# NOTES

## Eccentric Circumpolar Vortices in a Barotropic Atmosphere[1]

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Recent investigations (LA SEUR, 1954) have shown that on many occasions the circulation of the middle troposphere possesses a marked eccentricity; i.e., the center of symmetry of the flow is found at a considerable distance from the geographic pole. At such times the intensity of the zonal component of the flow (calculated with respect to the geographic pole) is low, while the meridional component appears large. Harmonic analysis of the motion reveals that most of the energy of the meridional motion is concentrated in the first longitudinal harmonic (wavelength 360 degrees of longitude). If, on the other hand, the motion is analyzed with respect to the circulation pole (centroid of the streamline field), it is found that the purely zonal component in the coordinate system thus defined is often as great or even greater than in those cases where the motion is symmetric about the geographic pole. Thus the zonal index calculated in geographical coordinates may at times give a rather misleading picture of the character of the circulation; what appears to be a flow with a large meridional component may actually be a rather symmetric zonal vortex with respect to an eccentric pole.

As far as this writer is aware, no previous theoretical analyses of the behavior of eccentric vortices having a scale comparable to the planetary dimensions have been made. ROSSBY (1948, 1949) has treated the case of an axially symmetric vortex embedded in an atmosphere at rest with respect to a rotating earth; the model so arrived at is compatible with the observed behavior of cyclones and anticyclones with diameters small compared to the planetary diameter. In the Rossby model the vortex is subject to an acceleration equal to the area integral of the Coriolis force over the horizontal extent of the vortex divided by the mass of air bounded between two surfaces of this horizontal extent which are separated by unit height. This resultant Coriolis acceleration is not balanced by any pressure gradient in the resting environment and hence produces a meridional component of translation of the vortex as a whole. This approach is obviously unsuitable for the study of a vortex which occupies the entire sphere, since no pressure-gradient-free region can surround such a vortex.

The balance of the present paper is devoted to the derivation of an equation governing the behavior of eccentric circumpolar vortices covering the entire sphere which are initially purely zonal with respect to the circulation pole, and to a brief discussion of the resulting motions for a few simple cases. The flow is assumed to satisfy the two-dimensional non-divergent barotropic vorticity equation.

The circulation pole is assumed to lie initially at geographic latitude $\phi_0$ and geographic longitude $\lambda_0 = 0$. The stream function $\psi$ for the motion is taken to be a single pure zonal spherical harmonic; hence the motion at $t = 0$ is prescribed everywhere on the sphere by the equation

$$\psi = A + BP_n(\sin \varphi) \tag{1}$$

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where $A$ and $B$ are independent of position, $P_n$ is the zonal harmonic (Legendre polynomial) of order $n$ with argument $\sin\phi$, and $\phi$ is latitude measured with respect to the eccentric pole. The instantaneous zonal angular velocity of the flow is given by

$$\dot{\lambda}=a^{-2}\sec\phi\,\partial\psi/\partial\phi=a^{-2}\sec\phi\,BP_n^1(\sin\phi) \quad (2)$$

One must now find solutions of the vorticity equation

$$\nabla^2(\partial\psi/\partial t)+\mathbf{v}\cdot\nabla(\nabla^2\psi+f)=0 \quad (3)$$

which satisfy (1) as an initial condition at $t=0$ and which are finite, single-valued and twice-differentiable over the sphere. It is possible to express the Coriolis parameter $f$ as a function of latitude and longitude in the eccentric system; this was, in fact, the first approach used by the writer. Such a solution is rather laborious, occupies excessive space in print, and is obtained in a form which masks the physical significance of the result. The same solution is obtainable much more simply by transforming (1) to geographic coordinates and solving (3) in the geographic system.

The transformation is readily accomplished by use of the "addition theorem" for zonal harmonics (JAHNKE and EMDE, 1943, p. 115). In the notation used in this paper, this theorem may be written

$$P_n(\sin\phi)=P_n(\sin\phi_g)\,P_n(\sin\phi_0)+ + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\sin\phi_0)\,P_n^m(\sin\phi_g)\cos m(\lambda_0-\lambda_g) \quad (4)$$

where $\phi_0$ and $\lambda_0$ are the geographic coordinates of the eccentric pole from which $\phi$ is measured, and $\phi_g$, $\lambda_g$ are the geographic coordinates of any point on the sphere. Since $\lambda_0=0$ initially, the stream function equation in geographic coordinates becomes

$$\psi=A+BP_n(\sin\phi_0)\,P_n(\sin\phi_g)+ + 2B\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\sin\phi_0)\,P_n^m(\sin\phi_g)\cos m\lambda_g \quad (5)$$

Inspection of (5) shows that all zonal and tesseral harmonics in the streamline field possess the same lower index $n$. Such an ensemble of harmonic waves is of the type discussed by NEAMTAN (1946). The property of such a set of waves which is of major interest here is that the resultant streamline field is a pattern in which the isolines of relative vorticity always coincide with the streamlines. This follows from the fact that the relative vorticity $\nabla^2\psi_n^m$ of any tesseral-harmonic stream function component is given by the associated Legendre equation

$$\nabla^2\psi_n^m+n(n+1)\,a^{-2}\psi_n^m=0,\qquad \psi_n^m=B_{mn}\,P_n^m\,e^{im\lambda_g} \quad (6)$$

so that, if $n$ is the same for all the $\psi_n^m$, the relative vorticity is given by

$$\nabla^2\psi=-n(n+1)\,a^{-2}\psi,\qquad \psi=\sum_{m=0}^{n}B_{mn}\,P_n^m(\sin\phi_g)\,e^{im\lambda_g} \quad (7)$$

and is simply proportional to the value of $\psi$ at each point. From this it follows that the velocity vector of the flow is always tangent to a line of constant relative vorticity and hence the advection of relative vorticity vanishes everywhere. This implies that there can be no generation of new zonal or tesseral harmonics by interaction between the existing harmonics, hence $n$ can never take on any value different from the initial one. Thus the solution of the vorticity equation (3) for a set of waves of this type will be closed with respect to the index $n$ (and hence with respect to the upper index $m$ also), since the term $\mathbf{v}\cdot\nabla f$ does not introduce nonlinear interactions.

Substituting (7) into (3) and setting $\mathbf{v}\cdot\nabla(\nabla^2\psi)=0$ and $\mathbf{v}\cdot\nabla f=2\Omega a^{-2}\partial\psi/\partial\lambda_g$ gives the vorticity equation applicable to sets of harmonic waves of the type given in (5)

$$2\Omega\,\partial\psi/\partial\lambda_g-n(n+1)\,\partial\psi/\partial t=0 \quad (8)$$

This is immediately recognizable as the equation for one-dimensional wave propagation in a linear non-dispersive medium. All harmonics propagate without change of amplitude in the negative $\lambda_g$-direction with the same angular speed $\omega=2\Omega/n(n+1)$. The solution of (8) which is satisfied by (5) as initial condition is

$$\psi = A + BP_n(\sin\phi_0)\,P_n(\sin\phi_g) +$$

$$+ 2B\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\sin\phi_0)\,P_n^m(\sin\phi_g)$$

$$\cos m\,[\lambda_g + 2\Omega t/n\,(n+1)] \qquad (9)$$

But this is identical in form to (4) with $-\lambda_g$ replaced by $-[\lambda_g+2\Omega t/n(n+1)]$. Hence (9) may be transformed back to an eccentric coordinate system with a *moving* pole located at geographic coordinates $\phi_0$, $\lambda_0 - 2\Omega t/n\,(n+1)$. The stream function in this coordinate system is the original one, $\psi = A + BP_n\,(\sin\phi)$, with $\phi$ being measured from the moving pole. It has thus been shown that an eccentric circumpolar vortex in a barotropic atmosphere, for which the stream function is expressible as a pure zonal spherical harmonic, retains its shape and intensity unchanged while the eccentric pole executes a retrograde precession about the geographic pole at an angular speed $2\Omega/n(n+1)$ which depends only upon the rotation speed of the planet and the number of wind speed maxima in the vortex. The sense of rotation of the vortex and the magnitude of the wind speed maxima (which are determined solely by the coefficient $B$) do not enter into the equation for the precession rate. In this respect the vortices considered in this paper differ markedly from those treated by Rossby in the previously-referenced papers; the sense and speed of the motion were major determining factors in the subsequent displacements of the vortices considered by Rossby. These differences are in no way surprising, since the physical models are highly dissimilar. It is of interest to note that the area integral of the Coriolis force over the entire sphere vanishes identically for the vortices treated in this paper.

The precession speeds and the latitudes and number of wind maxima are listed in Table I for the first few values of $n$. Directions of the wind are those obtained with negative values of $B$. Latitudes refer to the eccentric coordinate system.

**Table I. Properties of some Zonal Harmonic Eccentric Vortices.**

| n (No. Wind Maxima) | Latitudes and Directions of Maxima | Precession Period of Pole, Days |
|---|---|---|
| 1 | Equator, W. (Solid Rotation) | 1 |
| 2 | 45°, W; —45°, E. | 3 |
| 3 | ±59°, W; Equator, E. | 6 |
| 4 | 67°, W; 20.5°, E; —20.5°, W; —67°, E. | 10 |
| 5 | ±70°, W; ±34°, E; Equator, W. | 15 |

It is, of course, rather unlikely that the actual streamline field in the atmosphere can be fitted closely at any time by a single zonal harmonic. It is apparent from Table I that a zonal harmonic vortex gives a streamline field that is either an even or an odd function of latitude. In general in the atmosphere one expects a rough symmetry of the wind field with respect to the equator, hence the odd functions ($n$ even) are not too likely to be found in the real atmosphere. The even functions display the desired symmetry, but unfortunately always yield a wind maximum at the equator, in disagreement with the facts in the real atmosphere. It is therefore probable that more than one value of $n$ would be required to describe the actual zonal wind field. In such a case the vortices associated with each value of $n$ would interact nonlinearly to generate new harmonics. However, in cases of square-law nonlinearity (such as the relative-vorticity advection term in the vorticity equation) one always finds in the response function the same harmonics which would have been present in the absence of nonlinearity (although changed in amplitude) in addition to the new harmonics generated through the nonlinearity. Therefore one should expect to find evidence of eccentric circulations precessing at the indicated rates in the atmosphere even though they may be hidden in a mixture of other wave motions. For example, if a part of the total complex motion is a solid rotation about an eccentric axis ($n=1$), a diurnal wave in the stream function should be observable at each station. The corresponding variation in the geopotential is readily deducible by taking the divergence of the vector equation of motion so as to obtain the "balance equation" in the form

$$\eta\,\nabla^2\psi + \nabla\eta\cdot\nabla\psi = \nabla^2(G + {}^1/_2\,\mathbf{v}\cdot\mathbf{v}) = \nabla^2 Q \quad (10)$$

where $\eta$ is the absolute vorticity and $G$ represents the geopotential (to avoid confusion with the symbol $\phi$ already used for the latitude). Substituting from (9) into (10), with $n=1$, yields a Poisson equation which is readily solvable for $Q$ since the lefthand side of (10) consists of a finite number of spherical harmonics. The geopotential is obtained by subtracting from $Q$ the squared-velocity term, yielding

$$G = K - B^2/3a^2 + [{}^2/{}_3 \Omega B P_1 \; (\sin\phi_0) - B^2/3a^2 \; P_2 \; (\sin\phi_0)] \; P_2 \; (\sin\phi_g) + [\Omega/3 \; B P^1{}_1 \; (\sin\phi_0) - B^2/9a^2 \; P^1{}_2 \; (\sin\phi_0)] \; P^1{}_2 \; (\sin\phi_g) \cos(\lambda g + \Omega t) - B^2/36a^2 \; P_2^2 \; (\sin\phi_0) \; P_2^2 \; (\sin\phi_g) \cos 2 \; (\lambda_g + \Omega t)$$

which shows that the oscillation of geopotential at a station possesses a diurnal and a semidiurnal component. It is well-known (WILKES, 1949) that the surface pressure (or 1,000-mb geopotential) does in fact exhibit such oscillations, and that the semidiurnal component varies with latitude approximately in accordance with a $P_2^2$-distribution. These pressure oscillations are usually attributed to a tidal or thermal forcing and involve a divergent atmospheric model. The analysis in this paper indicates, however, that a non-divergent atmosphere with purely two-dimensional flow may also exhibit pressure oscillations with diurnal and semidiurnal periods. It is therefore reasonable to inquire whether or not the mechanism treated in this paper, i.e., a solid rotation of the atmosphere about a slightly eccentric pole, might contribute a part of the observed diurnal pressure variation.

The data indicate that the amplitudes of the diurnal and semidiurnal components of the "solar" pressure oscillation are of the same order of magnitude at most stations; the semidiurnal wave is usually somewhat the larger of the two. Substituting various values of $B$ into (11) shows that for all solid rotation angular speeds up to 100 deg long day$^{-1}$ the terms linear in $B$ are at least one order of magnitude larger than the quadratic terms (i.e., the motion is geostrophic to the extent of 90 per cent or more) and that therefore the semidiurnal amplitude factor is less than two per cent of the diurnal amplitude factor. It thus appears that the observed oscillations cannot be accounted for by an eccentric solid-rotation vortex unless the semidiurnal component is amplified by a factor of at least 100 through a resonance phenomenon. Such a resonance phenomenon involves a divergent atmosphere and hence falls outside the scope of the model used in this paper. However, it is imaginable that a divergence field might exist which is small enough to leave the two-dimensional flow pattern nearly unaffected while permitting the necessary resonant amplification.

Another factor must be considered, however. It is observed that the phases of the oscillations in the real atmosphere vary only slightly over the year with respect to solar time. Since the factor $\Omega$ appearing in the dynamical equations is the *sidereal* rotation rate of the earth, it follows that the periods of the pressure oscillations appearing in (11) are one sidereal day and one-half sidereal day respectively. The phases with respect to solar time of the oscillations described by (11) should therefore shift through 360 degrees during one solar year. Since only a small phase change during the year is actually observed, one must conclude that the contribution to the diurnal pressure variation from an eccentric vortex of solid rotation must be much smaller than the solar tidal or thermal influence.

The antisymmetric vortex with $n=2$ might conceivably play a part in the 20—30 km region. Evidence presented by SCHERHAG (1948) and KOCHANSKI and WASKO (1956) supports the notion of a circulation at these levels which is anticyclonic in the summer hemisphere and cyclonic in the winter hemisphere. Eccentricity of this flow should give rise to a 3-day cycle in the stream function at these upper levels.

The symmetric case with $n=3$, with its associated 6-day cycle, is at best only a rough approximation to the flow in the middle troposphere, but might be present as one component of a more complex flow. The case $n=5$ (with sense opposite to that given in Table I) might possibly be found in the lower levels of the troposphere, where the eccentricity would give rise to a 15-day periodicity in the pressure or geopotential.

As already pointed out by LA SEUR (1954), it must be recognized that on many occasions when eccentricity is well-marked the circulation pole is quasi-stationary. Obviously such situations cannot be explained in terms of the simple model and must be regarded as the result of orographic or thermal forcing, as discussed by FRENZEN (1956) and LA SEUR (1956).

It is of some interest to note that the analysis given above is not restricted to purely zonal spherical-harmonic streamline patterns but may be extended to include any wave pattern consisting of one or more tesseral-harmonic components plus one zonal-harmonic component in the eccentric coordinate system, provided that all of the harmonics have the same lower index. If the stream function in the eccentric system be written in the form

$$\psi = A + BP_n(\sin\phi) + \sum_{m=1}^{n} C_m P_n^m(\sin\phi)\, e^{im\lambda} \tag{12}$$

in which the $C_m$ may be complex, it is seen at once that this equation satisfies eq. (7). The direct transformation to geographic coordinates cannot be carried out on eq. (12) in a simple manner because the "addition theorem" of eq. (4) is not applicable to the associated Legendre polynomials. The author has been unable to locate or deduce the law of transformation for these functions. Fortunately, however, the proof of the extended analysis does not require the transformation but rests on the establishment of the invariance of certain quantities under the transformation.

It has just been stated that the stream function in eq. (12) satisfies the condition that

$$\nabla^2\psi = -n(n+1)\, a^{-2}\psi \tag{7}$$

Now $\psi$ is an absolute scalar field and hence $\nabla^2\psi$ is also an absolute scalar field; the numerical value of the Laplacian at each point is therefore independent of the choice of the coordinate system, as is the numerical value of the stream function itself. Therefore only the one value of the lower index $n$ may enter into the specification of the Laplacian, regardless of the coordinate system adopted. The geostrophic velocity vector $\mathbf{v}$ is an absolute vector, since it is computed from the surface gradient of the absolute scalar $\psi$. The quantity $\mathbf{v}\cdot\nabla(\nabla^2\psi)$ is therefore also an absolute scalar. Since it is clear from eq. (7) that this scalar vanishes in the eccentric coordinate system, and since it has been shown to be an absolute scalar, it must vanish in any other coordinate system. It has thus been proved that, whatever be the law of transformation of the associated Legendre polynomials, it cannot result in the introduction of polynomials with a lower index different from the value chosen for use in eq. (12). The scalar $\mathbf{v}\cdot\nabla f$ is likewise an absolute scalar, hence eq. (8) applies to the stream function of eq. (12) without modification. Since the conclusion regarding the precession rate of the pole is reached by (a) a coordinate transformation, followed by (b) solution of eq. (8) for the propagation velocity in the geographic system, and (c) performing the coordinate transformation which is the inverse of (a), the proof given above that eq. (8) is applicable to the set of harmonics in eq. (12) suffices; the law of transformation need not be specified.

The precession rate of the eccentric pole, $-2\Omega/n(n+1)$, is the same as the propagation speed relative to the geographical coordinates of harmonic waves with lower index $n$; hence the wave pattern is stationary in the eccentric system. The wave velocity as measured by an observer on the ground is seen to be unaffected by the choice of an eccentric coordinate system; only the amplitudes and phases of the various harmonics are altered by the choice of coordinates. The use of an eccentric pole may nevertheless be desirable on occasion in order to minimize the number and/or the amplitudes of the harmonic waves which might be involved in a particular computation.

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