The Diurnal and the Semidiurnal Atmospheric Solar Tide

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Abstract

World-wide oscillations generated by the sun are examined by applying the hydrodynamical equations to a rotating spherical model atmosphere. The oscillations are described as small perturbations on a static state in which the atmosphere is at rest relative to the earth's surface. Let $\dot{q}$ denote the heat supplied per unit mass and unit time. Let p and α denote perturbations in pressure and specific volume of a particle of air and let the displacement of the particle from its equilibrium position be given by θ, ψ, z, i.e. the variations of its colatitude, longitude and height, respectively. Consider the simplest oscillation of 24-hour period rotating with the sun:

$$\dot{q} = C_Q \sin(\Psi + \eta_Q), \quad x = C_x \sin(\Psi + \eta_x),$$

where Ψ denotes local time reduced to angle and x denotes any one of the variables $p, \alpha, z, \psi, \theta$. The amplitudes and phase angles are variable with latitude and height. $\dot{q}$, p, α, z, ψ are symmetric with respect to the equator.

It is shown that, if C_Q, η_Q are given as analytic functions of latitude and height, then the quantities C_x, η_x are determined by the equations of motion, the equation of continuity and the first law of thermodynamics.

Considering in like manner the simplest oscillation of 12-hr. period rotating with the sun:

$$\dot{q} = C_Q \sin(2\Psi + \eta_Q), \quad x = C_x \sin(2\Psi + \eta_x)$$

we find that, when C_Q, η_Q are given, the C_x, η_x satisfying the above mentioned equations contain an arbitrary function of height, $f(Z)$.

The absence in the first case of an arbitrary function which might be adapted to boundary conditions, may partly explain the weak development of the 24-hr. wave as a global phenomenon in spite of the marked diurnal period in the heat supply $\dot{q}$.

A special case of non-linear equations is considered in order to study certain relations between the 24-hr. and the 12-hr. wave. A heat supply of 24-hr. period of the form $\dot{q} = C_Q \sin(\Psi + \eta_Q)$ can generate a wave of 12-hr. period.

1. Introduction

The sun and the moon generate in the atmosphere certain oscillations of world-wide character resembling the oceanic tides. The expression "atmospheric tides" is widely used for such oscillations although they are generated to a large extent by the thermal action of the sun, whereas the oceanic tides are due to purely gravitational causes.

Among the atmospheric tidal waves caused by the sun we shall consider here a progressive wave W rotating with the sun (i.e. moving with the speed of local time from east to west). It will be assumed that the sun is in its equinoc-

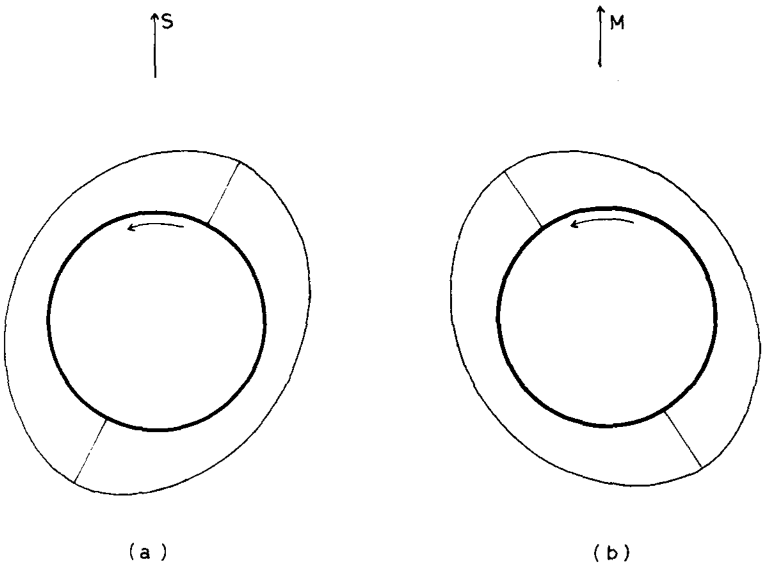

Fig. 1, a. Isobaric surface of W_2. b. Oceanic tidal wave.

tial position and that W is symmetric with respect to the equator and composed of two harmonic components W_1 and W_2 having periods of 24 and 12 hours, respectively. In the atmosphere we find waves of this kind and also a lunar tide L. The wave W_2 is well known. It is often called "the semidiurnal barometric variation". In amplitude it exceeds by far the other oscillations. Briefly we may indicate this by writing $W_2 > W_1$ and $W_2 > L$.

Fig. 1a) indicates an isobaric surface of W_2 in equatorial section. S is the direction to the sun. The surface has no motion relative to the sun. An observer at sea level finds maximum pressure two hours before noon at all latitudes.

If W_2 were excited gravitationally like L, it would be weaker than L. The fact that W_2 by far exceeds L in amplitude led Laplace to assume that the thermal action of the sun must outweigh its gravitational action in the generation of W_2. This is also supported by comparing the position of the crest of the wave in W_2 and in the oceanic tide, as shown in Fig 1a), and b), where M is the direction to the moon. On the other hand the assumption of a thermal excitation seems to be in disaccord with the fact $W_2 > W_1$, since the temperature variation is diurnal rather than semidiurnal. In order to explain this discrepancy Lord Kelvin offered the suggestion that the atmosphere has a period of free oscillation near to 12 hours so that W_2 is amplified by resonance.

In more recent years important contributions to the theory have been rendered by Chapman, Taylor, Pekeris, Bjerknes, Wilkes, Haurwitz, Siebert and others.

The theory of Lord Kelvin has been accepted by most authors, although it involves certain difficulties. In order to obtain the requisite degree of selective resonance it is necessary to postulate a free period of almost exactly 12 hours. Weekes and Wilkes (1947) have computed oscillations in atmospheric models having the required 12 hr. free period combined with vertical temperature distributions conforming to the facts at present known about the upper atmosphere. It is important to note that the free period is quite sensitive to the exact form chosen for the temperature variation with height between 70 and 90 km. This is a weakness of the theory since it seems unlikely that the temperature distribution should always be maintained in the exact form required for a 12-hr. free period.

The theory of atmospheric tides was originally based on and developed from the theory of oceanic tides and attention was, therefore, focussed on the gravitational semidiurnal effects and the possibility of a thermal semidiurnal effect. The predominating oscillation W_2 was investigated, but very little attention was devoted to W_1. In the present paper W_1 will be examined with a view to finding a possible explanation of its weak development. Further the relation between W_1 and W_2 will be considered.

The solar wave of 8-hour period nearly vanishes at the equinoxes and will be neglected in the present paper.

2. The model atmosphere and the fundamental equations

The model atmosphere is assumed to consist of an ideal gas having a stable stratification at all levels and being originally at rest on a spherical rotating globe. On this "static state" is superposed the forced oscillation $W = W_1 + W_2$ described in the previous paragraph. W_1 is generated thermally by the sun, while W_2 is generated thermally and gravitationally. The oscillations are supposed to be sufficiently small to permit the use of equations of perturbation. The mutual attraction of the air particles is neglected.

Let R denote distance from the earth's centre, Ψ local time reduced to angle and Θ co-latitude. An individual air particle M will be identified by Lagrangian coordinates a, b, c, chosen in the following manner. In the static state the coordinates R, Ψ, Θ of M take the values R_0, Ψ_0, Θ_0. We put $R_0 = a$, $\Psi_0 = b + \Omega t$, $\Theta_0 = c$, where b is the geographical longitude of M in the static state, Ω the angular velocity of the earth's rotation relative to the sun and t time. c may be limited to the interval $(0, \pi)$. Let P and S denote pressure and specific volume of M. Their values in the static state will be denoted by p_0 and α_0, respectively. In the state of oscillation R, Ψ, Θ, P, S are functions of a, b, c, and t. For the time differentiation following an individual particle we write $\frac{\partial}{\partial t} f(a, b, c, t) = \dot{f}$. Other symbols used in the following are:

c_p, c_v = specific heat at constant pressure, volume,
$\varkappa$ $= c_p/c_v$
g = acceleration of gravity,
q = heat per unit mass entering the air,
ε $= (\varkappa - 1)\dot{q}$, $\nu = b + \Omega t$, $\mu = \cos c$,
ϱ_0 $= 1/\alpha_0$ = density in the static state,
T_0 = absolute temperature in the static state,
Φ = potential of extraneous force.

In spherical coordinates the general equations of motion in Lagrangian form may be written as follows (V. BJERKNES 1930).

$$\begin{aligned}(\ddot{R} - R\dot{\Theta}^2 - R \sin^2\Theta\, \dot{\Psi}^2) \nabla R &+ \\ + \frac{\partial}{\partial t}(R^2 \sin^2\Theta\, \dot{\Psi}) \nabla \Psi &+ \\ + \left(\frac{\partial}{\partial t}(R^2\dot{\Theta}) - R^2 \sin\Theta \cos\Theta\, \dot{\Psi}^2\right) \nabla \Theta &+ \\ + S \nabla P + \nabla \Phi + \boldsymbol{F} &= 0,\end{aligned} \tag{1}$$

where $\nabla = \left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}\right\}$, and $\boldsymbol{F}$ is a frictional term. In order to apply this equation to the model atmosphere we introduce the apparent gravity g, writing in the usual manner:

$$\Phi - \frac{1}{2}\Omega^2 R^2 \sin^2\Theta = gR + \text{const.} \tag{2}$$

Putting $\dot{\Psi} = \Omega + \dot{\psi}$ and having regard to (2) we may write Eq. (1) in the form

$$\begin{aligned}\left[\ddot{R} - R\dot{\Theta}^2 - R \sin^2\Theta\, (2\Omega\dot{\psi} + \dot{\psi}^2)\right] \nabla R &+ \\ + \frac{\partial}{\partial t}\left[R^2 \sin^2\Theta\, (\Omega + \dot{\psi})\right] \nabla \Psi &+ \\ + \left[\frac{\partial}{\partial t}(R^2\dot{\Theta}) - R^2 \sin\Theta \cos\Theta\, (2\Omega\dot{\psi} + \dot{\psi}^2)\right] &\times \\ \times \nabla \Theta + S \nabla P + g \nabla R + \boldsymbol{F} &= 0\end{aligned} \tag{3}$$

This equation is valid in the vicinity of the earth, where g can be considered as constant.

The equation of continuity and the first law of thermodynamics may be written

$$R^2 \sin\Theta\, \frac{D(R, \Psi, \Theta)}{D(a, b, c)} = S\varrho_0 a^2 \sin c, \tag{4}$$

$$\varepsilon = (\varkappa - 1)\dot{q} = \varkappa P\dot{S} + S\dot{P}. \tag{5}$$

The effect of dissipation has not been entered explicitly in (5), but the quantity $\dot{q}$ may be taken to include dissipation. By hypothesis $\dot{q}$, $\dot{S}$, $\dot{P}$ are zero in the static state. In the state of perturbation $\varepsilon = \varepsilon(a, b, c, t)$ will be considered as given.

In order to deduce equations of perturbation we introduce into the equations (3, 4, 5) the following expressions:

$$\left.\begin{array}{l} R = a + z, \quad \Psi = \nu + \psi, \quad \Theta = c + \theta, \\ P = p_0 + p, \quad S = \alpha_0 + \alpha, \end{array}\right\} \quad (6)$$

and add a "tide-producing potential" φ. We assume that $z, \psi, \theta, p, \alpha, \varepsilon, \varphi$ are small quantities, functions of a, b, c and t. The equations contain linear terms and product-terms in these functions. Considering for instance the first term in (3), $\ddot{R} \nabla R$, we find that one component is $\ddot{R} \frac{\partial R}{\partial a} = \ddot{z} \left(1 + \frac{\partial z}{\partial a} \right)$ which contains one linear term and one product-term ("second-order term"). As regards the deduction of the linear terms reference is made to V. BJERKNES (1930).

When deducing from (3) the equations of perturbation we shall at the same time assign to $\boldsymbol{F}$ a definite value. The horizontal velocity relative to the earth is determined by $\dot{\psi}$ and $\dot{\theta}$ and the horizontal acceleration by $\ddot{\psi}$ and $\ddot{\theta}$. We introduce a simplified frictional force ("Rayleigh friction") proportional to the horizontal velocity. This can be done as follows: We replace $\ddot{\psi}$ by $(\ddot{\psi} + \beta\dot{\psi})$ and $\ddot{\theta}$ by $(\ddot{\theta} + \beta\dot{\theta})$ where β is a "friction coefficient" which may be assumed to be negligible except in a layer near the surface.

Collecting now all linear terms on the left side of the equations and all product-terms on the right, we may write the equations of perturbation as follows:

$$\ddot{z} - 2\Omega a \sin^2 c\, \dot{\psi} + \alpha_0 \frac{\partial p}{\partial a} + g \frac{\partial z}{\partial a} - g\varrho_0 \alpha + \frac{\partial \varphi}{\partial a} = G_1, \quad (7)$$

$$a^2 \sin^2 c\, (\ddot{\psi} + \beta\dot{\psi}) + 2\Omega a \sin^2 c\, \dot{z} + 2\Omega a^2 \sin c \cos c\, \dot{\theta} + \alpha_0 \frac{\partial p}{\partial b} + g \frac{\partial z}{\partial b} + \frac{\partial \varphi}{\partial b} = G_2, \quad (8)$$

$$a^2 (\ddot{\theta} + \beta\dot{\theta}) - 2\Omega a^2 \sin c \cos c\, \dot{\psi} + \alpha_0 \frac{\partial p}{\partial c} + g \frac{\partial z}{\partial c} + \frac{\partial \varphi}{\partial c} = G_3, \quad (9)$$

$$\frac{2}{a} z + \frac{\partial z}{\partial a} + \frac{\partial \psi}{\partial b} + \frac{\partial \theta}{\partial c} + \operatorname{cotg} c \cdot \theta - \varrho_0 \alpha = G_4, \quad (10)$$

$$\varepsilon - \varkappa p_0 \dot{\alpha} - \alpha_0 \dot{p} = G_5. \quad (11)$$

where the G_i are sums of product-terms which are small of the second order of magnitude when the terms on the left are of the first order. It is seen that the equations contain the following components of $\boldsymbol{F}$:

$$F_a = 0, \quad F_b = a^2 \sin^2 c\, \beta\dot{\psi}, \quad F_c = a^2 \beta\dot{\theta}.$$

The tide-producing potential of the sun has the form $\varphi_r = KR^2 \left(\frac{1}{3} - \cos^2 \vartheta \right)$ where K is constant and ϑ is the angular distance from the sun. Since the sun is in the equatorial plane we have $\cos \vartheta = \sin \Theta \cos \Psi$ and we may write

$$\varphi_r = gH \left(\frac{1}{3} - \frac{1}{2} \sin^2 \Theta - \frac{1}{2} \sin^2 \Theta \cos 2\Psi \right) \quad (12)$$

where H may be considered as constant (LAMB 1932, p. 359). Putting in (12) $\Theta = c + \theta$, $\Psi = \nu + \psi$ we get linear terms and product-terms. The latter may be included in G_2, G_3. Writing only the linear part we may consider φ_r as the real part of

$$\varphi = -\frac{1}{2} gH \sin^2 c\, (1 + e^{2i\nu}) + \frac{1}{3} gH. \quad (13)$$

3. Forced oscillations of 24-hour period

Putting in Eqs. (7 to 11) $G_i = 0$ we obtain linearized equations which will be used in this paragraph to study certain properties of the oscillation W_1.

We consider first the general form of a 24-hour wave travelling with the speed of local time. Each of the significant variables in (6) must be a function of the form $F(a, \nu, c)$. They are assumed to be finite for all values of ν, c and $a \geqq a_0$, (a_0-radius of the globe). Further they must be periodic in b with period 2π, which involves periodicity in ν with the same period and in t with period $t_0 = 2\pi/\Omega = 24$ hours. Bearing this in mind and having regard to the theorem that any function of the position (b, c) on a sphere can be expanded in a series of surface-harmonics we may write

$$F(a, \nu, c) = \sum_0^\infty S_n \quad (14)$$

with

$$S_n = C_n P_n(\mu) + \sum_{s=1}^{s=n} K_{s,n} e^{isv} P_n^s(\mu), \qquad (15)$$

where $\mu = \cos c$, P_n denote the Legendre polynomials and P_n^s the Legendre associated functions of the first kind. It is important to note that Legendre functions of the second kind must be excluded since they are infinite for $c=0$ ($\mu=1$). The quantities C_n, $K_{s,n}$ are functions of a. We shall assume that the $K_{s,n}$ are complex quantities. The real part of $K_{s,n}\, e^{isv} P_n^s$ may then be written as $B_s \sin(sv+\eta_s)$ where B_s and η_s depend on a and c. Each term in (15) thus represents a partial wave which generally has a "tilt" with height and with latitude.

The expression (14) represents the most general oscillation with 24-hr. period, rotating with the sun. The particular oscillation W_1 is obtained by taking only the term with $s=1$ in (15). The variables p, α, z, ε, ψ must be symmetric with respect to the equator. (15) can be made symmetric by discarding the terms containing the asymmetric functions P_{2n+1} and P_{2n}^1. We then have for W_1:

$$x = \sum_0^\infty C_{2n}^{(x)} P_{2n}(\mu) + e^{iv} \sum_0^\infty K_{2n+1}^{(x)} P_{2n+1}^1(\mu) \qquad (16)$$

$(x = p,\ \alpha,\ z,\ \psi,\ \varepsilon)$.

The remaining variable, θ, is asymmetric. Its form is determined by the equations (8, 9, 10) having regard to (16).

Abbreviating (13) and (16) we may write $\varphi = \varphi_s + A_\varphi e^{2iv}$, $x = x_s + A_x e^{iv}$ and in like manner $\theta = \theta_s + A_\theta e^{iv}$. The potential φ has a 12-hr. period and affects the oscillation W_2 which will be examined later. We shall consider the total effect of φ in connection with W_2. When treating W_1 we therefore put in (7, 8, 9) $\varphi=0$. Assuming that $\varepsilon_s=0$ we may then satisfy (7 to 11) with $G_i=0$, $\varphi=0$ by choosing x_s, $\theta_s=0$ together with suitable values of A_x, A_θ.

We now therefore neglect the first term in (16) and seek solutions of the form

$$x = A_x e^{iv},\ (x = p,\ z,\ \alpha,\ \psi,\ \varepsilon),\quad \theta = A_\theta e^{iv}. \qquad (17)$$

Noting that the Legendre associated functions have the form

$$P_{2n+1}^1(\mu) = \sin c\ (k_0 + k_2\mu^2 + \ldots k_{2n}\mu^{2n})$$

and introducing these expressions in the last term of (16) we obtain expansions of the form:

$$A_x = x_1 \sin c + x_3 \sin^3 c + \ldots + \\ + x_{2n+1} \sin^{2n+1} c + \ldots \qquad (18)$$

$$A_\theta = \cos c\,(\theta_2 \sin^2 c + \theta_4 \sin^4 c + \\ + \ldots + \theta_{2n} \sin^{2n} c + \ldots) \qquad (19)$$

Series of this kind were used by MARGULES (1892, 93). The form (19) of A_θ follows from the equations (8, 9, 10) having regard to (18). The coefficients x_i, θ_i are functions of a.

The formal mathematical operations leading from (16) to (18, 19) do not ensure the uniform convergence of these series. It is known from mathematical analysis that an analytic function $f(c)$ with period 2π can be expanded in a series of the form

$$\left.\begin{aligned} f(c) &= A_0 + A_1 \sin c + A_2 \sin^2 c + \ldots + \\ &+ \cos c\,(B_0 + B_1 \sin c + B_2 \sin^2 c + \ldots \end{aligned}\right\} \qquad (20)$$

It has been assumed that the longitude b varies in the interval $(-\pi, \pi)$ and c in the interval $(0, \pi)$. However, the equations (7 to 11) would not be altered if b had the interval $(0, \pi)$ and c the interval $(0, 2\pi)$. We may, therefore, regard (18, 19) as special cases of (20) and conclude that the former are valid at least when the A_x, A_θ are analytic functions of c.

It is seen that (18) contains only odd powers and (19) only even powers of $\sin c$. This is due to the properties of Legendre functions and would not be immediately apparent if we had started from (20) instead of (15, 16).

It is of interest to observe that all amplitudes in (17) vanish for $c=0$ since $P_{2n+1}^1(1)=0$, whence it follows that a particle M_0 which, in the static state, is situated on the earth's axis, will remain there and will undergo no change in height, pressure and density. This is also confirmed by the following considerations: For particles on the earth's axis b is arbitrary and the values of p, α, z, ε must therefore be independent of b when $c=0$. According to (17) this is only possible if the amplitudes $A_{p,\alpha,z,\varepsilon}$ vanish for $c=0$. Let us next consider A_ψ and A_θ. An expansion of the form (16), which has been applied to $x=\psi$, is evidently valid also for $y = \sin^2 c \cdot \psi$ by a suitable choice of the coefficients. Taking again only the last term of (16) we are led to the following series

$$A_\gamma = \sin^2 c\, A_\psi = \psi_{-1} \sin c + \psi_1 \sin^3 c + \ldots$$

If $\psi_{-1} \neq 0$ this gives an infinite value of A_ψ for $c=0$ (but a finite linear velocity $\psi \sin c$). However, the linearized equations of perturbation were based on the assumption that product-terms could be neglected, e.g. $\dot{\psi}^2$ was neglected in comparison with $2\Omega\dot{\psi}$. Therefore the equations would not be valid if ψ became infinite at any point. We must have then $\psi_{-1}=0$ and we thus come back to the expansion (18) for A_ψ. Similarly we might at first thought assume an expansion of the form

$$\sin c A_\theta = \cos c\,(\theta_0 \sin c + \theta_2 \sin^3 c + \ldots)$$

If $\theta_0 \neq 0$ this would make the term $\operatorname{cotg} c \cdot \theta$ in (10) infinite for $c=0$, all the other terms on the left in (10) remaining finite. We must have, therefore, $\theta_0=0$, which leads back to (19). Thus it has been shown that A_ψ and A_θ vanish for $c=0$.

By means of the expressions (17, 18, 19) and the equations (7 to 11) we shall now prove the following proposition.

> If $\varepsilon = A_\varepsilon e^{i\nu}$ is given, then the variables $p, z, \alpha, \psi, \theta$ of the form $Ae^{i\nu}$ can be determined by the equations of motion, the equation of continuity and the first law of thermodynamics, provided that the series (18, 19) converge uniformly. (21)

In (7 to 11) we put $G_i=0$, $\varphi=0$ and introduce (17), whereby $e^{i\nu}$ drops out and equations between the A_x, A_θ are obtained. Introducing then the series (18, 19) ($A_p = p_1 \sin c + \ldots$, $A_z = z_1 \sin c + \ldots$ etc.) we easily find that each of the equations (7, 8, 10, 11) may be written in the form

$$\sum_0^\infty F_{2n+1}^{(i)} \sin^{2n+1} c = 0, \quad (i = 1, 2, 3, 4), \tag{22}$$

and equation (9) in the form

$$\cos c \sum_0^\infty F_{2n} \sin^{2n} c = 0, \tag{23}$$

where $F_{2n+1}^{(i)}$ and F_{2n} are functions of a. Obviously they must all be zero. We write first the equations $F_1^{(i)}=0$, $F_0=0$ by collecting in (7, 8, 10, 11) the terms containing the first power of $\sin c$ and in (9) the terms in $\cos c$. Denoting differentiation with respect to a by a prime we obtain:

$$\text{from 7: } F_1^{(1)} = -\Omega^2 z_1 + \alpha_0 p_1' + g z_1' - g\varrho_0\alpha_1 = 0, \tag{24}$$

$$\text{from (8) and (9): } F_1^{(2)} = F_0 = \alpha_0 p_1 + g z_1 = 0, \tag{25}$$

$$\text{from (10): } F_1^{(3)} = \frac{2}{a} z_1 + z_1' + i\psi_1 + 3\,\theta_2 - \varrho_0\alpha_1 = 0, \tag{26}$$

$$\text{from (11): } F_1^{(4)} = \varepsilon_1 - i\Omega\,(\varkappa p_0 \alpha_1 + \alpha_0 p_1) = 0. \tag{27}$$

Since $F_1^{(1)}$ and F_0 are identical we have obtained only four independent equations for $n=0$. Using the equation obtained by differentiating (25) we may replace (24) by:

$$[F_1^{(1)},\ F_1^{(2)}] = \Omega^2 z_1 + \alpha_0' p_1 + g\varrho_0\alpha_1 = 0 \tag{28}$$

If ε_1 is given it is always possible to determine α_1, p_1, z_1 from (25, 27, 28). The determinant of these equations may be written (using the equation of state $p_0\alpha_0 = R_0 T_0$).

$$\left.\begin{aligned} D_1 &= g^2 - g\varkappa p_0 \alpha_0' + \Omega^2 \varkappa p_0 \alpha_0 \\ &= \varkappa R_0 \left[-g\left(T_0' + \frac{g}{c_p}\right) + \Omega^2 T_0 \right] \end{aligned}\right\} \tag{29}$$

If we neglect the last term, which is very small compared with the others, it is seen that $D_1=0$ when $T_0' = -g/c_p$ (approximately 10^{-2} centigrade per metre), i.e. when the atmosphere is in the indifferent static equilibrium. As we have assumed static stability we have everywhere $D_1 \neq 0$.

The remaining two variables ψ_1, θ_2 are connected by Eq. (26). Another equation between them can be found as follows: From (8) and (9) we compute

$$F_3^{(2)} = a^2\Omega\,(-\Omega + i\beta)\,\psi_1 + 2\Omega^2 a i z_1 + 2\Omega^2 a^2 i\theta_2 + i\,(\alpha_0 p_3 + g z_3) = 0, \tag{30}$$

$$F_2 = a^2\Omega\,(-\Omega + i\beta)\,\theta_2 - 2\Omega^2 a^2 i\psi_1 + 3\,(\alpha_0 p_3 + g z_3) = 0 \tag{31}$$

Multiplying (30) by $3i$ and adding we obtain

$$[F_3^{(2)}, F_2] = \left(5i + 3\frac{\beta}{\Omega}\right)\psi_1 + \left(7 - i\frac{\beta}{\Omega}\right)\theta_2 + \frac{6}{a}z_1 = 0 \quad (32)$$

The equations (26, 32) determine ψ_1 and θ_2 (z_1 and α_1 being already determined).

Let (p_1, α_1, z_1, ψ_1, θ_2) be called "the first set" of coefficients, ($p_{2n-1}, \dots \psi_{2n-1}, \theta_{2n}$) "the n^{th} set" ($n \geqq 1$). We have seen that the first set is determined when ε_1 is known. If all sets up to the n^{th} have been found the $(n+1)^{\text{th}}$ set ($p_{2n+1} \dots \psi_{2n+1}, \theta_{2n+2}$) can be determined. The first set was found by putting the expressions $F_1^{(2)}$, $F_1^{(3)}$, $F_1^{(4)}$, $[F_1^{(1)}, F_1^{(2)}]$ and $[F_1^{(2)}, F_2]$ equal to zero. In like manner the $(n+1)^{\text{th}}$ set will be found by putting the analogous expressions

$$F_{2n+1}^{(2)},\ F_{2n+1}^{(3)},\ F_{2n+1}^{(4)},\ [F_{2n+1}^{(1)}, F_{2n+1}^{(2)}],\ [F_{2n+3}^{(2)}, F_{2n+2}] \quad (33)$$

equal to zero. (The meaning of the brackets will be defined below). We have

$$F_{2n+1}^{(1)} = -\Omega^2 z_{2n+1} - 2\Omega^2 ia\psi_{2n-1} + \alpha_0 p'_{2n+1} + g z'_{2n+1} - g\varrho_0\alpha_{2n+1} = 0, \quad (34)$$

$$F_{2n+1}^{(2)} = a^2(-\Omega^2 + i\beta\Omega)\psi_{2n-1} + 2\Omega^2 a i z_{2n-1} + 2\Omega^2 a^2 i(\theta_{2n} - \theta_{2n-2}) + i(\alpha_0 p_{2n+1} + g z_{2n+1}) = 0, \quad (35)$$

$$F_{2n} = a^2(-\Omega^2 + i\beta\Omega)\theta_{2n} - 2\Omega^2 a^2 i\psi_{2n-1} + (2n+1)(\alpha_0 p_{2n+1} + g z_{2n+1}) = 0, \quad (36)$$

$$F_{2n+1}^{(3)} = \frac{2}{a}z_{2n+1} + z'_{2n+1} + i\psi_{2n+1} + (2n+3)\theta_{2n+2} - (2n+2)\theta_{2n} - \varrho_0\alpha_{2n+1} = 0, \quad (37)$$

$$F_{2n+1}^{(4)} = \varepsilon_{2n+1} - i\Omega(\varkappa p_0\alpha_{2n+1} + \alpha_0 p_{2n+1}) = 0 \quad (38)$$

Differentiating (35) we find an expression for $(\alpha_0 p'_{2n+1} + g z'_{2n+1})$. Introducing this into (34) we obtain

$$[F_{2n+1}^{(1)}, F_{2n+1}^{(2)}] = \Omega^2 z_{2n+1} + \alpha'_0 p_{2n+1} + g\varrho_0\alpha_{2n+1} = L_n, \quad (39)$$

where L_n is a known quantity depending on ψ_{2n-1}, z_{2n-1}, θ_{2n}, θ_{2n-2} and their first derivatives. From (36, 38, 39) we can find z_{2n+1}, p_{2n+1}, α_{2n+1}. The determinant of these equations is equal to D_1 in (29) which is different from zero in the stable atmosphere.

It remains to find ψ_{2n+1}, θ_{2n+2}. If in (35, 36) we change n to $(n+1)$ we get the equations $F_{2n+3}^{(2)} = 0$, $F_{2n+2} = 0$. Eliminating here the expression $(\alpha_0 p_{2n+3} + g z_{2n+3})$ we obtain the equation

$$[F_{2n+3}^{(2)}, F_{2n+2}] = \left((2n+5)i + (2n+3)\frac{\beta}{\Omega}\right)\psi_{2n+1} + \left(4n + 7 - \frac{\beta}{\Omega}i\right)\theta_{2n+2} + (4n+6)\left(\frac{1}{a}z_{2n+1} - \theta_{2n}\right) = 0. \quad (40)$$

From (37) and (40) we can always determine ψ_{2n+1} and θ_{2n+2} since the determinant $D_2 = 4(n+1)(n+2)\left(i + \frac{\beta}{\Omega}\right)$ is different from zero.

Thus the $(n+1)^{\text{th}}$ set of coefficients can be determined when the previous sets are known and as the first set is known it follows that all coefficients in (18, 19) can be determined when ε is given. If the series thus obtained converge uniformly the amplitudes in (17) are determined. The proposition (21) is thereby proved.

It will be assumed that ε is an analytic and regular function of the position on the sphere. It can be shown by means of Eqs. (7—11) that the other significant variables must then also be analytic, although not necessarily regular everywhere. (This point will be considered more closely in a later paragraph.) In special cases singularities may occur, whereby the variables become infinite for certain values of c, and no oscillation of the form W_1 is possible. Generally, however, all variables will be analytic and regular, whence it follows that they can be expanded in convergent series of the form (18, 19) and determined by ε as stated in proposition (21).

It is important to note that the value thus found for z will generally not satisfy the boundary condition $z = 0$ for $a = a_0$. We may affirm, therefore, that, if ε is given as an analytic and regular function, an oscillation of the form (17) is generally not possible.

Since the above considerations are based on the expansions (18, 19) derived from (16) it is of interest to inquire whether the latter formula can yield expansions for the significant variables other than (18, 19).

The formula (16) can be applied not only to the functions x but also to other symmetric functions of the form $y = A_y e^{i\nu}$. Applying the second part of (16) to $y = x \sin^{2n} c$ we are led back to series of the form (18) for A_x (after deleting terms which are infinite for $c=0$). For $y = x \sin^{2n+1} c$ the case is different. Consider for instance $y = x \sin c$. Here A_y takes the form (18) and consequently x takes the form

$$\bar{x} = (x_2 \sin^2 c + \dots) e^{i\nu}. \qquad (41)$$

It is possible to satisfy the equations (7 to 11) by expressions of this form (with $\bar{x} = p, \alpha, z, \varepsilon, \psi$) together with $\bar{\theta} = \cos c\, (\theta_3 \sin^3 c + \dots) e^{i\nu}$, but here the following important point should be noted. The x in (17) are analytic functions of b and c in all points of the sphere, whereas the $\bar{x}$ in (41) have a singularity for $c=0$. Considering for instance the term $f = x_2 \sin^2 c\, e^{i\nu}$ we have $\dfrac{\partial^2 f}{\partial c^2} = 2x_2 \cos 2c\, e^{i\nu}$, which is discontinuous for $c=0$, changing its value from $h = 2x_2 e^{i\nu}$ to $-h$, when b is replaced by $(b+\pi)$. (Generally the function $\sin^n c\, e^{im\nu}$ is analytic for $c=0$ when and only when $(m+n)$ is an even number.) Although solutions like (41) are mathematically possible there is no reason to suppose that ε, which determines the given heat supply, should have this form. It seems natural to assume that ε is analytic and regular everywhere, i.e. of the form (18), which leads to the same form for p, z, α, ψ.

4. Forced oscillations of 12-hour period

The most general oscillation of 12-hr. period rotating with the sun is obtained from Eqs. (14, 15) by discarding the terms with $s=1$. The simple oscillation W_2 is obtained by discarding all terms except those with $s=2$. In order to have symmetry with respect to the equator we further delete in (15) those Legendre functions which are asymmetric. We thus obtain

$$x = \sum_0^\infty C_{2n}^{(x)} P_{2n}(\mu) + e^{2i\nu} \sum_0^\infty K_{2n}^{(x)} P_{2n}^2(\mu), \qquad (42)$$

$$(x = p,\ \alpha,\ z,\ \varepsilon).$$

(It is seen from (7) that $\psi \sin^2 c$ must have the same form as z, p and α). The asymmetric variable θ might be expressed by a corresponding formula with asymmetric Legendre functions in the second part.

Abbreviating these formulae we may write $x = x_s + a_x e^{2i\nu}$, $\theta = \theta_s + a_\theta e^{2i\nu}$ and from (13) $\varphi = \varphi_s + A_\varphi e^{2i\nu}$. The term $\varphi_s = gH\left(\dfrac{1}{3} - \dfrac{1}{2}\sin^2 c\right)$ causes a slight permanent modification of the static state so that, for instance, the static pressure will be $(p_0 + p_s)$ instead of p_0, etc. This modification is of secondary interest and will not be studied here. We put, therefore, $\varphi = A_\varphi e^{2i\nu}$ and consider only the timevariable part of (42) seeking solutions of (7 to 11) of the form

$$y = a_y e^{2i\nu}, \quad (y = p,\ \alpha,\ z,\ \varepsilon,\ \psi,\ \theta). \qquad (43)$$

Writing the Legendre functions in the form

$$P_{2n}^2(\mu) = = (1-\mu^2)(k_0 + k_2\mu^2 + \dots + k_{2n-2}\mu^{2n-2})$$

and introducing this in (42), last term, we obtain

$$a_x = x_2 \sin^2 c + x_4 \sin^4 c + \dots \qquad (44)$$

$(x = p, \alpha, z, \varepsilon)$. Since $a_\psi \sin^2 c$ must have this form, we have

$$a_\psi = \psi_0 + \psi_2 \sin^2 c + \psi_4 \sin^4 c + \dots \qquad (45)$$

The form of a_θ is found from (8, 9, 10):

$$a_\theta = \cos c\, (\theta_1 \sin c + \theta_3 \sin^3 c + \dots). \qquad (46)$$

It is of interest to note that, for $c=0$, A_ψ in (18) is zero, whereas a_ψ in (45) is different from zero. The amplitude of linear velocity, $a_\psi \sin c$, is zero for $c=0$.

We shall prove the following proposition:

$$\left.\begin{array}{l}\text{If } \varepsilon = a_\varepsilon e^{2i\nu} \text{ and } \varphi = A_\varphi e^{2i\nu} \text{ are given}\\ \text{then the variables } p, z, \alpha, \psi, \theta \text{ of the form}\\ a_y e^{2i\nu} \text{ which satisfy the equations of}\\ \text{motion, the equation of continuity and the}\\ \text{first law of thermodynamics, contain an}\\ \text{arbitrary function } f(a).\end{array}\right\} (47)$$

Putting in (7 to 11) $G_i = 0$ and $\varphi = -\dfrac{1}{2} gH \sin^2 c\, e^{2i\nu}$ according to (13) and having regard to (44, 45, 46) we may write the equations (7, 8, 11) in the form

$$\sum_{0}^{\infty} F^{(i)}_{2n+2} \sin^{2n+2} c = 0, \quad (i = 1, 2, 4) \tag{48}$$

while (9) and (10) have the form

$$\cos c \sum_{0}^{\infty} F_{2n+1} \sin^{2n+1} c = 0, \qquad \sum_{0}^{\infty} F^{(3)}_{2n} \sin^{2n} c = 0, \tag{49}$$

respectively. Starting with $n = 0$ we get:

$$\text{from (7)} \ F^{(1)}_2 = -4\Omega^2 z_2 - 4\Omega^2 a i\psi_0 + \alpha_0 p_2' + g z_2' - g\varrho_0 \alpha_2 = 0, \tag{50}$$

$$\text{from (8)} \ F^{(2)}_2 = (-4\Omega^2 + 2i\Omega\beta)\, a^2 \psi_0 + 4\Omega^2 a^2 i\theta_1 + 2i(\alpha_0 p_2 + g z_2) - igH = 0, \tag{51}$$

$$\text{from (9)} \ F_1 = (-4\Omega^2 + 2i\Omega\beta)\, a^2 \theta_1 - 4\Omega^2 a^2 i\psi_0 + 2(\alpha_0 p_2 + g z_2) - gH = 0, \tag{52}$$

$$\text{from (10)} \ F^{(3)}_0 = i\psi_0 + \theta_1 = 0, \tag{53}$$

$$\text{from (11)} \ F^{(4)}_2 = \varepsilon_2 - 2i\Omega(\varkappa p_0 \alpha_2 + \alpha_0 p_2) = 0. \tag{54}$$

It is seen that $iF^{(2)}_2 + F_1 = 0$ is identical with (53) so that we have only four independent equations. Introducing in (52) $\theta = -i\psi_0$ we obtain

$$\alpha_0 p_2 + g z_2 = \frac{1}{2} gH - \beta\Omega a^2 \psi_0. \tag{55}$$

Differentiating this and introducing the expression thus obtained into (50) we get

$$4\Omega^2 z_2 + \alpha_0' p_2 + g\varrho_0 \alpha_2 = -4\Omega^2 a i\psi_0 + \Omega \frac{d}{da}(\beta a^2 \psi_0). \tag{56}$$

If ε_2 and ψ_0 are given the equations (54, 55, 56) determine p_2, α_2, z_2. The determinant of these equations is

$$D_3 = -\varkappa R g\left(T_0' + \frac{g}{c_p}\right) + 4\Omega^2 R T_0, \tag{57}$$

which, like D_1 in (29), is different from zero when the atmosphere is statically stable.

Thus the "first set" of coefficients (p_2, z_2, α_2, θ_1, ψ_0) is determined by (53, 54, 55, 56) when ε and ψ_0 are given. It will be shown that, when all sets up to the n^{th} (p_{2n}, z_{2n}, α_{2n}, θ_{2n-1}, ψ_{2n-2}) are known then the $(n+1)^{\text{th}}$ set is determined. The procedure is very similar to that given by Eqs. (34 to 40) and will be indicated briefly here. We use the five equations

$$F^{(1)}_{2n+2}, \ F^{(2)}_{2n+2}, \ F_{2n+1}, \ F^{(3)}_{2n}, \ F^{(4)}_{2n+2} = 0. \tag{58}$$

It is easily verified that the equations

$$i(n+1) F^{(2)}_{2n+2} + F_{2n+1} = 0, \quad F^{(3)}_{2n} = 0$$

always determine ψ_{2n}, θ_{2n+1}. These being found, we derivate the third equation in (58) and introduce the expression thus obtained for $(\alpha_0 p_{2n+1}' + g z_{2n+1}')$ into the first. We obtain

$$[F^{(1)}_{2n+2}, \ F_{2n+1}] = 4\Omega^2 z_{2n+2} + \alpha_0' p_{2n+2} + g\varrho_0 \alpha_{2n+2} = L_{2n+2}, \tag{59}$$

where L_{2n+2} is known. This equation together with F_{2n+1}, $F^{(4)}_{2n+2} = 0$ determine p_{2n+2}, z_{2n+2}, α_{2n+2}. Their determinant is D_3 (57).

Thus the $(n+1)^{\text{th}}$ set has been determined by means of the previous sets, whence it follows that all coefficients in (44, 45, 46) can be found when ε, φ, and ψ_0 are given. $\psi_0 = \psi_0(a)$ must be a small quantity. Apart from this condition ψ_0 may be chosen arbitrarily. Proposition (47) is thereby proved.

It is of interest to observe that the boundary condition $z = 0$ for $a = a_0$ can always be satisfied by a suitable choice of the function $\psi_0(a)$ as follows:

Let ε and ψ_0 be analytic functions of a and c. We have seen that the "first set" ($p_2, \alpha_2, z_2, \theta_1, \psi_0$) is determined by ($\psi_0, \psi_0', \varepsilon_2, H$). The second set is determined by the first set and its first derivatives and ε_4, i.e. by (ψ_0, ψ_0', ψ_0'', ε_2, ε_2', ε_4, H). Proceeding in this manner we find in the n^{th} set for instance

$$z_{2n} = f_{2n}(\psi_0, \ \psi_0', \dots \psi_0^{(n)}, \ \varepsilon_2, \ \varepsilon_2', \dots \dots \varepsilon_2^{(n-1)} \dots \varepsilon_{2n}, \ H)$$

The conditions $z_{2n} = 0$ for $a = a_0$ render an infinite number of linear equations

$$L_n(\psi_{00}, \ \psi_{00}', \dots \psi_{00}^{(n)}) = M_n$$

where the subscript 00 indicates the values for $a = a_0$ and the M_n are known. If the constant

ψ_{00} is chosen arbitrarily, ψ'_{00} is found from $L_1 = M_1$, ψ''_{00} is found from $L_2 = M_2$ etc. Since we have

$$\psi_0(a) = \psi_{00} + \psi'_{00}(a - a_0) + \frac{1}{2}\psi''_{00}(a - a_0)^2 + \dots$$

it is seen that $\psi_0(a)$ is determined, save for an arbitrary constant.

5. Further remarks on the linear theory of the 24-hr. and the 12-hr. solar wave

In the foregoing paragraphs we have sought particular solutions of the hydro-dynamical equations, of the form $Ae^{im\nu}$, by direct insertion in the given system of first-order linear equations (7 to 11). An alternative procedure would be to base the theory on a linear partial differential equation of the second order, deduced from the system (7 to 11) by suitable eliminations. This procedure presents certain features of interest which will be indicated briefly here.

Let p_e, α_e denote deviations from the equilibrium values of pressure and specific volume at a geometrical point P. At a certain moment a particle $M(a, b, c)$ with pressure $(p_0 + p)$ is situated at P at the level $(a + z)$, where the equilibrium values are $p_0 + p'_0 z$, $\alpha_0 + \alpha'_0 z$. We then have

$$p = p_e + p'_0 z, \quad \alpha = \alpha_e + \alpha'_0 z \tag{60}$$

Introducing these expressions in (7 to 11), putting $G_i = 0$ and $y = A_y e^{im\nu}$, $(y = p_e, \alpha_e, z, \psi, \theta, \varepsilon, \varphi)$, we obtain a linear system from which it is possible to eliminate z, ψ, θ, α_e whereby we obtain an equation of the form

$$A_{11}\frac{\partial^2 p_e}{\partial a^2} + 2A_{12}\frac{\partial^2 p_e}{\partial a \partial c} + A_{22}\frac{\partial^2 p_e}{\partial c^2} + A_1\frac{\partial p_e}{\partial a} +$$
$$+ A_2\frac{\partial p_e}{\partial c} + Ap_e = B_1\frac{\partial \varepsilon}{\partial a} + B_2\frac{\partial \varepsilon}{\partial c} + B\varepsilon + L(\varphi) \tag{61}$$

where $L(\varphi)$ is a linear form in φ and its first and second derivatives. The coefficients of (61) are generally complex. It can be shown that we have

$$A_{11} = \beta_m^2 - 4\Omega^2\cos^2 c, \; A_{12} = \frac{4\Omega^2}{a}\sin c\cos c,$$
$$A_{22} = \frac{1}{m\Omega a^2}(K\beta_m - 4\Omega^3 m\sin^2 c) \tag{62}$$

where

$$\beta_m = -m\Omega + i\beta, \; K = g\frac{d}{da}\ln\vartheta_0 - m^2\Omega^2 \tag{63}$$

(ϑ_0 = potential temperature in the static state).

It has been assumed that the given periodic heat supply, represented by ε, is an analytic function of a and c. The right side of (61) is therefore analytic, and p_e must be analytic. Since the other variables can be expressed as linear functions of p_e, φ, ε and their derivatives, they must all be analytic. Thus the statement in paragraph 3 is confirmed.

It is further of interest to note that, in the case of no friction ($\beta = 0$) the coefficients of (61) are real. It can then be seen from (62) that the equation (61) is of the hyperbolic type when $m = 2$ (12-hr. oscillation). When $m = 1$ (24-hr. oscillation) Eq. (61) is hyperbolic at low, elliptic at high latitudes, the latitudes of transition being very near to $c = 60°$, $c = 120°$.

6. Relation between the 24-hr. and the 12-hr. solar wave. Non-linear equations

In the following we shall start by considering the solutions (17) and (43) (W_1 and W_2) of the equations (7 to 11) without regard to boundary conditions. The effect of the boundaries will be discussed at a later stage.

In the previous paragraphs linearized equations have been used to study each wave component separately. Although this method is useful for the study of small oscillations it cannot lead to a complete theory, however small the amplitudes may be. Important features of the motion are lost in the linear theory. For instance, a heat supply of the form $\dot{q} = A_q \sin(\nu + \eta_q)$ means that the heat entering a unit mass during one half-period is completely rejected during the next half-period and no energy seems to be left to overcome friction. Nevertheless the equations yield a solution in the form of an undamped oscillation. This apparent contradiction is solved by noting that the loss of energy per unit mass and unit time due to friction is of the second order of magnitude. It does not appear in the linearized equations but it is included in the expressions G_i of (7 to 11).

Further, the linear theory can give no information on the relation between the waves

W_1 and W_2, although a relation obviously must exist.

In the following the second-order terms G_i in (7 to 11) will be taken into account. They contain a great number of product-terms, which it is unnecessary to write out here. For the purpose in hand it will suffice to know the form of the G_i.

The values of the significant variables found in the previous paragraphs may be considered as a first approximation to the correct values, which may be written:

$$x = x^{(1)} + x^{(2)}, \quad (x = p, \alpha, z, \psi, \theta, \varepsilon) \tag{64}$$

where the $x^{(1)}$ are the values satisfying the linearized equations, while the $x^{(2)}$ are "corrections" of the second order of magnitude. It is immediately apparent that every product-term containing a quantity $x^{(2)}$ is of the third (or higher) order of magnitude. For instance $p\alpha = p^{(1)}\alpha^{(1)}$ + third-order terms. If we neglect third-order terms, the G_i may thus be considered as depending only on the $x^{(1)}$ and their derivatives.

In connection with the linearized equations we have used complex variables for convenience. When treating product-terms we must revert to real variables. Considering the 24-hr. oscillation W_1 we may then write $x^{(1)} = B_x \cos \nu + C_x \sin \nu$, whence it follows that the G_i, being sums of product-terms have the form:

$$G_i = H_i + M_i \cos 2\nu + N_i \sin 2\nu, \tag{65}$$

where the coefficients are functions of a and c which can be computed if the $x^{(1)}$ have been determined.

Writing the equations (7 to 11) in the abbreviated form $L_i = G_i$ ($i = 1, 2, 3, 4, 5$) and introducing $x = x^{(1)} + x^{(2)}$, $(x = p, \alpha, z, \psi, \theta, \varepsilon)$, we may write $L_i = L_i^{(1)} + L_i^{(2)}$, where $L_i^{(1)}$ denotes the expressions on the left of the equations with $p^{(1)}, \alpha^{(1)} \ldots \varepsilon^{(1)}$ in the place of $p, \alpha \ldots \varepsilon$ and $L_i^{(2)}$ the corresponding expressions with $p^{(2)}, \alpha^{(2)} \ldots \varepsilon^{(2)}$. ($L_i^{(2)}$ does not contain φ.) According to the definition of the first-order variables $x^{(1)}$ we have $L_i^{(1)} = 0$, and consequently in virtue of (65):

$$L_i^{(2)} = H_i + M_i \cos 2\nu + N_i \sin 2\nu. \tag{66}$$

In (7 to 11) the individual terms on the left side were of the first-order of magnitude. Replacing $p, \alpha \ldots \varepsilon$ by $p^{(2)}, \alpha^{(2)} \ldots \varepsilon^{(2)}$ we have obtained the system (66) where each term is of the second order.

At first thought one might expect all the variables $x^{(2)}$ in (66) to be of the same form as the expressions on the right. However, this would not satisfy the second equation. The equations (66) can be satisfied by putting:

$$\left.\begin{aligned} \psi^{(2)} &= k_\psi t + h_\psi + m_\psi \cos 2\nu + n_\psi \sin 2\nu \\ y^{(2)} &= h_y + m_y \cos 2\nu + n_y \sin 2\nu \end{aligned}\right\} \tag{67}$$

$$(y = p, \alpha, z, \theta, \varepsilon)$$

where the coefficients are functions of a and c. The periodic and the non-periodic part of (67) can be found separately. If we replace in (7 to (11) ψ by $k_\psi t + h_\psi$ and $(p, \alpha \ldots \varepsilon)$ by $(h_p, \ldots h_\varepsilon)$ and delete the periodic part of the G_i we obtain a special case of the system (66). We write here only the first of the five equations:

$$2\Omega a \sin^2 c\, k_\psi + \alpha_0 \frac{\partial h_p}{\partial a} + g \frac{\partial h_z}{\partial a} - g\varrho_0 h_\alpha = H_1. \text{(68}$$

It can be shown from Eq. (1) that, owing tc the symmetry of the oscillation, the quantities $H_{1,\ 2,\ 4,\ 5}$, must be of the form $f(\sin c) = a_1 \sin c + a_3 \sin^3 c + \ldots$ and H_3 of the form $\cos c\ f(\sin c)$. Correspondingly we can find h_θ in the latter form and the other variables in the form $f(\sin c)$.

The non-periodic terms $k_\psi t$, h_ψ may be considered as permanent modifications of the static state. Thus the pressure will be $p_0 + p_s + h_p$, where p_s (defined in paragraph 4) is of the first order, h_p of the second order of magnitude. The rotation velocity of the atmosphere will be $\Omega + k_\psi$. On this modified static state the oscillations of first and second order are superposed.

Turning now to the periodic part of the solution we neglect in (67) the terms $k_\psi t$, h_ψ, h_y and delete in (66) the terms H_i. We then have a forced oscillation of 12-hour period. Again it is convenient to write the periodic solution in the complex form $\overline{m}_x e^{2i\nu}$ and the right side of (66) in the form $\overline{M}_i e^{2i\nu}$, where $\overline{m}_x$ and $\overline{M}_i$ are complex.

In the general case the "heat function" has the form

$$\varepsilon = A_\varepsilon^{(1)} e^{i\nu} + \left(a_\varepsilon^{(1)} + a_\varepsilon^{(2)}\right) e^{2i\nu} + h_\varepsilon^{(2)} \tag{69}$$

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where the indices (1), (2) indicate the order of magnitude. (The amplitudes are complex, $h_\varepsilon^{(2)} = h_\varepsilon$ is real.) It is of interest to consider the following particular case

$$\varepsilon = A_\varepsilon^{(1)} e^{i\nu} + h_\varepsilon, \text{ and } \varphi = 0. \qquad (70)$$

Even in this case we generally have M_i, N_i, $\overline{M}_i \neq 0$ and the equations (66) still render a forced 12-hr. oscillation which vanishes only in the case $A_\varepsilon^{(1)} = 0$. Thus the heat supply of 24-hr. period (70) generates a 12-hr. oscillation of the second order. (A similar phenomenon is known from tidal theory where a tidegenerating force of 12-hr. period generates both a 12-hr. and a 6-hr. wave.)

It is important to note that h_ε cannot be given arbitrarily but is determined by $A_\varepsilon^{(1)}$. If we introduce in Eq. (5) $P = p_0 + p^{(1)} + p^{(2)}$, $S = \alpha_0 + \alpha^{(1)} + \alpha^{(2)}$, the equation may be written (neglecting third-order terms):

$$L_5^{(2)} = \varkappa p^{(1)} \dot{\alpha}^{(1)} + \alpha^{(1)} \dot{p}^{(1)}.$$

Introducing the real values $p^{(1)} = D_p \sin(\nu + \eta_p)$, $\alpha^{(1)} = D_\alpha \sin(\nu + \eta_\alpha)$ we get

$$L_5^{(2)} = \Omega D_p D_\alpha \left[\frac{1}{2}(\varkappa - 1)\sin(\eta_p - \eta_\alpha) + \right.$$
$$\left. + \frac{1}{2}(\varkappa + 1)\sin(2\nu + \eta_p + \eta_\alpha)\right]$$

and we thus have

$$h_\varepsilon = H_5 = \frac{1}{2}(\varkappa - 1)\Omega D_p D_\alpha \sin(\eta_p - \eta_\alpha). \qquad (71)$$

D_p, D_α are determined by $A_\varepsilon^{(1)}$ according to (47). The corresponding heat supply per unit mass and unit time $\dot{q} = h_\varepsilon/(\varkappa - 1)$ is the source of energy which is necessary to maintain the undamped oscillation against friction. We note that we cannot have $\eta_p = \eta_\alpha$ everywhere, since no energy would then be available. Pressure and density must therefore be out of phase, at least in the lower layers. This is connected with a lag of phase with height, which is an important feature, as shown by J. Bjerknes (1948) for the wave W_2.

We shall now consider the boundary conditions. A heat supply of the form (69) must generate some kind of wave motion in the model atmosphere. At first thought one might expect this motion to be of the simple form $W_1 + W_2$. However, this cannot be the case, since W_1 has been found to be incompatible with the boundary condition at the ground. The motion that actually takes place cannot be computed by the method described in the foregoing paragraphs. What probably happens is that the first term in (69) tends to create the wave of the form W_1, but that this breaks down and the resulting motions of the particles interact in such a manner as to strengthen the component W_2, the available energy determined by h_ε being transferred from W_1 to W_2.

If second-order terms are neglected the air particles in a simple wave of the form W_1 or W_2 have elliptic orbits. In W_1, for instance, the horizontal linear displacements of a particle would be of the form

$$a \sin c \cdot \psi = A_1^{(1)} \cos \nu + A_2^{(1)} \sin \nu,$$
$$a\theta = B_1^{(1)} \cos \nu + B_2^{(1)} \sin \nu.$$

Taking the case (70) and including in the solution the second-order terms of 12-hr. period obtained from (66) we get

$$a \sin c\psi = A_1^{(1)} \cos \nu + A_2^{(1)} \sin \nu +$$
$$+ A_1^{(2)} \cos 2\nu + A_2^{(2)} \sin 2\nu,$$

and a similar expression for $a\theta$, and the orbit becomes a slightly deformed ellipse as indicated in Fig. 2a).

An oscillation set up by the heat supply (70) far away from the earth's surface would have orbits like Fig. 2a). We have seen, however, that such an oscillation is incompatible with the boundary condition at the ground. It seems possible that this may cause the flow of energy to be diverted to the second-order part of the oscillation (of 12-hr. period). This would cause the orbit to take a form like Fig. 2b) or even 2c) which is very close to a pure 12-hr. oscillation (the particle passing from A to B during 12 hours and continuing from B to A in the following 12 hours). In this process of transition the equations (66) would cease to be valid, since the 12-hr. component would no longer be of the second order of magnitude, and a mathematical treatment would have to revert to the general non-linear equations (3, 4, 5).

The 12-hr. oscillation of the case (70) would of course be further amplified in the case (69) by the second part of ε and by the potential φ,

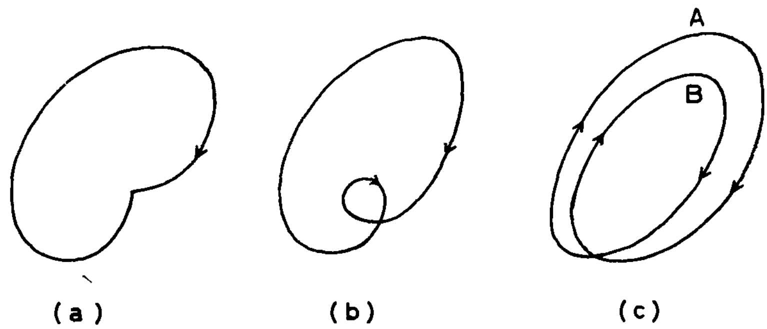

Fig. 2. Paths of an air particle in the oscillation $W = W_1 + W_2$.

but these terms are small and cannot alone account for the predominance of W_2 (save for the case of a high degree of selective resonance).

During 24 hours a flow of heat passes through the air masses, each particle absorbing heat during one half-period and rejecting heat during the following half-period. The hypothetical diversion of the heat flow in such a manner as to increase the loop of the particle orbits may be illustrated roughly by a simple mechanical model shown in Fig. 3.

A fluid is circulating in a closed system of tubes situated in a vertical plane. The part ABC is heated while the rest is cooled. The motion indicated by the arrows is stationary and maintains a constant flow of heat from the warm to the cold source. The arrangement is such that a particle making the big circuit BCEFB completes this in the time t_0, while a particle making the small circuit ABCDA takes the time $1/2\ t_0$. Suppose now that the big circulation is partly hindered, for instance by an obstacle in the tube EF. The warm and cold sources being unaltered, the small circulation must then be accelerated, and if its circulation time shall remain equal to $1/2\ t_0$ the tube AD must be moved to a position A'D'. Thus the small circuit gains importance relative to the big circuit.

Although the analogy between the systems of Fig. 2 and Fig. 3 is obviously incomplete it indicates the possibility of an increase of the loop in Fig. 2 because the "big circuit" of Fig. 2a) is hindered by the boundary condition, while no such hindrance prevents the development of the 12-hr. wave causing the loop in Fig. 2b), c).

7. The upper boundary condition

In the model atmosphere we may envisage three main possibilities as regards the upper boundary condition, namely (i) infinite extension of the atmosphere, (ii) a rigid spherical upper boundary, (iii) a free surface.

Let us consider first the infinite model atmosphere. Let S_1 denote a sphere, concentric with the earth, with radius $a = a_1$ chosen such that all wave-generating sources are inside S_1

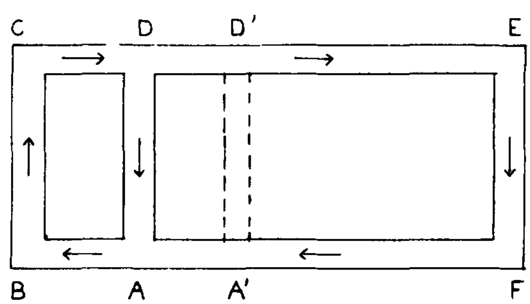

Fig. 3. Circulating fluid.

and that the atmosphere outside S_1 does not take part in the earth's rotation. In the outer atmosphere the density is

$$\varrho(a) = \frac{T_1}{T}\varrho_1 \exp\left(-\int_{a_1}^{a}\frac{g}{RT}da\right)$$

where R is the gas constant, ϱ and T density and temperature. Their values at S_1 are ϱ_1, T_1. g is proportional to a^{-2}. If $T > T_2 > 0$ the integral remains finite when $a \to \infty$, and therefore we have $\varrho > \varrho_2 > 0$ (T_2 and ϱ_2 constants). We may assume for simplicity that ϱ approaches a constant value. It is known that, in waves propagating outwards from a centre in a homogeneous medium the particles have a velocity amplitude proportional to $1/a\, e^{ika}$(k real constant), thus approaching zero when $a \to \infty$. It may be concluded that, whatever the wave-generating energy sources inside S_1 may be, the waves propagating into the outer atmosphere will always satisfy the boundary condition at infinity. The oscillation W_2 will, therefore, always be possible in this case.

Let us next consider the cases (ii) and (iii) of a finite atmosphere. We have seen from the linear theory of paragraph 4 that, if ε is given arbitrarily in the form $a_\varepsilon e^{2i\nu}$, the oscillation W_2 is determined by the fundamental equations and the lower boundary condition. The boundary condition at the upper rigid or free surface will therefore generally not be satisfied. This result seems to conflict with the classical theory of tides in a homogeneous incompressible ocean where the conditions at the lower and upper boundary can always be satisfied. The reason for this discrepancy may be seen by applying the equations (7 to 11) to the latter case. We then put $\alpha_0 = \text{const.}$, $\alpha_0' = 0$ and replace Eq. (11) by $\alpha = 0$. The classical theory is quasistatic, which entails the deletion of the two first terms in (7). Frictional terms are also deleted. Accordingly the equation (54) is replaced by $\alpha_2 = 0$ and the two first terms of (50) are deleted. The system (50 to 54) with $\beta = 0$ then reduces to the following four equations $\alpha_0 p_2' + g z_2' = 0$, $\alpha_0 p_2 + g z_2 = \frac{1}{2} gH$, $i\psi_0 = \theta_1$, $\alpha_2 = 0$. Here we may choose both $\psi_0(a)$ and $z_2(a)$ arbitrarily. With two arbitrary functions we can satisfy both boundary conditions. It is of interest to note that this result depends on the quasistatic approximation, which has not been applied in the above study of W_1 and W_2.

Turning now from the model to the real atmosphere we meet with two main difficulties in determining the boundary condition at the upper limit. One difficulty is that of taking account of the physical conditions which prevail in the upper regions and the other is due to the great particle velocities which the oscillation causes there. As regards the latter PEKERIS (1951) has pointed out that the linearized theory of atmospheric oscillations ceases to be valid at heights greater than 80 to 100 kilometres because at those heights it is no longer permissible to neglect the terms which are quadratic in the velocities.

It seems then that the above considerations concerning the upper boundary condition in the model atmosphere cannot be applied to the real atmosphere, for which no satisfactory mathematical treatment of the upper boundary condition seems to be possible at present.

Wilkes has pointed out that, owing to molecular viscosity and molecular thermal conductivity, one must think of the air at high level (120 km) as having distinctly sticky properties as far as large-scale oscillations of long period are concerned, so that any oscillatory energy which finds its way up to these heights will be degraded to heat. On the other hand he has shown that the greater part of the energy in the oscillation W_2 is probably reflected before reaching this height. These results seem to indicate that the upper boundary condition may be a condition intermediate between that of the infinite and that of the finite model atmosphere.

8. Concluding remarks

The theory of the solar waves considered here is incomplete. The main obstacles in the way of a satisfactory theory seem to be the mathematical difficulties involved in the application of non-linear forms of the hydrodynamical equations, and difficulties associated with the upper boundary condition. In reference to the latter point it is of interest to observe that the propositions (21) and (47) are independent of the upper boundary condition. Further they are independent of assumptions concerning resonance. In any case the formation of the semidiurnal wave W_2 is favoured at the expense of the diurnal wave W_1. If, in addition, there exists a free oscillation with a period not far from 12 hours, this will contribute to the growth of W_2. It may not, however, be necessary to assume a very high degree of selective resonance.

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