and (10) give

$$\dot{x}^2 + (C - \alpha x^2)^2 = v_0^2 \tag{11}$$

Let us introduce the angle φ between the x-axis and the tangent to the orbit. Then

$$\sin \varphi = \frac{C - \alpha x^2}{v_0} \tag{12}$$

Let us distribute orbits with a given value of C homogeneously in the y-direction and distribute particles homogeneously along each curve. Then it is possible to get a density distribution of particles, which is independent of the y-coordinate but shows a characteristic variation with the x-coordinate. The particle density is proportional to

$$(v_x)^{-1} = (v_0 \cos\varphi)^{-1}.$$

Let us introduce orbits with different values of C and let us introduce a certain density A (not particle density). Then we can write the density of particles at a certain point x as

$$n(x) = \int \frac{A\left(\dfrac{C}{v_0}\right)}{v_0 \cos\varphi} dC \qquad (13)$$

The integration has to be extended over all orbits which pass through the point at which we want to know the density.

From equation (12) we get, if x is kept constant

$$\cos\varphi\, d\varphi = \frac{dC}{v_0} \qquad (14)$$

If we then introduce φ as the new variable, equation (13) can be written

$$n(x) = \int A\left(\frac{\alpha x^2}{v_0} + \sin\varphi\right) d\varphi \qquad (15)$$

$\left(\dfrac{\alpha x^2}{v_0} + \sin\varphi\right.$ is the argument of A and not a factor$\left.\right)$

Equation (15) is correct as long as the orbits have trochoidal character which is the case when

$$\frac{C}{v_0} > 1 \qquad (16)$$

and we shall restrict the discussion to that case.

If we now interest ourselves in those particles $I\,dS\,d\varphi$ which arrive perpendicularly at a surface dS within the angular interval $d\varphi$ we find that

$$I(x, \varphi) = u_0 A\left(\frac{\alpha x^2}{v_0} + \sin\varphi\right) \qquad (17)$$

where u_0 is the ordinary particle velocity.

Since the density in the physical problem does not satisfy (15) when x is below a certain value we shall try to get a satisfactory solution of (15) when (16) is satisfied, without discussing the convergence when $x \to 0$.

When we want to apply this to a particular case we may simply try to find a function A which makes $n(x)$ a linear function. We shall try to obtain that condition at least asymptotically.

To get n we integrate A over a small interval in x^2 and hence we can take $n\pi = A$ as a first approximation. If we introduce

$$A_\nu = c_\nu \left\{\frac{1 - k^2 \sin^2\left(\dfrac{\varphi}{2} - \dfrac{\pi}{4}\right)}{k^2}\right\}^{\frac{1}{2}-\nu} \qquad (18)$$

where

$$k^2 = \frac{2}{2 + \dfrac{\alpha x^2}{v_0}} \qquad (19)$$

we find that A_0 is the first approximation we have just defined. Except for a factor independent of k the bracket is one plus the same argument as in (15), somewhat rewritten. It has been found suitable to introduce the A_ν in (18) as higher approximations.

The space variation is via k. If $\alpha > 0$ we get $0 < k^2 < 1$. Now (15) gives for the n_ν which corresponds to the A_ν

$$\left.\begin{aligned} n_0 &= 2c_0 \cdot k^{-1} \cdot \mathbf{E} \\ n_1 &= 2c_1 \cdot k \cdot \mathbf{K} \\ n_2 &= 2c_2 \cdot \frac{k^3}{(k')^2} \cdot \mathbf{E} \\ n_3 &= 2c_3 \cdot \frac{k^5}{(k')^4} \cdot \left[\mathbf{E} + \frac{k^2}{3}(\mathbf{B} - 2\,\mathbf{E})\right] \end{aligned}\right\} \qquad (20)$$

etc.

Here $\mathbf{K}$, $\mathbf{E}$ and $\mathbf{B}$ are elliptic integrals as defined by JAHNKE-EMDE (1945).

Let us try to get a linear relationship between n and x when x is large. If we observe that $\dfrac{k'}{k} = x\sqrt{\dfrac{\alpha}{2v_0}}$ and that $k' \approx 1$ when x is large, we can see that n_0 shows such a character. By adding n_1, n_2 etc. with proper coefficients we can obtain a solution the deviation of which from linearity starts with a term k^{2m}, where we by adding a sufficiently large number of terms can arbitrarily specify the value of m.

If we put $c_0 = \dfrac{2}{\pi}\sqrt{\dfrac{v_0}{2\alpha}}$ and $c_0 = -4c_1 = -64c_2 = 256\,c_3$ etc. we find that

$$n_0 + n_1 + n_2 + n_3 = x(1 + o(k^8)) \qquad (21)$$

Hence only terms of 8th order and higher in k enter in the resulting expression.

If we now develop I in a Fourier series

$$I = I_0 + I_1 \cos \psi + I_2 \cos 2\psi + I_3 \cos 3\psi + \ldots \tag{22}$$

$$\text{with } \psi = \varphi - \frac{\pi}{2}$$

equation (17) immediately shows that all sine terms vanish identically. We find

$$\left.\begin{aligned} I_0 &= \frac{u_0 n}{\pi} \\ I_1 &= \frac{2u_0 x}{\pi} \cdot \frac{k^2}{8} \cdot \\ &\quad \cdot \left(1 + 2k^2 + \frac{125}{128} \cdot k^4 + \frac{119}{128} \cdot k^6 + \ldots\right) \\ I_2 &= -\frac{2u_0 x}{\pi} \cdot \frac{k^4}{128} \left(1 + 2k^2 + \frac{91}{32} \cdot k^4 \ldots\right) \end{aligned}\right\} \tag{23}$$

We also need to know the results in the case when the density does not vary with the space coordinates. In that special case we get

$$\left.\begin{aligned} n &= \pi A_0 \\ I &= u_0 A_0 \end{aligned}\right\} \tag{24}$$

Let us now prescribe the value of n, $\frac{dn}{dx}$, $g = \frac{eB}{m}$ and $\frac{dg}{dx}$ at a certain but arbitrary point x_1, the point at which we want to study the relation between n and I. Let us use the index one for those values. As a measure of the inhomogeneity in the magnetic field we introduce $\chi = \left(\frac{4g'v_0}{g^2}\right)_1$ and we then find

$$k^2 = \frac{\chi}{1 + \chi} \tag{25}$$

As a measure of the inhomogeneity in the density we may introduce

$$\varkappa = \left(\frac{n'v_0}{ng}\right)_1 \tag{26}$$

where the prime denotes a space derivative. With these notations we can write

$$n = n_1 + n_1'(x - x_1) + 0(\chi^4) \tag{27}$$

$$\left.\begin{aligned} I_0 &= \frac{u_0 n_1}{\pi} \\ I_1 &= \frac{u_0 n_1}{\pi} \varkappa \left(1 - \frac{3}{128} \cdot \chi^2 - \ldots\right) \\ I_2 &= \frac{u_0 n_1}{\pi} \frac{\varkappa}{16} \chi \left(1 + \frac{123}{32} \cdot \chi^2 + \ldots\right) \end{aligned}\right\} \tag{28}$$

The analysis shows that if the density is independent of the space coordinate, then the intensity distribution is isotropic irrespective of whether the magnetic field is inhomogeneous or not. Further we find that the effect of the inhomogeneity in the magnetic field on the I_1 component is a second order effect. The component I_2 is, in the first approximation, proportional to both $\varkappa$ and χ, i.e. proportional to both the inhomogeneity in the magnetic field and the inhomogeneity in the density. The deviation is of the second order in χ.

The relation between the intensity and density is given by an integral equation. The homogeneous equation has an infinity of solutions and therefore we can prescribe $n = n(x)$ and further the angular dependence of I at one arbitrary point. In the present paper we wanted all points to be on the same footing, as far as possible, and therefore the discussion could be simplified as this paper has shown.

It is possible to use the solution of the homogeneous equation (15) to construct the solution of the inhomogeneous equation. The solution will then be a linear term in x and a remainder, which is expressible in Fresnel integrals and trigonometric functions. In the present connection the method we have used seems to be the most useful one of the two solutions.

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